CHAPTER VI

MATHEMATICAL UNIQUENESS OF TRIGONOMETRIC SERIES

6.1. Definitions and Notations. Let $f$ be $L$-integrable and periodic with period $2\pi$, and let the Fourier series of $f$ be

$$
\frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right).
$$

Then the series conjugate to it is

$$
\sum_{k=1}^{\infty} \left( b_k \cos kx - a_k \sin kx \right) = \sum_{k=1}^{\infty} g_k(x),
$$

and the derived series is

$$
\sum_{k=1}^{\infty} \left( \frac{d}{dx} (b_k \cos kx - a_k \sin kx) \right).
$$

We write

$$
f_x(t) = f(t, t) = \begin{cases} 
  f(x+t) - f(x-t), & 0 < t \leq \pi, \\
  g(x), & t = 0
\end{cases}
$$

where

$$
g(x) = \left\{ f(x+\epsilon) - f(x-\epsilon) \right\},
$$

and

$$
\lambda_x(t) = \frac{f_x(t)}{4 \sin \frac{t}{2}}.
$$

6.2. Introduction. The purpose of this chapter is to generalize the following theorem of UAG [22] by using $\alpha$-regular matrices, as we know that the notion of $\alpha$-regularity generalizes
the notion of almost regularity. We also prove an analogous result for conjugate vectors.

**Theorem 1.** (see [22]). Let \( A = (a_{nk}) \) be an almost regular infinite matrix of real numbers. Then, for every \( x \in [-\pi, \pi] \) for which \( h_x(t) \) is of bounded variation on \([0, \pi]\),

\[
\lim_{n \to \infty} \frac{1}{p} \sum_{k=0}^{p-1} a_{nk} \sin \left( (k+\frac{1}{2}) t \right) = h_x(t),
\]

uniformly in \( n \), if and only if

\[
\lim_{n \to \infty} \frac{1}{p} \sum_{k=0}^{p-1} a_{nk} \sin \left( (k+\frac{1}{2}) t \right) = 0,
\]

for every \( t \in [0, \pi] \), uniformly in \( n \).

6.1. We establish the following theorems.

**Proof 1.** Let \( A = (a_{nk}) \) be a \( \sigma \)-regular matrix. Then, for each \( x \in [-\pi, \pi] \) for which \( h_x(t) \in L^1 \) \([0, \pi]\)

\[
\lim_{p \to \infty} \frac{1}{p} \sum_{k=0}^{p-1} a(\sigma^p(n), k) \sin(k x) = h_x(t),
\]

uniformly in \( n \), if and only if

\[
\lim_{p \to \infty} \frac{1}{p} \sum_{k=0}^{p-1} a(\sigma^p(n), k) \sin(k x) = 0,
\]

for every \( x \in [0, \pi], \) uniformly in \( n \).
THEOREM 2. Let \( A = (a_{jk}) \) be a \( \sigma \)-regular matrix. Then for each \( x \in [0, 2\pi] \) for which \( f(x) \in W [0, 2\pi] \),

\[
(2.1) \quad \lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{p} a(\sigma^j(n), k) c_k(x) = c^{-1} g(x)
\]

uniformly in \( n \), if and only if

\[
(2.2) \quad \lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{p} a(\sigma^j(n), k) \cos k t = 0,
\]

for all \( t \in [0, \pi] \), \( t > 0 \), uniformly in \( n \), where by

\( f(x) \in W [0, 2\pi] \), we mean \( f(x) \) is of bounded variation on \([0, 2\pi] \).

6.4. Lemma. We need the following lemma for the proof of our theorems.

Lemma 1 (see [23]). Necessary and sufficient conditions for the matrix \( A \) to be \( \sigma \)-regular are

1. \( \| A \| < \infty \),

2. \( \lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{p} a(\sigma^j(n), k) = 0 \), uniformly in \( n \),

for each \( k \),

3. \( \lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{p} a(\sigma^j(n), k) = 1 \), uniformly in \( n \).
Lemma 2 (Jordan's Convergence criterion for Fourier series, see [2]).

If \( f(x) \) is of bounded variation in some interval \((a,b)\), then its Fourier series converges at every point of this interval. Its sum is \( f(x) \) at a point of continuity and 
\[
\frac{[f(x+c) - f(x-c)]}{2}
\]
at a point of discontinuity. Finally, if \((a',b')\) lies entirely inside the interval \((a,b)\), where \( f(x) \)
is continuous, then the Fourier series converges uniformly in \((a',b')\).

6.5. Proof of Theorem 1. Partial sum of the Carried Fourier series of the function \( f \) is given by

\[
x_k(x) = \frac{1}{\pi} \int_0^\pi f_x(t) \left( \frac{k}{\pi} \tan \frac{\pi t}{2} \right) dt
\]

\[
= \frac{1}{\pi} \int \frac{1}{2} \sin \left( k + \frac{1}{2} \right) t \sin \frac{t}{2} dt
\]

\[
= I_k + \frac{2}{\pi} \int_0^\pi \sin \left( k + \frac{1}{2} \right) t \cos \frac{t}{2} h_x(t)
\]

where

\[
I_k = \frac{1}{\pi} \int_0^\pi h_x(t) \frac{\sin \left( k + \frac{1}{2} \right) t}{\tan \frac{t}{2}} dt.
\]

Since \( h_x(t) \in W [0,\pi] \) and \( h_x(t) \to h_x(\pi) \) as \( t \to \pi \), \( h_x(t) \cos \frac{1}{2} t \) has the same properties. Therefore, by Lemma 2
\[ I_k = h_x (0) \text{, as } k \to \infty. \]

Now
\[
\frac{1}{p+1} \sum_{j=0}^{\infty} a (\sigma^j (n), k) a_k ^j (x) = \frac{1}{p+1} \sum_{j=0}^{\infty} a (\sigma^j (n), k) I_k
\]
\[ \quad + \int_0^1 \left[ \frac{1}{p+1} \sum_{j=0}^{\infty} a (\sigma^j (n), k) \sin \left( x + \frac{1}{2} t \right) \right] \, \text{d} K (t). \]

\[ = J_1 + J_2 \text{, say.} \]

Since \( n \) is \( \sigma \)-regular, by virtue of condition (3) of Lemma 1,

\[ J_1 = h_x (\infty) \text{, as } p \to \infty, \text{ uniformly in } n. \]

Thus we have to show that (ii) holds if and only if

\[ J_2 = 0 \text{, as } p \to \infty, \text{ uniformly in } n. \]

Now, \( f \) being \( \sigma \)-regular, i.e. for all \( x \in C \), \( f \in \mathcal{C} \). But \( \mathcal{C} \subset \mathcal{C} \). Therefore \( Ax \in \mathcal{C} \). Since there exists a constant \( B \) such that for all \( n \),

\[ \sup \left( \frac{1}{p+1} \sum_{j=0}^{\infty} a (\sigma^j (n), k) \right) < B. \]
Therefore

\[(1.4) \quad \left| \frac{1}{p+1} \sum_{j=0}^{p} a(\gamma_j(n), k) \sin \left( k \cdot \frac{1}{p+1} \right) \right| \leq \left| \frac{1}{p+1} \sum_{k=1}^{p} \sin \left( k \cdot \frac{1}{p+1} \right) \right| \cdot \left| \frac{1}{p+1} \sum_{j=0}^{p} a(\gamma_j(n), k) \right| \leq \left| \frac{1}{p+1} \sum_{k=1}^{p} \sin \left( k \cdot \frac{1}{p+1} \right) \right| \cdot \frac{1}{p+1} \sum_{j=0}^{p} a(\gamma_j(n), k) \leq \frac{1}{p+1} \sum_{j=0}^{p} \frac{1}{p+1} \sum_{k=0}^{p} a(\gamma_j(n), k) \leq \frac{1}{p+1} \sum_{j=0}^{p} \frac{1}{p+1} \sum_{k=0}^{p} a(\gamma_j(n), k) \leq B \quad \text{for all } p \quad \text{and } n \quad \text{(by (1.3))}.\]

Hence, by a theorem on the weak convergence of sequences in the Banach spaces of all continuous functions defined on a finite closed interval (see Reference [1], pp. 134 - 135), it follows that (1.4) holds if and only if \( J_2 = 0 \), as \( p \to \infty \) and (1.2) holds.

Since (1.4) is satisfied, it follows that \( J_2 = 0 \), if and only if (1.2) holds.

This completes the proof of Theorem 1.

6.6. Proof of Theorem 2. The partial sum of the series conjugate to the Fourier series of the function \( f \), is given by
\[ s_k(x) = \frac{1}{\pi} \int_0^\pi f_x(t) \sin kt \, dt \]

\[ = \frac{g(x)}{k} + \frac{1}{\pi} \int_0^\pi \cos kt \, d f_x(t), \]

Therefore

\[ \frac{1}{p+1} \sum_{j=0}^{p} \sum_{k=1}^{\infty} a(\alpha^j(n), k) \zeta_j(x) = I_1 + I_2, \]

where

\[ I_1 = \frac{1}{p+1} \sum_{j=0}^{p} \sum_{k=1}^{\infty} a(\alpha^j(n), k) g(x) / \pi, \]

and

\[ I_2 = \frac{1}{\pi} \int_0^\pi \left[ \frac{1}{p+1} \sum_{j=0}^{p} \sum_{k=1}^{\infty} a(\alpha^j(n), k) \cos kt \right] d f_x(t). \]

By virtue of condition (3) of Lemma 4,

\[ I_1 \to g(x) / \pi, \quad \text{as } p \to \infty, \quad \text{uniformly in } n. \]

Thus, we have to show that

\[ (2.3) \quad I_2 = \frac{1}{\pi} \int_0^\pi d f_x(t) K_{pa}(t) \to 0, \quad \text{uniformly in } n, \quad \text{as } p \to \infty, \]

where

\[ (2.4) \quad K_{pa}(t) = \frac{1}{p+1} \sum_{j=0}^{p} \sum_{k=1}^{\infty} a(\alpha^j(n), k) \cos kt. \]

Now, it is easy to show that condition (2.3) is equivalent to the following condition
\[(2.5) \quad \frac{1}{n} \sum_{j=0}^{n-k} a(c_j(n), k) \cos kt \leq K, \quad 0 < r < \infty, \quad r > 1, \quad n \geq 1, \quad 0 < \delta < \pi, \quad \text{for all } n \text{ and } p.\]

Therefore, arguing as in Theorem 1, we have

\[(2.6) \quad |x_{mn}(t)| = \left| \frac{1}{n} \sum_{j=0}^{n-k} a(c_j(n), k) \cos kt \right| < K, \quad 0 < r < \infty, \quad n \geq 1, \quad 0 < \delta < \pi, \quad \text{for all } n \text{ and } p.\]

Hence, by a theorem on the weak convergence of sequences in the Banach spaces of all continuous functions \([0, 2\pi] \rightarrow \mathbb{R}\), it follows that (2.5) holds if and only if

(i) \[|x_{mn}(t)| \leq K, \quad 0 < r < \infty, \quad n \geq 1, \quad 0 < \delta < \pi, \quad \text{for all } n \text{ and } p, \]

and

(ii) (2.6) holds.

Since, by virtue of condition (2.6), (i) is true. It follows that (2.5) holds if and only if (2.6) holds, and (2.5) is equivalent to (2.6), hence (2.1) holds if and only if (2.5) holds.

This completes the proof of Theorem 2.
6.7. COROLLARY.

Let $A = (a_{mk})$ be almost regular matrix. Then, for each $x \in [0, 2\pi]$ for which $f(x) \in MW [0, 2\pi]$,

$$\lim_{p \to \infty} \frac{1}{p^{p+1}} \sum_{j=0}^{p-1} a_{m+j,k} e^{ikf(x)} = 0,$$

uniformly in $n$, if and only if

$$\lim_{p \to \infty} \frac{1}{p^{p+1}} \sum_{j=0}^{p-1} a_{m+j,k} \cos kt = 0,$$

for all $0 < k < \infty$, uniformly in $n$. 