Chapter III

MATHE TRANSFORMATIONS BETWEEN SOME SEQUENCE SPACES

3.1. Definitions and Notations. Let \( p = \{ p_k \} \) be a sequence of strictly positive numbers with \( \sup_k p_k < \infty \). Then we shall be interested in the following classes of sequences 
\( x = \{ x_k \} \) of complex numbers (see [25], [10] and [11])

\[
\ell(p) = \left\{ x : \sum_k |x_k|^{p_k} < \infty \right\},
\]

\( w_0(p) = \left\{ x : \sum_k |x_k|^{\frac{1}{p_k}} < \infty \right\},
\]

\( w(p) = \left\{ x : \frac{1}{\alpha} \sum_{k=1}^{\infty} |x_k|^{\frac{1}{p_k}} \rightarrow 0 \text{ for some } \alpha \in \mathbb{C} \right\}.
\]

If \( \bar{p} = \sup_k p_k \) and \( p' = \max(1, \bar{p}) \), the space \( \ell(p) \) is linear metric space paranormed by 
\( \| x \| = \left( \sum_k |x_k|^{p_k} \right)^{1/p'} \)

and the set \( w(p) \) is a linear metric space with metric function

\( \| x \|_{w'} = \sup_{j \leq 2^{j+1}} \left\{ \sum_{k \in z_j} |x_k|^{p_k} \right\}^{1/p'} \),

where \( z_j \) is sum over \( 2^j \leq k \leq 2^{j+1} \), and it has been proved that \( w(p) \) is complete [7].

If \( p_k = p \) for every \( k \), we have \( \ell(p) = \ell_p \), \( w_0(p) = \ell_1 \)
and \( w(p) = w_p \). \( \ell_p \) and \( w_p \) are banach space for \( 1 \leq p < \infty \) and complete \( p \)-normed space for \( 0 < p < 1 \).
3.2. Introduction. In this chapter, we characterize the
matrices of the classes \((\ell(p), V_\circ), (V_\circ(p), V_\circ), (w(p), V_\circ)\),
which extend the classes \((\ell(p), c), (w(p), c)\) and \((w(p), c)\) due to
Lascaris and Randix [7] and Randix [12], and contain
the results due to Haddy [17] as special cases. Using the same
notations as in Chapter II, for every nonnegative integer \(n\) and
\(n \geq 1\),

\[
\sum_n (x) = \sum_{k=1}^{\infty} t(n,k,x) x_k,
\]

where

\[
t(n,k,x) = \sum_{j=0}^{\infty} a_j (c_j(x), x) / (x+1).
\]

3.3. We establish the following theorems.

THEOREM 1(a). (see [15], Theorem 5.1). Let \(1 < p_k < \infty\) for
every \(k\). Then \(x \in (\ell(p), V_\circ)\) if and only if

(i) there exists an integer \(n \geq 1\) such that, for every \(n\),

\[
\sup_{m \geq 1} |t(n,k,m)| \leq x_k \to k < \infty, (p_k^{-1} + q_k^{-1} = 1); \quad \text{and}
\]

(ii) \(a_k(x) \in V_\circ\), for each \(k\).

In this case, the \(\circ\)-limit of \(\sum_{k} x_k \circ k\) for every \(x \in \ell(p)\).
(b) Let \( 0 < p_k \leq 1 \) for every \( k \). Then \( \alpha \in \left( l(p), v_\infty \right) \) if and only if

(i) \( \sup_{n,k} |t(n, k, m)|^{p_k} < \infty \),

(ii) \( a(k) \in v_\infty \), for each \( k \).

THEOREM 2. (see [6]), \( \alpha \in \left( l(p), v_\infty \right) \) if and only if

(i) for every integer \( n \geq 1 \)

\[ C_n = \sup_{n,k} |t(n, k, m)|^{1/p_k} < \infty, \text{ for every } n \]

and

(ii) \( a(k) \in v_\infty \), for each \( k \).

In this case, the \( \alpha \)-limit of \( \lambda x \) is same as in Theorem 1.

THEOREM 3. (see [16]). Let \( 0 < p_k \leq 1 \), then

\( \alpha \in \left( w(p), v_\infty \right) \) if and only if

(i) there exists an integer \( n \geq 1 \), such that for every \( m \)

\[ n_m = \sup_{m} \max_{n \geq 1} \left( 2^{n-1} \right)^{1/p_k} |t(n, k, m)| \leq \infty, \]

(ii) \( a(k) \in v_\infty \), for each \( k \),

and

(iii) \( a \in v_\infty \).
In this case, the $\omega$-limit of $\{x_n\}$ is, for every $x \in \omega(p)$,

$$\lim_{n \to \infty} \left[ u_n - \sum_{k} u_k x_k \right] = 0.$$  

Theorem 4(a). (see [16]). Let $1 \leq p < \infty$, then

$\phi \in \omega_{\omega} \iff$

1. $\phi(r) = \sup_{r \in \mathbb{R}} \left\{ \frac{\|t_n(p)\|}{\|t_n(p)\|^p} \right\}^{1/p} < \infty$, for every $n$, $\|\cdot\|^p \cdot \|\cdot\|^q = 1$.

2. $\phi(k) \in \omega_{\omega}$, for each $k$;

and

3. $\phi \in \omega_{\omega}$.

In this case, the $\omega$-limit of $\{x_n\}$ is the same as in Theorem 3.

(b) Let $0 < p < \infty$. Then $\phi \in \omega_{\omega}$ if and only if conditions (1), (ii) with $\omega$-limit are, and (iii) with $\omega$-limit of Theorem 4(a) hold.

3.4. **Proof of Theorem 1(a).** Suppose that $\phi \in \omega_{\omega}$. Necessity of (ii) is obvious, since $\phi \in \omega_{\omega}$, since $f_{mn}(x)$ exists for each $n$ and $x \in \omega(p)$. Therefore, $\left\{ f_{mn} (x) \right\}$ is a sequence of continuous real functionals on $\omega(p)$ and further on $\omega(p)$.

$$\sup_{n} |f_{mn} (x)| < \infty.$$
Now, condition (i) follows by arguing with uniform boundedness principle.

Conversely, suppose that the condition (i) and (ii) are satisfied and \( x \in L^p(\mu) \). Now we have, for every \( r > 1 \),

\[
\sum_{k=1}^{\infty} |t(n, k, m)| \alpha_k r^{-\alpha_k} \leq \sup \left\{ \left\langle t(n, k, m) \right\rangle \alpha_k r^{-\alpha_k} \right\}
\]

and therefore

\[
\sum_{k=1}^{\infty} |u_k| \alpha_k r^{-\alpha_k} = \lim \lim \sum_{k=1}^{\infty} |t(n, k, m)| \alpha_k r^{-\alpha_k} \leq \sup \left\{ \left\langle t(n, k, m) \right\rangle \alpha_k r^{-\alpha_k} \right\} < \infty.
\]

Thus the series \( \sum_{k=1}^{\infty} u_k x_k \) converges for each \( m \) and \( x \in L^p(\mu) \). For a given \( \varepsilon > 0 \) and \( x \in L^p(\mu) \), choose \( k_0 \) such that

\[
(iii) \quad \left( \frac{1}{r} \sum_{k=k_0+1}^{\infty} |x_k|^r \right)^{1/r} < \varepsilon,
\]

where \( M = \sup |x_k| \). Since (ii) holds, therefore there exists \( n_0 \) such that, for every \( n > n_0 \),

\[
\left| \sum_{k=1}^{n} \left( t(n, k, m) - u_k \right) \right| < \varepsilon,
\]

by equation (iii) it follows that

\[
\left| \sum_{k=k_0+1}^{\infty} \left( t(n, k, m) - u_k \right) \right| < \varepsilon
\]

is arbitrary small, therefore
\[\lim_{m \to \infty} t(n, k, m) x_k = \sum_{k} u_k x_k,\]

uniformly in \(n\), and hence the proof is complete.

(b) The case \(0 < p_k < 1\) has similar proof.

5.5. Proof of Theorem 2. Suppose that \(\lambda \in \mathcal{C}_0(p, \mathbb{R})\).

Since \(\lambda \in \mathcal{C}_0(p)\), therefore the necessity of (i) is obvious.

We suppose that (i) is not true. Then there exists \(S > 1\), such that \(\mathcal{C}_p = \infty\). Therefore, by Theorem 1, we have the matrix

\[C = (c_{nk}) = (a_{nk}^{1/p_k}) \in L_1, \mathbb{R}_+\], i.e. \(a_{nk}^{1/p_k}\) for all \(k \in \mathbb{Z}\). Now, the sequence \(y = (y_k) = (a_{nk}^{-1/p_k}) \in \mathcal{C}_0(p)\). Set \(y = x \in \mathcal{C}_0(p)\), which contradicts the fact that \(\lambda\) is \((\mathcal{C}_0(p), \mathbb{R}_+)\)-matrix. Hence (i) is true.

Conversely, suppose that the conditions (i) and (ii) are satisfied and \(x \in \mathcal{C}_0(p)\). Then

\[\left| \frac{1}{p_k} t(n, k, m) x_k \right| \leq \frac{1}{p_k} |x_k|^{1/p_k} \left| t(n, k, m) \right|^{1/p_k} \leq M\]

Now, arguing as in Theorem 1, we have

\[\lim_{m \to \infty} t(n, k, m) x_k = \sum_{k} u_k x_k,\]

uniformly in \(n\), which completes the proof.
3.6. **Proof of Theorem 3.** Suppose that $A \in (w(p), V)$ and that $z \in w(p)$. Since $e^k$ and $e$ are in $w(p)$, necessity of (ii) and (iii) is obvious. Now, we know that

$$\lim_{k \to \infty} t(n,k,m) x_k$$

converges for each $m$ and $x \in w(p)$, therefore $(t(n,k,m))_{k} \in w^t(p)$ and

$$\max \{ z^{-1} f_{n,m}(x) \mid t(n,k,m) \mid < \infty \}$$

for each $m$ (see [7]). Furthermore, \[ \{ f_{n,m} \}_{n} \] is a sequence of continuous linear functionals on $w(p)$ such that $\lim_{n \to \infty} f_{n,m}(x)$ exists uniformly in $m$. Therefore, by Banach-Steinhaus theorem, necessity of (i) follows immediately.

Conversely, suppose that the conditions (1'), (ii) and (iii) hold and $z \in w(p)$. We know that $(t(n,k,m))_{k}$ and $(v_k)$ are in $w^t(p)$, whence the series \( \sum_{k} t(n,k,m) x_k \) and \( \sum_{k} v_k x_k \) converges for each $m$. Put

$$c(n, k, m) = t(n, k, m) - v_k.$$  

Then

$$t(n,k,m) x_k = v_k x_k + \sum_{k} c(n,k,m) + \sum_{k} c(n,k,m) (x_k - x)$$

by (ii) for an integer $k_0 > 0$ we have
\[
\lim_{n \to \infty} \frac{r}{k} c(n, k, m) (x_k - l) = 0,
\]
where \( l \) is the limit of \( x \) for \( x \in \mathbb{R}^n \). Now, since
\[
\sup_{r>0} \max_{n \geq 1} \frac{r}{k} |c(n, k, m)| \leq 2 T_n
\]
and
\[
\lim_{n \to \infty} \frac{r}{k} \prod_{i=1}^{m} |t(n, k, m)| \frac{1}{x_k^i} = 0,
\]
then
\[
\lim_{n \to \infty} \frac{r}{k} t(n, k, m) x_k = (\lim_{k} \frac{r}{k} u_k) x_k = (\lim_{k} \frac{r}{k} u_k) x_k.
\]

3.7. **Proof of Theorem 4(a).** Suppose that \( l \in (w_p, \ell_p) \).

Since \( l \) and \( c \) are in \( w_p \), (ii) and (iii) must hold. For the necessity of (i), suppose that \( f_{mt}(x) = \frac{r}{k} t(n, k, m) x_k \) exists for each \( n \) whenever \( x \in w_p \). Then, for each \( n \) and \( k \geq 0 \),

define \( f_{mt}(x) = \frac{r}{k} t(n, k, m) x_k \). Then \( \{ f_{mt} \}_{m} \) is a sequence of continuous linear functionals on \( w_p \), since

\[
|f_{mt}(x)| \leq \frac{1}{k} \prod_{i=1}^{m} \left| t(n, k, m) \right|^q \left( x_k^i \right)^{1/q} \left( \frac{r}{k} |x_k|^p \right)^{1/q}
\]

\[
\leq \frac{r^{p/q}}{k} \prod_{i=1}^{m} \left| t(n, k, m) \right|^q \left( \frac{r}{k} |x_k|^p \right)^{1/q} \| x \|
\]

It follows from Theorem 1.10, that for each \( n \)
\[
\lim_{j \to \infty} f_{\text{rep}}(x) = f_{\text{vm}}(x)
\]

is in the dual space \( w_0' \), whence there exists \( K_{\text{vm}} \) such that

\[
(1) \quad |f_{\text{vm}}(x)| \leq K_{\text{vm}} \|x\|.
\]

For each \( m \), we take any integer \( j > 0 \) and define \( x \in w_0' \) in \([12], \text{ Theorem 7, pp 173}\), then we get

\[
\sum_{r=0}^{\infty} 2^{r/j} \left( \frac{1}{r} \sum_{k=0}^{r} |t(n, k, m)|^q \right)^{1/q} \leq K_{\text{vm}} < \infty.
\]

For each \( m \)

\[
(2) \quad \sum_{r=0}^{\infty} 2^{r/j} \left( \frac{1}{r} \sum_{k=0}^{r} |t(n, k, m)|^q \right)^{1/q} \leq K_{\text{vm}} < \infty.
\]

Now

\[
|f_{\text{vm}}(x)| \leq \sum_{r=0}^{\infty} 2^{r/j} \left( \frac{1}{r} \sum_{k=0}^{r} |t(n, k, m)|^q \right)^{1/q} \leq \sum_{r=0}^{\infty} 2^{r/j} \left( \frac{1}{r} \sum_{k=0}^{r} |t(n, k, m)|^q \right)^{1/q} \left( \|x_k\|^p \right)^{1/p} < \infty.
\]

Therefore \( f_{\text{vm}}(x) \) is absolutely and uniformly convergent for each \( m \), we have

\[
|f_{\text{vm}}(x)| \leq \sum_{r=0}^{\infty} 2^{r/j} \left( \frac{1}{r} \sum_{k=0}^{r} |t(n, k, m)|^q \right)^{1/q} \|x\|,
\]
so that

\[(4) \quad K_{rm} \leq \sup_{r=0}^{\infty} \left( \frac{r}{r+1} \right)^{1/q} \left( \frac{t(n, k, m)}{r+1} \right)^{1/q}.\]

By virtue of (2) and (4)

\[K_{rm} = \sup_{r=0}^{\infty} \left( \frac{r}{r+1} \right)^{1/q} \left( \frac{t(n, k, m)}{r+1} \right)^{1/q}.\]

Finally, by Theorem 1.19, for every \(n\), the existence of \(\lim_{m} f_{rm}(x)\) on \(w_{p}\) implies that

\[
\sup_{m} K_{rm} = \sup_{m} \sup_{r=0}^{\infty} \left( \frac{r}{r+1} \right)^{1/q} \left( \frac{t(n, k, m)}{r+1} \right)^{1/q} < \infty,
\]

which completes the proof of (i).

Conversely, suppose that the conditions are satisfied and \(x \in w_{p}\). Since, by virtue of (5), \(f_{rm}(x)\) is absolutely and uniformly convergent for each \(m\). Note that (i) and (ii) together imply that

\[
\sup_{r=0}^{\infty} \left( \frac{r}{r+1} \right)^{1/q} \left( \frac{t(n, k, m)}{r+1} \right)^{1/q} \leq M(n) < \infty,
\]

and by Hölder's inequality, \(\sum_{k} x_{k} < \infty\). Now, arguing as in Theorem 3, it follows that \(x \in (w_{p}, V_{o})\).

(b). If \(x \in (w_{p}, V_{o})\), then \(\omega\)-limit \(x_{k} \rightarrow o = \omega\)-limit \(e_{k}\).

\(\omega\)-limit \(A_{n} = 1 = \omega\)-limit \(a_{k}\) and condition (i) hold. Conversely, suppose that conditions are satisfied. Then, by Theorem 4(a),
A \in (w_0, v_0) \) and \( \lim_{x \to \infty} (x) = 1 = \lim \frac{\mu}{\nu} x, \) since \( \nu = \tau \) and \( \nu_k = 0 \) for each \( k. \) Hence \( A \) is the matrix of the class \((w_0, v_0)_{\text{reg}}.\)

3.3. **Remark.** For \( 0 < p < \infty, \) it is quite obvious that

\[
(w_0, v_0)_{\text{reg}} \cap (w_0, v_0) = \emptyset,
\]

since the conditions \( \sigma - \lim \nu = \infty \) and

\[
\lim_{n \to \infty} \sum_{k=1}^{n} a(\sigma - j(n), k) = 0
\]

are incompatible.