CHAPTER 4

DECISION ANALYSIS USING EVIDENCE THEORETIC AND FUZZY SET THEORETIC FRAMEWORKS *

4.0 Introduction

In modeling real world problems involving uncertainty, the use of different uncertainty frameworks is essential. In the past, Probability Theory was the only available mathematical tool for quantifying all types of uncertainty. But in recent days, Zadeh's Fuzzy set Theory[88] has been found useful to deal quantitatively with uncertainty due to linguistic vagueness and imprecision, and Shafer's Evidence Theory [75, 5] has been proved to be useful to deal with non probabilistic uncertainty which arise due to limitations of evidence, and also to deal with probabilistic situations in which reliable estimates of the probabilities involved are not known completely.

Application of Fuzzy Set Theory and Statistical Decision Theory to the decision problems involving both fuzziness and randomness lead to a specific formulation of statistical fuzzy decision problems and this area was explored by several authors [21, 67, 68, 74]. Strat [80], proposed a decision analysis methodology for a

* Some of the results of this Chapter are to appear in the paper 'Equivalence class representation of fuzzy numbers-Application to Evidence Theory and Decision Analysis', The Journal of Fuzzy Mathematics, IFMI, U.S.A.
problem in decision theory, using Shafer's belief functions and the notion of expected
utility intervals. In [78], Smets introduced 'transferable belief model' (TBM)- a model
which represents quantified beliefs based on basic probability assignment functions.
In this approach, we use only the available information, but not more. We only
postulate the existence of basic probability assignment values, assigned to certain
subsets of the universe, each expressing the support given specifically to that subset
induced by the evidence, and that could be transferred to more specific subsets of the
universe if new pieces of evidence become available. A tool for decision making in
TBM is the pignistic probability function derived from the corresponding basic
probability assignment function.

In [18], Denœux gave an application of TBM to a decision problem in decision
theory, where utilities are represented in terms of fuzzy numbers and beliefs about
the crisp states of nature are represented by a fuzzy belief structure.

In this chapter, we modify the Denœux approach by extending the decision
problem with crisp states and 'without new information' to the case with fuzzy states
and 'with fuzzy new information'. Imprecision in the beliefs and degrees of beliefs
about the fuzzy states of nature is represented by a fuzzy number valued basic
probability assignment with fuzzy focal elements and imprecision in the utility values
is represented by fuzzy number valued utilities. We incorporate the new factual
information about the future states, in the decision process, by updating the prior basic probability assignment function using fuzzified Dempster's rule of conditioning. Equivalence class concept of fuzzy numbers is introduced for the ordering of the fuzzily evaluated expected utilities. The case with new hypothetical information is also discussed.

4.1 Preliminaries

The following are adopted from [71].

4.1.1 General decision problem in statistical decision theory

Basic data which define a statistical decision problem are the following.

1. Space of terminal acts \( A = \{ a \} \)

   The decision maker wishes to select a single act or decision 'a' from some domain \( A \) of potential acts or decisions.

2. State space \( \Theta = \{ \emptyset \} \)

   The decision maker believes that the consequence of adopting terminal act 'a' depends on some 'state of nature' which he cannot predict with certainty.

3. Family of experiments \( E = \{ e \} \)

   To obtain further information about the 'states' the decision maker may select a single experiment \( e \) from a family \( E \) of potential experiments.
4. Sample space $Z = \{ z \}$

Each potential outcome of a potential experiment $e$ will be labeled by an $z$ with domain $Z$.

5. Utility evaluation $U(e, z, a, \theta)$ on $E \times Z \times A \times \Theta$

The decision maker assigns a utility $U(e, z, a, \theta)$ to performing $e$, observing a particular $z$, taking a particular action ‘$a$’ and then finding that a particular $\theta$ obtains.

6. Probability assessment $p_{0, z}(\ldots/e)$ on $\Theta \times Z$

The general decision problem is, given $E, Z, A, \Theta, U$ and $p_{0, z}/e$, how should the decision maker choose an $e$ and then having observed $z$, choose an ‘$a$’ in such a way as to maximize the expected utility.

The following are from [78].

4.1.2 The transferable belief model (TBM)

The transferable belief model is a mathematical idealized model for representing the quantified beliefs held by an agent at a given time on a given frame of discernment. It concerns the same concepts as considered by the Bayesian model[3], except it doesn’t rely on probabilistic quantification, but on basic probability assignments. The TBM is based on a two level model: (1) a credal level where beliefs are entertained and quantified by basic probability assignment functions (2) a pignistic level (pignus = a bet(in Latin)) where beliefs can be used to make decisions and
are quantified by probability functions. The credal level precedes the pignistic level. At any time, beliefs are entertained and updated at the credal level, the pignistic level appears only when a decision needs to be made. When a decision must be made, beliefs at the credal level induce a probability measure at the pignistic level.

4.1.3 Pignistic Probability function (Bet Probability Function)

Let \( m \) be a basic probability assignment on \((X, \mathcal{P}(X))\). Then the pignistic probability function \( \text{BetP} \) derived from \( m \) is given by

\[
\text{BetP}(x) = \frac{\sum_{A : x \in A} m(A)}{|A|}
\]

\[
= \frac{\sum_{A \subset X : A \neq \emptyset} m(A)|x \cap A|}{|A|}
\]

where \(||\) denotes cardinality.

\( \text{BetP} \) is the basic tool for decision making in TBM.

4.1.4 Dempster's rule of conditioning [86]

Let \( m \) be a basic probability assignment on some powerset \( \mathcal{P}(X) \). Then the conditional basic probability assignment of \( A \) given that the truth is in \( B \), where \( A \cup B, \in \mathcal{P}(X) \) is given by,

\[
m_B(A) = K \Sigma \{ m(C) : B \cap C = A \}
\]

where \( K \) is the renormalisation constant independent of \( A \).
4.2 Decision analysis with crisp utility, crisp basic probability assignment, and factual fuzzy information

4.2.1 Formulation and analysis

We define a decision problem with fuzzy states of nature $\Theta_s$, $s = 1, \ldots, l$, all of which are defined on a universe of discourse $\Theta = \{\theta_1, \theta_2, \ldots, \theta_l\}$. Let the available information at hand about the future states be represented by a basic probability assignment function $m$ on the fuzzy powerset $\mathcal{F}(\Theta)$ with fuzzy focal elements $\tilde{F}_k^*, k = 1, \ldots, r$. Let the set of fuzzy actions or alternatives be $\tilde{A} = \{\tilde{A}_1, \ldots, \tilde{A}_n\}$. If we get any factual information regarding $\Theta$ from any reliable source, we can update the current basic probability assignment function $m$.

Here we conduct an error free experiment 'e' to get additional information about the future state. Let the resulted fuzzy outcome be $\tilde{Z}_i$ which induces a fuzzy subset $\tilde{F}_i$ of $\Theta$ such that $p(\tilde{F}_i / \tilde{Z}_i) = 1$, and $\tilde{F}_i$ serves to be an additional information about the future state. Since the experiment is actual and error free, the new information can be considered factual, and hence we incorporate the new information in the decision process by updating the current basic probability assignment $m$ on $\mathcal{F}(\Theta)$ to $m_1$ using Dempster's conditioning.

The updated $m_1$ is given by

$$m_1(\tilde{F}) = K \sum_j [m(\tilde{F}_j^* : \tilde{F}_j^* \cap \tilde{F} - \tilde{F})]$$  \hspace{1cm} (3)

where $K$ is the renormalisation constant independant of $\tilde{F}$. 

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Let the fuzzy focal elements of \( m_i \) be \( \tilde{F}_i \), \( i = 1, ..., r \).

We use the updated basic probability assignment function \( m_i \) to construct the bet probability function required for decision making.

The bet probability of \( \tilde{\theta}_s \) induced by \( m_i \), denoted \( \text{Bet} \ P_i(\tilde{\theta}_s) \) is

\[
\text{Bet} \ P_i(\tilde{\theta}_s) = K^1 \sum_{i=1}^{s} \frac{m_i(\tilde{F}_i) |\tilde{F}_i \cap \tilde{\theta}_s|}{|\tilde{F}_i|}, \quad s = 1, ..., l_0
\]

where \( K^1 \) is the renormalisation constant independent of \( \tilde{\theta}_s \).

Then the expected utility of selecting the alternative \( \tilde{A}_k \) is given by

\[
U^*(\tilde{A}_k) = \sum_s U(\tilde{A}_k, \tilde{\theta}_s) \text{Bet} P_i(\tilde{\theta}_s)
\]

where \( U(\tilde{A}_k, \tilde{\theta}_s) \) is the utility associated with \( (\tilde{A}_k, \tilde{\theta}_s) \)

\[
\text{Let } \max_k U^*(\tilde{A}_k) = U^*(\tilde{A}_i)
\]

Then the optimum decision will be to choose the act \( \tilde{A}_i \) which maximizes the expected utility.

4.2.2 Remark

In (4), \( K^1 \) becomes 1 with fuzzy intersection as the 'algebraic product' and the fuzzy states \( \tilde{\theta}_s \) as orthogonal fuzzy states.
\[ \sum_{s=1}^{b} \tilde{\theta}_s(\theta_k) = 1, \quad k=1, \ldots, l \quad (7) \]

**Proof:**

We have \[ \sum_{s=1}^{b} \text{Bet}_r(\tilde{\theta}_s) = 1 \]

i.e., \( k' \left[ \sum_{s=1}^{b} \sum_{i=1}^{\eta_0} m_i(\tilde{F}_u) \left( \sum_k \tilde{F}_u(\theta_k) \tilde{\theta}_s(\theta_k) \right) / \sum_k \tilde{F}_u(\theta_k) \right] = 1 \)

i.e., \( k' \left[ \sum_{i=1}^{\eta_0} m_i(\tilde{F}_u) \sum_{s=1}^{\lambda_0} \left( \sum_k \tilde{F}_u(\theta_k) \tilde{\theta}_s(\theta_k) \right) \right] = 1 \)

i.e., \( k' \left[ \sum_{i=1}^{\eta_0} m_i(\tilde{F}_u) \sum_k \tilde{F}_u(\theta_k) \right] = 1 \) \(( \because \sum_{s=1}^{\lambda_0} \tilde{\theta}_s(\theta_k) = 1)\)

i.e., \( k' \left[ \sum_{i=1}^{\eta_0} m_i(\tilde{F}_u) \right] = 1 \)

i.e., \( k' \cdot 1 = 1 \) \(( \because \sum_{i=1}^{\eta_0} m_i(\tilde{F}_u) = 1)\)

\( \therefore k' = 1 \)
4.3 Decision analysis using fuzzy number valued utility, fuzzy number valued basic probability assignment and factual fuzzy information

4.3.1 Formulation and Analysis

In the decision analysis that we have described, let the utility and initial basic probability assignment be represented by fuzzy numbers. We confine them into corresponding equivalence classes as described in Chapter 3. Then the resulting updated fuzzy number valued basic probability assignment \( \tilde{m}_i \) and the fuzzy number valued bet probability function \( \tilde{\text{Be}}\tilde{P}_i \) are also get represented by equivalence classes of fuzzy numbers. The renormalisation constants \( K \) and \( K^1 \) in (3) and (4) should be represented by equivalence classes of fuzzy numbers such that

\[
\sum_i \tilde{m}_i(\tilde{F}_i) = \tilde{1}
\]

and

\[
\sum_s \tilde{\text{Be}}\tilde{P}_i(\tilde{\theta}_s) = \tilde{1}
\]

The resulting fuzzy expected utility of each action \( \tilde{A}_k \) is an equivalence class of fuzzy numbers given by

\[
\tilde{U}^*(\tilde{A}_k) = \sum_s \tilde{U}(\tilde{A}_k, \tilde{\theta}_s) \cdot \tilde{\text{Be}}\tilde{P}_i(\tilde{\theta}_s) ; \quad k = 1, ..., n
\]

where \( \tilde{U}(\tilde{A}_k, \tilde{\theta}_s) \) is the equivalence class representing the fuzzy number valued utility associated with \((\tilde{A}_k, \tilde{\theta}_s)\). To find the alternative which leads to the maximum expected utility, we have to specify some total ordering among the
equivalence classes representing the fuzzy expected utilities. According to the results defined in Chapter 3, two equivalence classes of fuzzy numbers are comparable if the \( \alpha \)-cuts of the respective equivalence classes are comparable for every \( \alpha \in (0, 1] \). This method facilitates the comparison of the fuzzy expected utilities and we admit indeterminacy when two equivalence classes are incomparable.

4.3.2 Remark

In the above decision analysis, the equivalence class concept was introduced for the typical representation of normalised fuzzy number valued basic probability assignment functions. To carry out further arithmetical operations in the decision process, fuzzy number valued utilities were also represented by equivalence classes of fuzzy numbers. Consequently, the expected utilities were also represented by equivalence classes of fuzzy numbers. As the multiplication of equivalence classes of fuzzy numbers is not conformal with the usual multiplication of fuzzy numbers, the equivalence class corresponding to an expected utility need not represent the actual fuzzy number representing the expected utility. But since we require only the rank ordering of the expected utilities, the equivalence class representation is good enough for the purpose.
4.4 Decision analysis with hypothetical information
in the case of a family of experiments

In this section we extend the decision problem described in section 4.2 to the
case of a family of experiments.

From a family of experiments \( \{e\} \), each providing additional information
regarding the future states, suppose we have to select the experiment having
maximum expected utility. We shall denote a fuzzy outcome of the experiment \( 'e' \) by
\( \tilde{Z}_{et} \) with probability of occurrence \( p(\tilde{Z}_{et}) \). Assume that, each fuzzy outcome \( \tilde{Z}_{et} \) of \( e \)
determines a fuzzy subset \( \tilde{F}_{et} \) of \( \Theta \) such that \( p(\tilde{F}_{et} / \tilde{Z}_{et}) = 1 \) and \( \tilde{F}_{et} \) serves to be an
additional information about the future state. Since this new information is only
hypothetical and not factual, we may derive \( \text{BetP} \) from the non updated basic
probability assignment function corresponding to the future states and condition \( \text{BetP} \)
on the new information using classical probabilistic conditioning. Assuming
\( U(e, \tilde{Z}_{et}, \tilde{A}_k, \tilde{\theta}_s) \) as the utility associated with \( (e, \tilde{Z}_{et}, \tilde{A}_k, \tilde{\theta}_s) \), the expected utility
of selecting the alternative \( \tilde{A}_k \) is then given by

\[
U^*(e, \tilde{Z}_{et}, \tilde{A}_k) = \sum_s U(e, \tilde{Z}_{et}, \tilde{A}_k, \tilde{\theta}_s) \text{BetP}(\tilde{A}_k / \tilde{Z}_{et})
\]  

(8)

where \( \text{BetP}(\tilde{A}_k / \tilde{Z}_{et}) = K' \frac{\text{BetP}(\tilde{A}_k \cap \tilde{F}_{et})}{\text{p}(\tilde{Z}_{et})} \)

(9)

where \( K' \) is the renormalization constant such that \( \sum_s \text{BetP}(\tilde{A}_k / \tilde{Z}_{et}) = 1 \)

Let \( U^*(e, \tilde{Z}_{et}) = \max_k U^*(e, \tilde{Z}_{et}, \tilde{A}_k) \)

(10)

Then the expected utility of \( 'e' \) is given by
After calculating $U^*(e)$ for each experiment 'e', choose the 'e' for which $U^*(e)$ is the greatest. After selecting e and observing the outcome $\tilde{Z}_{et}$ which actually materialises, the optimum decision will be to choose the act 'a' that maximises the expected utility.

The above procedure can be extended to the case with fuzzy number valued utilities and fuzzy number valued basic probability assignments.

4.5 Value of Information

The difference between the maximum expected utilities, with and without new information can be taken as a measure for assessing the value of new information [71, 74].

\[ V(e) = U^*(e) - U^*(e_0) \]

where $U^*(e)$ is the maximum expected utility with new information obtained by conducting the experiment e and $U^*(e_0)$ is the maximum expected utility without any new information ($e_0$-denotes the null experiment).

When $U^*(e)$ and $U^*(e_0)$ are represented by equivalence classes of fuzzy numbers, then $V(e)$ is also get represented by equivalence class of fuzzy numbers.