Chapter IV

COMPLEX PARTIALLY SPEACE AND INFINITE PRODUCTS

4.1. Definitions and Notations. Recently, HMDA [3] has extended the spaces \( \hat{c} \) and \( \hat{c}_0 \) to \( \hat{c}(p) \) and \( \hat{c}_0(p) \) respectively just as \( l_{\infty} \), \( c \) and \( c_0 \) were extended to \( l_{\infty}(p) \), \( c(p) \) and \( c_0(p) \).

If \( p = \{ p_n \} \) such that \( p_n > 0 \) and \( \sup_n p_n < \infty \), then

\[
\hat{c}(p) = \left\{ x : \lim_{n \to \infty} d_{p_n}(x - l) = 0 \right\},
\]

uniformly in \( n \) for some \( l \),

and

\[
\hat{c}_0(p) = \left\{ x : \lim_{n \to \infty} d_{p_n}(x) = 0 \right\},
\]

where

\[
d_{p_n}(x) = \sum_{k=1}^{\infty} p_k x_k^n.
\]

In particular, if \( p_n = p > 0 \) for every \( n \), we have \( \hat{c}(p) = \hat{c} \)

and \( \hat{c}_0(p) = \hat{c}_0 \).

It is quite natural to expect that the space \( V_p \) of bounded sequences all of whose \( \sigma \)-means are equal, can be extended to \( V_p(p) \) just as \( \hat{c} \) space of almost convergent sequences was extended to \( \hat{c}(p) \). Now, if \( p = \{ p_n \} \) is a sequence of strictly
positive integers such that \( \sup_{m} p_{m} < \infty \), we define

\[
V_{\sigma} (p) = \left\{ x : \lim_{n} \left( t_{n}(x - i) \right)^{p_{n}} = 0, \text{ uniformly in } x \right\},
\]

where

\[
t_{n}(x) = (z_{n} \plus{} \epsilon z_{n} + \ldots + \epsilon^{n} z_{n}) / (\operatorname{det} \cdot),
\]

and

\[
\sigma = \sigma - \lim_{n} x.
\]

Also

\[
V_{\sigma \sigma} (p) = \left\{ x \in V_{\sigma} (p) : \sigma - \lim_{n} x = 0 \right\}.
\]

Then \( p_{m} = p \) for every \( m \), we have \( V_{\sigma} (p) = V_{\sigma} \) and \( V_{\sigma \sigma} (p) = V_{\sigma \sigma} \)

i.e. the set of all sequences \( x \in V_{\sigma} \) such that \( \sigma - \lim_{n} x = 0 \).

Furthermore, if \( \sigma (n) = n + 1 \), \( V_{\sigma} (p) = \hat{\sigma} (p) \) and \( V_{\sigma \sigma} (p) = \hat{\sigma} (p) \).

4.2. Introduction. In this chapter, we prove that \( V_{\sigma} (p) \) and \( V_{\sigma \sigma} (p) \) are complete linear topological spaces and also determine necessary and sufficient conditions to characterize the matrices in the class \( (\sigma_{e}(p), V_{\sigma \sigma} (p)) \) which generalizes the following result of Nanda [38].

**Theorem A.** \( \lambda \in (\sigma_{e}(p), \hat{\sigma}_{e}(p)) \) if and only if

1. there exists an integer \( N > 1 \) such that, for every \( n \)
\[
\sup_n \left\{ \frac{1}{n+1} \sum_{k=0}^{n} a_{n,1,k} |N^{-1/\beta_k}|^P \right\}^P < \infty,
\]

(2) \[
\lim_n \frac{1}{n+1} \sum_{k=0}^{n} a_{n,1,k} |P_\beta| = 0, \text{ uniformly in } n.
\]

4.3. To establish the following results.

**Theorem 1.** \( V_{00}(p) \) and \( V_{0C}(p) \) with \( \inf \beta_k > 0 \) are complete linear topological spaces paranormed by \( \| \cdot \| \), where

\[
h(x) = \sup_{x \in X} \| t_{\beta_k}(x) \|^P_x,
\]

and

\[
\| \cdot \| = \max (1, \beta), \quad \beta = \sup \beta_k.
\]

**Theorem 2.** \( \Lambda \in (c_0(p), V_{00}(p)) \) if and only if

(1) there exists an integer \( r > 1 \) such that

\[
\nu_n = \sup_n \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \left| t(n,k,m) |N^{-1/\beta_k}|^P \right\}^P < \infty, \text{ for every } n,
\]

and

(2) \[
\lim_n \frac{1}{n+1} \sum_{k=0}^{n} t(n,k,m) |P_\beta| = 0, \text{ uniformly in } n,
\]

where

\[
t(n,k,m) = \sum_{j=0}^{m} a(\mathcal{O}^{j}(n), k) / (n+1).
\]
4.4. **Proof of Theorem 1.** Since $p_n/n \leq 1$, we have, for
every $n$ and $\lambda$ (see Lemma [12]),

\[
(1) \quad |t_{mn}(x-y)|^{p_n/n} \leq |t_{mn}(x)|^{p_n/n} + |t_{mn}(y)|^{p_n/n},
\]

and for every $\lambda \in \mathbb{C}$ (see Lemma [13]),

\[
(2) \quad |\lambda|^{p_n/n} \leq \max \{1, |\lambda|\}.
\]

It follows that $V_{\varphi_0}(p)$ and $V_{\varphi}(p)$ are linear spaces. It is
easy to see that $h(c) = 0$ and $h(x) = h(-x)$, The subadditivity
of $h$ follows from (1) by taking sup over $\Lambda$ with respect to $x$ and $n$.
Further from (2), it follows that

\[
h(\lambda x) = \sup_{m,n} |t_{mn}(\lambda x)|^{p_n/n} \leq \max \{1, |\lambda|\} \sup_{m,n} |t_{mn}(x)|^{p_n/n}.
\]

Therefore $\lambda = e$, $x = e \implies \lambda x = e$ and if $\lambda$ is fixed, $x = e$
$\implies \lambda x = e$. If $x \in V_{\varphi_0}(p)$ is fixed, given $\varepsilon > 0$ there exists
an integer $n_0$ such that

\[
(3) \quad \sup_{x \in V_0} |\lambda t_{mn}(x)|^{p_n/n} < \varepsilon/2, \text{ for every } n,
\]

and we can choose $\delta > 0$, such that, for $|\lambda| < \delta$, we have

\[
(4) \quad \sup_{x \in V_0} |\lambda t_{mn}(x)|^{p_n/n} < \varepsilon/2, \text{ for every } n.
\]

Thus, from (3) and (4), we get.
\[ |\lambda| < \varepsilon \implies h(\lambda x) \leq \varepsilon. \]

It follows that \( V_{\infty}(p) \) is a linear topological space paranormed by \( h \). Now, it is enough to show that, for fixed \( x \in V_{\infty}(p) \),
\[ \lambda \rightarrow 0 \implies \lambda x \rightarrow 0. \]

Let \( \inf p_m = p' > 0 \). Then we have
\[ h(\lambda x) \leq \max( |\lambda|, |\lambda|^{p'} ) h(x). \]

It follows that \( V_{\infty}(p) (p' > 0) \) is a linear topological space paranormed by \( h \).

Let \( \{ x^i \} \) be a Cauchy sequence in \( V_{\infty}(p) \). Then \( \{ x^i_k \} \) is a Cauchy sequence in \( C \) for each \( k \) and so \( x^i_k \rightarrow x_k \), for each \( k \). But \( x = \{ x_k \} \), given \( \varepsilon > 0 \), there exists \( N_0 \) such that, for
\[ i, j > N_0, \]
\[ | t_{m/n} (x^i - x^j) |^{p'/n} < \varepsilon/5, \quad \text{for all } n \text{ and } m. \]

Taking limit as \( j \rightarrow \infty \), we get
\[ | t_{m/n} (x^i - x) |^{p'/n} \leq \varepsilon/5, \quad \text{for all } n \text{ and } m. \]

Therefore \( h(x^i - x) \rightarrow 0 \) and \( x \in V_{\infty}(p) \).
If \( \{x^i\} \) be a Cauchy sequence in \( V_\theta(p) \), then, as above there exists \( x \) such that \( x^i \to x \). We now show that \( x \in V_\theta(p) \).

Since \( x^i \in V_\theta(p) \), there exists \( \tau^i \in \mathbb{C} \) such that

\[
|t_{mn}(x^i - x^n)|^{\frac{1}{p_m}} \leq c/5,
\]

for all \( m \) and \( n \). From (5), (7) and (1) we get,

\[
|t_{mn}(I^i e - I^n e)|^{\frac{1}{p_m}} \leq 3/5 c.
\]

Thus \( \{x^i\} \) is Cauchy sequence in \( \mathbb{C} \) and therefore, there exists \( \tau \in \mathbb{C} \) such that

\[
|t_{mn}(I^i e - e)|^{\frac{1}{p_m}} \leq 3/5 c.
\]

Now, by (1), (6), (7) and (8), we have

\[
|t_{mn}(x - e)|^{\frac{1}{p_m}} \leq c.
\]

This completes our proof.

**4.3. Proof of Theorem 2.** Suppose that \( A \subseteq (c_0(p), V_{c_0}(p)) \).

Since \( e^k \in c_0(p) \), necessity of (ii) is obvious. Put \( a_{mn}(x) = |t_{mn}(A x)|^{\frac{1}{p_m}} \). Now \( \{a_{mn}\}_m \) is a sequence of continuous linear functionals such that \( \lim a_{mn}(x) \) exists. Therefore, by uniform boundedness principle, for \( c < z < 1 \), there exists a
sphere $S([0, a]) \subset o_0(p)$ with $a = (c, c, c, ...)$ and a constant $v$ such that $e_{x_k}(x) \leq x$, for each $x$ and $x \in S([0, a])$. Define for each $r$, $x(r) = \{ x_k \}$:

$$x_k(r) = \begin{cases} \lfloor v/k \rfloor \ \text{sgn} \ (t(n, k, x)) \ , & \text{for } 0 \leq k \leq x, \\ 0, & \text{elsewhere.} \end{cases}$$

Note, $x(r) \in \mathbb{R}[0, a]$ and

$$\left\{ \frac{v}{k} \bigg| t(n, k, x) \bigg| n^{-1/p_k} \right\}^{p_k} \leq x,$n
for each $n$ and each $r$, where $p = 6^{1/3}$. It follows that (1) holds.

Conversely, suppose that the conditions (1) and (ii)
hold and $x \in o_0(p)$. Given $c > 0$, there exists $k_0$ such that
for $K$ and $n$ both larger than $k_0$,

$$v^{1/p_k} \ | x_k | \ < \ (c^{1/p_n})^{1/p_k}.$$

We have, for $n = \max \{ 1, 2^{K-1} \}$, where $K = \sup p_k$ (see ADDCOV [10]),

$$\left| t_{x_k}(x) \right|^{p_k} \ \leq \ v (J_1 + J_2).$$
where

\[ J_1 = \sum_{k \leq k_0} t(n, k, n) x_k^p \]

and

\[ J_2 = \sum_{k > k_0} t(n, k, n) x_k^p \]

Since (ii) holds, there exists an integer \( k_0 \) such that, for \( n > k_0 \)

\[ |t(n, k, n)| < \epsilon^{1/p} \]

Therefore, for such \( n \),

\[ J_1 \leq \sum_{k \leq k_0} |x_k|^p \]

\[ < \epsilon \left( \sum_{k \leq k_0} |x_k|^p \right) \]

and, for \( n > k_0 \),

\[ J_2 \leq \sum_{k > k_0} |t(n, k, n)| x_k^p \]

\[ < \epsilon \]

It follows that \( A \in \mathcal{C}_0(n), V_{\sigma_0}(p) \).

4.6. PROPOSITION. It is quite easy to see that \( V_{\sigma_0}(p) \) and \( V_{\sigma_0}(p) \) are 1-convex.

Since, for \( \epsilon < \delta < 1 \),

\[ U = \{ x : M(x) \leq \delta \} \]
is an absolutely 1-convex set, for let $a, b \in U$ and

$$|\lambda| + |\mu| \leq 1,$$

then

$$h(\lambda a + \mu b) \leq (|\lambda| + |\mu|)^{\frac{1}{n}} \leq 0.$$

This terminates the proof.