Chapter - I

Introduction
Chapter 1

Introduction

1.1 Fuzzy Sets

The idea of a fuzzy set was first proposed in 1960 by Lotfi Zadeh [106] as a means of handling uncertainty that is due to imprecision or vagueness rather than to randomness. While mathematicians have been involved with the development of fuzzy sets from the very beginning, the concept of fuzzy set has in recent years received wider consideration from the mathematical community. Many interesting mathematical problems are coming to the fore and the mathematical foundations of the subject are now becoming more firmly established.

Fuzzy sets are considered with respect to a nonempty base set $X$ of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to non-membership, $0 < u(x) < 1$ to partial membership and $u(x) = 1$, to full membership. According to Zadeh, a fuzzy subset of $X$ is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \rightarrow [0,1]$. The function $u$ itself is often used synonymously for the fuzzy set.
For example, the function $u : \mathbb{R}^1 \to [0, 1]$ with

$$u(x) = \begin{cases} 
0 & \text{if } x \leq 1 \\
\frac{1}{99}(x - 1) & \text{if } 1 < x \leq 100 \\
1 & \text{if } 100 < x
\end{cases}$$

provides an example of a fuzzy set of real numbers $x \gg 1$. See Figure 1.1.

![Figure 1.1: Fuzzy set of real numbers $x \gg 1$.](image)

There are of course many other reasonable choices for membership grade function. The only possibilities for membership function of an ordinary subset $A$ of $X$ are non-membership and full membership. Such a set can be identified with the fuzzy set on $X$ given by its characteristic function $\chi_A : X \to [0, 1]$, that is, with

$$\chi_A(x) = \begin{cases} 
0 : x \notin A \\
1 : x \in A.
\end{cases} \quad (1.1.1)$$
1.2 Definitions and Basic Properties

The following definitions and basic properties are as established in [25, 27, 33, 35, 39, 45, 46, 48, 54, 96, 108].

Let \( P_K(\mathbb{R}^n) \) denote the family of all nonempty, compact, convex subsets of \( \mathbb{R}^n \). Addition and scalar multiplication in \( P_K(\mathbb{R}^n) \) are defined as usual. Let \( A \) and \( B \) be two nonempty bounded subsets of \( \mathbb{R}^n \). The distance between \( A \) and \( B \) is defined by the Hausdorff metric

\[
d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\},
\]

where \( \| \cdot \| \) denotes the usual Euclidean norm in \( \mathbb{R}^n \). Then it is clear that \( (P_K(\mathbb{R}^n), d) \) becomes a complete metric space.

We denote the Hausdorff semimetric by \( \rho(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\| \). It is clear that

\[
\rho(A, B) = 0 \iff A \subset \overline{B}
\]

and

\[
\rho(A, C) \leq \rho(A, B) + \rho(B, C),
\]

where \( A, B, C \) are nonempty bounded subsets of \( \mathbb{R}^n \) and \( \overline{B} \) denotes the closure of \( B \). Also

\[
d(A, B) = \max \{\rho(A, B), \rho(B, A)\}
\]

and

\[
d(A, B) = 0 \iff \overline{A} = \overline{B}.
\]

A fuzzy subset of \( \mathbb{R}^n \) is defined in terms of a membership function which assigns to each point \( x \in \mathbb{R}^n \) a grade of membership in the fuzzy set. Such a membership function \( u : \mathbb{R}^n \to [0, 1] \) is used synonymously to denote the corresponding fuzzy set.
The $\alpha$-level set $[u]^\alpha$ of a fuzzy set $u$ on $X$ is defined as

$$[u]^\alpha = \{ x \in X : u(x) \geq \alpha \} \text{ for each } \alpha \in (0, 1),$$

while its support $[u]^0$ is the closure in the topology of $X$ of the union of all the level sets, that is,

$$[u]^0 = \bigcup_{\alpha \in [0,1]} [u]^\alpha.$$

An inclusion property follows immediately from the above definitions.

**Property 1.2.1.** For all $0 \leq \alpha \leq \beta \leq 1$

$$[u]^\beta \subseteq [u]^\alpha \subseteq [u]^0.$$

In general, some level sets of a fuzzy set may be empty. Indeed, in the trivial case of $u(x) \equiv 0$ for all $x \in \mathbb{R}^n$, even the support is empty; $u$ here is the empty fuzzy set. Here we shall restrict our attention to the normal fuzzy sets which satisfy

**Assumption 1.2.1.** $u$ maps $\mathbb{R}^n$ onto $[0, 1]$.

Obviously then $[u]^1 \neq \emptyset$ which is often used as an alternative definition of a normal fuzzy set. In view of Property 1.2.1, we have

**Property 1.2.2.** $[u]^\alpha \neq \emptyset$ for all $\alpha \in [0, 1]$.

For practical and theoretical purposes, it is convenient to further restrict the membership functions under consideration so that

**Assumption 1.2.2.** $[u]^0$ is a bounded subset of $\mathbb{R}^n$.

**Assumption 1.2.3.** $u$ is upper semicontinuous.

Hence each level set $[u]^\alpha$ and also $[u]^0$, by definition, is a closed subset of $\mathbb{R}^n$. Moreover they are all bounded since they are subsets of $[u]^0$ which is bounded and so
Property 1.2.3. \([u]^{\alpha}\) is a compact subset of \(\mathbb{R}^n\) for all \(\alpha \in [0,1]\).

Another immediate consequence of Assumption 1.2.3 is

Property 1.2.4. For any nondecreasing sequence \(\alpha_i \to \alpha\) in \([0,1]\),

\[ [u]^{\alpha} = \bigcap_{i \geq 1} [u]^{\alpha_i}. \]

In most cases, however, we will restrict our attention to a subset with the following additional property: a fuzzy set \(u\) is said to be fuzzy convex if

\[ u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \]

for all \(x, y \in [u]^0\) and \(\lambda \in [0,1]\). As a final restriction on the membership functions we often require

Assumption 1.2.4. \(u\) is fuzzy convex.

Let \(u\) be fuzzy convex and \(x, y \in [u]^{\alpha}\) for some \(\alpha \in (0,1]\) so that \(u(x) \geq \alpha\) and \(u(y) \geq \alpha\). Then

\[ u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \geq \alpha \]

for any \(\lambda \in [0,1]\), so that \(\lambda x + (1 - \lambda)y \in [u]^{\alpha}\). Hence \([u]^{\alpha}\) is a convex subset of \(\mathbb{R}^n\) for any \(\alpha \in (0,1]\). The support \([u]^0\) is also convex which follows from the fact

\[ d([u]^{\alpha}, [u]^0) \to 0 \text{ as } \alpha \to 0+ \]

and the completeness of the metric space \((P_\mathcal{K}(\mathbb{R}^n), d)\). Thus we have

Property 1.2.5. If \(u\) is fuzzy convex, then \([u]^{\alpha}\) is convex for each \(\alpha \in [0,1]\).

Let \(I = [a, b] \subseteq \mathbb{R}\) be a compact interval and let \(E^n\) denote the set of all \(u : \mathbb{R}^n \to [0,1]\) which satisfy the Assumptions 1.2.1-1.2.4, namely

(i) \(u\) is normal, that is, there exists an \(x_0 \in \mathbb{R}^n\) such that \(u(x_0) = 1\),
(ii) $u$ is fuzzy convex,

(iii) $u$ is upper semicontinuous,

(iv) $\{u\}^0 = cl \{ x \in \mathbb{R}^n : u(x) > 0 \}$ is compact.

For $0 < \alpha \leq 1$, denote $\{u\}^\alpha = \{ x \in \mathbb{R}^n : u(x) \geq \alpha \}$. Then from (i)-(iv), it follows that the $\alpha$-level set $\{u\}^\alpha \in P_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, then using Zadeh's extension principle [70], we can extend $g$ to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u,v)(z) = \sup_{z=g(x,y)} \min \{ u(x), v(y) \}.$$

It is well known that

$$\{\tilde{g}(u,v)\}^\alpha = g(\{u\}^\alpha, \{v\}^\alpha)$$

for all $u, v \in E^n$, $0 \leq \alpha \leq 1$ and continuous function $g$. Further we have

$$\{u + v\}^\alpha = \{u\}^\alpha + \{v\}^\alpha$$

$$\{k u\}^\alpha = k \{u\}^\alpha,$$

where $k \in \mathbb{R}$. The real numbers can be embedded in $E^n$ by the rule $c \mapsto \tilde{c}(t)$ where

$$\tilde{c}(t) = \begin{cases} 1 & \text{for } t = c, \\ 0 & \text{elsewhere}. \end{cases}$$

**Theorem 1.2.1.** [66] If $u \in E^n$, then

(1) $\{u\}^\alpha \in P_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$,

(2) $\{u\}^{\alpha_2} \subset \{u\}^{\alpha_1}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,

(3) if $\{\alpha_k\}$ is a nondecreasing sequence converging to $\alpha > 0$, then

$$\{u\}^\alpha = \bigcap_{k \geq 1} \{u\}^{\alpha_k}.$$
Conversely, if \( \{ A^\alpha : 0 \leq \alpha \leq 1 \} \) is a family of subsets of \( \mathbb{R}^n \) satisfying (1)-(3), then there exists an \( u \in E^n \) such that \([u]^\alpha = A^\alpha\) for \(0 < \alpha \leq 1\) and
\[
[u]_0^\alpha = \bigcup_{0 < \alpha \leq 1} A^\alpha \subset A^0.
\]

Define the metric \( D : E^n \times E^n \to \mathbb{R}^+ \cup \{0\} \) by \( D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha) \), where \( d \) is the Hausdorff metric defined in \( P_K(\mathbb{R}^n) \). Then it is easy to show that \( D \) is a metric in \( E^n \). In addition, using results of [86], we see that \((E^n, D)\) is a complete metric space. Also
\[
D(u + w, v + w) = D(u, v),
\]
\[
D(\lambda u, \lambda v) = |\lambda|D(u, v),
\]
\[
D(u + v, w + z) \leq D(u, w) + D(v, z), \quad \text{for all } u, v, w, z \in E^n \text{ and } \lambda \in \mathbb{R}.
\]

An arbitrary fuzzy number is represented by an ordered pair of functions \((u(\alpha), \bar{u}(\alpha))\), \(0 \leq \alpha \leq 1\), that satisfy the following requirements:

1. \( u(\alpha) \) is a bounded left continuous nondecreasing function over \([0,1]\).
2. \( \bar{u}(\alpha) \) is a bounded left continuous nonincreasing function over \([0,1]\).
3. \( u(\alpha) \leq \bar{u}(\alpha), \ 0 \leq \alpha \leq 1\).

If \( v \in E \), then the \( \alpha \)-level set
\[
[v]^\alpha = \{ s | v(s) \geq \alpha \}, \ 0 < \alpha \leq 1,
\]
is a closed bounded interval which is denoted by
\[
[v]_0^\alpha = [u^\alpha, \bar{u}^\alpha]
\]
and there exists a \( t_0 \in \mathbb{R} \) such that \( v(t_0) = 1 \).

Two fuzzy numbers \( v \) and \( u \) are called equal and written as \( v = u \), if \( v(s) = u(s) \) for all \( s \in \mathbb{R} \). It follows that
\[
v = u \iff [v]^\alpha = [u]^\alpha \text{ for all } s \in \mathbb{R}.
\]
For the arithmetic operations on fuzzy numbers we refer to [64]. The following results are well known and frequently used in the sequel.

Lemma 1.2.1. If \( u, v \in E \), then, for \( \alpha \in (0, 1) \),
\[
[u + v]^\alpha = [u^\alpha + v^\alpha, u^\alpha + v^\alpha],
\]
\[
[u \cdot v]^\alpha = \min\{u^\alpha v^\alpha, u^\alpha v^\alpha, u^\alpha v^\alpha, v^\alpha v^\alpha\}, \max\{u^\alpha v^\alpha, u^\alpha v^\alpha, u^\alpha v^\alpha, v^\alpha v^\alpha\})
\]
and
\[
[u - v]^\alpha = [u^\alpha - v^\alpha, u^\alpha - v^\alpha].
\]

Lemma 1.2.2. Let \([u^\alpha, v^\alpha]\), \(0 < \alpha \leq 1\), be a given family of non-empty intervals. If

(i) \([u^\alpha, v^\alpha] \supset [u^\beta, v^\beta]\) for \(0 < \alpha \leq \beta\) and

(ii) \(\lim_{k \to \infty} u^{\alpha_k}, \lim_{k \to \infty} v^{\alpha_k} = [u^\alpha, v^\alpha]\)

whenever \((\alpha_k)\) is a nondecreasing sequence converging to \(\alpha \in (0, 1)\), then the family

\([u^\alpha, v^\alpha], \ 0 < \alpha \leq 1\), represents the \(\alpha\)-level sets of a fuzzy number \(v \in E\). Conversely, if \([u^\alpha, v^\alpha], \ 0 < \alpha \leq 1\), are the \(\alpha\)-level sets of a fuzzy number \(v \in E\), then the conditions (i) and (ii) hold true.

A mapping \(y : I \to E\) is called a fuzzy process and its \(\alpha\)-level set is denoted by
\([y(t)]^\alpha = [y^\alpha(t), y^\alpha(t)]\), \(t \in I\), \(\alpha \in (0, 1)\).

Triangular fuzzy numbers are those fuzzy sets \(U \in E\) which are characterized by an ordered triple \((x^l, x^c, x^r) \in \mathbb{R}^3\) with \(x^l \leq x^c \leq x^r\) such that \([U]^0 = [x^l, x^r]\) and \([U]^1 = \{x^c\}\) then
\[
[U]^\alpha = [x^c - (1 - \alpha)(x^c - x^l), x^c + (1 - \alpha)(x^r - x^c)], \quad (1.2.1)
\]
for any \(\alpha \in [0, 1]\).

A tilde is placed over a symbol to denote a fuzzy set like \(\tilde{a}_1, \tilde{f}(t), \ldots\)
1.2.1 Measurability

Definition 1.2.1. A mapping $F : I \rightarrow E^n$ is strongly measurable, if, for all $\alpha \in [0, 1]$, the set-valued map $F_\alpha : I \rightarrow P_K(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_K(\mathbb{R}^n)$ has the topology induced by the Hausdorff metric $d$.

Lemma 1.2.3. [50] If $F$ is strongly measurable, then it is measurable with respect to the topology generated by $D$.

Proof. Let $\epsilon > 0$ and $u \in E^n$ be arbitrary. Then

$$T_1 = \{t : D(F(t), u) \leq \epsilon\} = \bigcap_{\alpha \in [0, 1]} \{t : d(F_\alpha(t), [u]^\alpha) \leq \epsilon\}.$$

But, for all $v \in E^n$, we have

$$\lim_{k \to \infty} d([v]^\alpha_k, [v]^\alpha) = 0,$$

whenever $(\alpha_k)$ is a nondecreasing sequence converging to $\alpha$. Thus by the triangle inequality for the metric $d$, we have

$$d(F_\alpha(t), [u]^\alpha) \leq \limsup d(F_{\alpha_k}(t), [u]^\alpha_k),$$

where $\alpha_k \nearrow \alpha$ and consequently

$$\{t : d(F_\alpha(t), [u]^\alpha) \leq \epsilon\} \supset \bigcap_{k \geq 1} \{t : d(F_{\alpha_k}(t), [u]^\alpha_k) \leq \epsilon\}.$$

Thus

$$T_1 = \bigcap_{k \geq 1} \{t : d(F_{\alpha_k}(t), [u]^\alpha_k) \leq \epsilon\},$$

where $\{\alpha_k : k = 1, 2, \ldots\}$ is any denumerable dense subset of $[0, 1]$. Hence $T_1$ is measurable. \hfill \Box

Lemma 1.2.4. If $F : I \rightarrow E^n$ is continuous with respect to the metric $D$, then it is strongly measurable.
Proof. Let \( \epsilon > 0 \) be arbitrary and \( t_0 \in I \). By continuity, there exists a \( \delta > 0 \) such that

\[
D(F(t), F(t_0)) < \epsilon \quad \text{whenever} \quad |t - t_0| < \delta.
\]

But, by the definition of \( D \), we have \( d(F_\alpha(t), F_\alpha(t_0)) < \epsilon \) for all \( |t - t_0| < \delta \); so \( F_\alpha \) is continuous with respect to the Hausdorff metric. Therefore \( F_\alpha^{-1}(U) \) is open and hence measurable for each open \( U \) in \( P_K(\mathbb{R}^n) \). \( \square \)

Lemma 1.2.5. Let \( F : I \to E^n \) be strongly measurable and \( F_\alpha(t) = [\lambda_\alpha(t), \mu_\alpha(t)] \) for \( \alpha \in [0, 1] \). Then \( \lambda_\alpha \) and \( \mu_\alpha \) are measurable.

1.2.2 Integrability

Dubois and Prade deal with integrals of fuzzy set-valued mappings and generalize the Riemann integrals over a closed interval to fuzzy mappings \([36, 37]\). Kaleva \([50]\) defines the integral of a fuzzy-valued function and establishes some of its properties. Many of the basic operations on integrals are shown in \([15, 51, 65, 103]\).

A mapping \( F : I \to E^n \) is said to be integrably bounded if there exists an integrable function \( h(t) \) such that \( \|x(t)\| \leq h(t) \) for all \( x \in F_\alpha(t) \).

Definition 1.2.2. A mapping \( F : I \to E^n \) is called levelwise continuous at \( t_0 \in I \) if the set-valued mapping \( F_\alpha(t) = [F(t)]^\alpha \) is continuous at \( t = t_0 \) with respect to the Hausdorff metric \( d \) for all \( \alpha \in [0, 1] \).

Definition 1.2.3. A mapping \( F : I \times E^n \to E^n \) is called levelwise continuous at the point \( (t_0, x_0) \in I \times E^n \) provided, for any fixed \( \alpha \in [0, 1] \) and arbitrary \( \epsilon > 0 \), there exists a \( \delta(\epsilon, \alpha) > 0 \) such that

\[
d([F(t, x)]^\alpha, [F(t_0, x_0)]^\alpha) < \epsilon
\]

whenever \( |t - t_0| < \delta(\epsilon, \alpha) \) and \( d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha) \) for all \( t \in I, x \in E^n \).
Theorem 1.2.2. [15] If $F : I \rightarrow E^n$ is strongly measurable and integrably bounded, then $F$ is integrable.

It is known that $\left[ \int_I F(t)dt \right]^0 = \int_I F_0(t)dt$.

Theorem 1.2.3. If $F : I \rightarrow E^n$ is strongly measurable and integrably bounded, then $F$ is integrable.

Theorem 1.2.4. Let $F : I \rightarrow E^n$ be integrable and $c \in I$. Then

$$\int_a^b F = \int_a^c F + \int_c^b F.$$ 

Definition 1.2.4. The integral of a fuzzy mapping $F : I \rightarrow E^n$ is defined levelwise by $\left[ \int_I F(t)dt \right]^\alpha = \int_I F_\alpha(t)dt$. The set of all $\int_I f(t)dt$ such that $f : I \rightarrow R^n$ is a measurable selection for $F_\alpha$ for all $\alpha \in [0, 1]$.

Theorem 1.2.5. Let $F, G : I \rightarrow E^n$ be integrable and $\lambda \in \mathbb{R}$. Then

(i) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,

(ii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,

(iii) $D(F(t), G(t))$ is integrable,

(iv) $D \left( \int_I F(t)dt, \int_I G(t)dt \right) \leq \int_I D(F(t), G(t))dt$.

Definition 1.2.5. The fuzzy integral $\int_a^b y(t)dt$, $0 \leq a \leq b \leq 1$, is defined by

$$\left[ \int_a^b y(t)dt \right]^\alpha = \left[ \int_a^b \underline{y}^\alpha(t)dt, \int_a^b \overline{y}^\alpha(t)dt \right].$$
1.2.3 Differentiability

Dubois and Prade [38] discuss the differentiation of fuzzy functions. In [50], Kaleva adopts the H-differentiability of Puri and Ralescu [85], which generalizes the Hukuhara differentiability of set-valued mappings and studies the properties of differentiable mappings.

Let $x, y \in E^n$. If there exists an $z \in E^n$ such that $x = y + z$, then we call $z$ the $H$-difference of $x$ and $y$, denoted by $x - y$.

**Definition 1.2.6.** A mapping $F : I \rightarrow E^n$ is differentiable at $t_0 \in I$, if there exists an $F'(t_0) \in E^n$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and are equal to $F'(t_0)$.

**Remark 1.2.1.** From the definition, it follows that if $F$ is differentiable, then the multivalued mapping $F_\alpha$ is Hukuhara differentiable for all $\alpha \in [0, 1]$ and

$$DF_\alpha(t) = [F'(t)]^\alpha. \quad (1.2.2)$$

Here $DF_\alpha$ denotes the Hukuhara derivative of $F_\alpha$.

**Definition 1.2.7.** If $F : I \rightarrow E^n$ is differentiable at $t_0 \in I$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point $t_0$.

A mapping $F : I \rightarrow E^n$ is called differentiable at $t_0 \in I$ if, for any $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at $t_0$ with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) | \alpha \in [0, 1]\}$ defines the fuzzy number $F(t_0) \in E^n$.

**Theorem 1.2.6.** Let $F : I \rightarrow E^n$ satisfy the assumptions:

(a) for each $t \in I$, there exists a $\beta > 0$ such that the $H$-differences $F(t + h) - F(t)$ and $F(t) - F(t - h)$ exist for all $0 \leq h < \beta$
(b) the set-valued mappings \( F_\alpha, \quad \alpha \in [0,1], \) are uniformly Hukuhara differentiable with derivatives \( DF_\alpha, \) that is, for each \( t \in I \) and \( \epsilon > 0, \) there exists a \( \delta > 0 \) such that

\[
d((F_\alpha(t + h) - F_\alpha(t))/h, DF_\alpha(t)) < \epsilon \]
and

\[
d((F_\alpha(t) - F_\alpha(t - h))/h, DF_\alpha(t)) < \epsilon \]

for all \( 0 \leq h < \delta \) and \( \alpha \in [0,1]. \) Then \( F \) is differentiable and the derivative is \( DF_\alpha(t) = [F'(t)]^\alpha. \)

**Theorem 1.2.7.** Let \( F : I \rightarrow E^1 \) be differentiable. Define \( F_\alpha(t) = [f_\alpha(t), g_\alpha(t)], \alpha \in [0,1]. \) Then \( f_\alpha \) and \( g_\alpha \) are differentiable and

\[
[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)].
\]

**Theorem 1.2.8.** Let \( F : I \rightarrow E^n \) be differentiable on \( I. \) If \( t_1, t_2 \in I \) with \( t_1 \leq t_2, \) then there exists a \( C \in E^n \) such that \( F(t_2) = F(t_1) + C. \)

**Theorem 1.2.9.** If \( F : I \rightarrow E^n \) is differentiable, then it is continuous.

**Theorem 1.2.10.** If \( F, G : I \rightarrow E^n \) are differentiable and \( \lambda \in \mathbb{R}, \) then \( (F + G)'(t) = F'(t) + G'(t) \) and \( (\lambda F)'(t) = \lambda F'(t). \)

**Theorem 1.2.11.** Let \( F : I \rightarrow E^n \) be continuous. Then, for all \( t \in I, \) the integral \( G(t) = \int_a^t F \) is differentiable and \( G'(t) = F(t). \)

**Theorem 1.2.12.** Let \( F : I \rightarrow E^n \) be differentiable and the derivative \( F' \) be integrable over \( I. \) Then, for each \( s \in I, \) we have

\[
F(s) = F(a) + \int_a^s F'.
\]

(1.2.5)
1.3 Fuzzy Initial Value Problem

When a physical problem is transformed into the deterministic initial value problem

\[
\frac{d\phi(t)}{dt} = f(t, \phi(t)), \\
\phi(0) = \omega_0,
\]

one can not be sure whether this modelling is perfect. The initial value may not be known exactly and the function \( f \) may contain unknown parameters; especially, if they are known through some measurements, they are necessarily subject to errors. The analysis of the effect of these errors leads to the study of the qualitative behavior of the solution of the above differential equation like continuous dependence and several kinds of stability problems.

If the nature of the error is random, then, instead of the deterministic equation, we get a random differential equation with a random initial value and random coefficients. If the underlying structure is not probabilistic, it may be appropriate to use the fuzzy numbers instead of real random variables. This leads to a fuzzy initial value problem [52, 71, 91, 101, 105].

Bede and Gal [19, 20] and Bede et al. [21] introduce a more general definition of the derivative for fuzzy mappings, enlarging the class of differentiable fuzzy mappings. Chalco-Cano and Roman-Flores [26], Diamond [31], Kloeden [56], Lakshmikantham [57, 58], Nieto [67], Rodriguez-Lopez [88, 89] and Park [75] solve the fuzzy differential equations. Similarly one can describe fuzzy boundary value problems [28, 87, 84], fuzzy integral equations [40, 41, 76, 78, 79] and stability problems [30, 49]. Numerical solutions of fuzzy differential equations are widely discussed in [1, 4, 9, 68, 102].
Example 1.3.1 Let \( a \in E^1 \) have level sets \( [a] = [\underline{a}, \overline{a}] \) for \( \alpha \in [0, 1] \) and suppose that a solution \( y : [0, b] \to E^1 \) of the fuzzy differential equation
\[
\frac{dy}{dt} = ay
\] (1.3.1)
on \( E^1 \) has the level sets \( [y(t)] = [\underline{y}(t), \overline{y}(t)] \) for \( \alpha \in [0, 1] \) and \( t \in [0, b] \).

The Hukuhara derivative \( \frac{dy}{dt}(t) \) has the level sets
\[
\left[ \frac{dy}{dt}(t) \right]^\alpha = \left[ \frac{dy}{dt}(t), \frac{dy}{dt}(t) \right]
\]
for \( \alpha \in [0, 1] \) and \( t \in [0, b] \). By the extension principle, the fuzzy set \( f(y(t)) = ay(t) \) has the level sets
\[
[ay(t)]^\alpha = \left\{ \min\{a^\alpha y^\alpha(t), a^\alpha \overline{y}(t), a^\alpha \overline{y}(t), a^\alpha \overline{y}(t)\} \right\}
\]
for all \( \alpha \in [0, 1] \) and \( t \in [0, b] \). Thus the fuzzy differential equation (1.3.1) is equivalent to the coupled system of ordinary differential equations
\[
\frac{dy}{dt}^\alpha = \min\{a^\alpha y^\alpha(t), a^\alpha \overline{y}(t), a^\alpha \overline{y}(t), a^\alpha \overline{y}(t)\}
\]
and the system of ordinary differential equations (1.3.2) reduces to
\[
\frac{dy}{dt} = -y
\]
and the system of ordinary differential equations (1.3.2) reduces to
\[
\frac{dy}{dt}^\alpha = -\overline{y}, \quad \frac{dy}{dt}^\alpha = -y
\]
for \( \alpha \in [0, 1] \), with the solution corresponding to an initial value \( y_0 \in E^1 \) with \( [y_0] = [\underline{y}_0, \overline{y}_0] \), for \( \alpha \in [0, 1] \), given by
\[
y^\alpha(t) = \frac{1}{2}(\underline{y}_0^\alpha - \overline{y}_0^\alpha)e^t + \frac{1}{2}(\underline{y}_0^\alpha + \overline{y}_0^\alpha)e^{-t}
\]
\[
\overline{y}(t) = \frac{1}{2}(\underline{y}_0^\alpha - \overline{y}_0^\alpha)e^t + \frac{1}{2}(\underline{y}_0^\alpha + \overline{y}_0^\alpha)e^{-t}, \quad \text{for} \quad \alpha \in [0, 1] \quad \text{and all} \quad t \geq 0.
Thus, for \( y_0 = \chi_{\{c_0\}} \), the solution \( y(t) = \chi_{\{c_0e^{-t}\}} \rightarrow \chi_{\{0\}} \) as \( t \rightarrow \infty \). In contrast, when \([y_0]^\alpha = [\alpha - 1, 1 - \alpha]\) for \( \alpha \in [0, 1] \), the solution has the level sets

\[
[y(t)]^\alpha = [(\alpha - 1)e^t, (1 - \alpha)e^t] = (1 - \alpha)e^t[-1, 1]
\]

for all \( \alpha \in [0, 1] \) and \( t \geq 0 \). In particular, \( \text{diam}[y(t)]^\alpha = 2(1 - \alpha)e^t \), so that the solution becomes fuzzier with increasing time.

Our next example shows that the fuzzification of the initial condition will alter the behavior of the solution of the equation.

**Example 1.3.2** Consider the fuzzy initial value problem

\[
\frac{dy}{dt} = -2y, \quad y(0) = (0; 1)_S,
\]

where \((c; d)_S\) is the symmetric triangular fuzzy number with the interval \([c, d]\) as its support and let \( y(t) \) be a fuzzy number valued function of time.

The \( \beta \)-level set of \( y(t) \) is the compact interval \( y_\beta(t) = [y_\beta(t), \overline{y}_\beta(t)] \) and note that

\(-2y_\beta = [-2\overline{y}_\beta, -2y_\beta] \) while \( y_\beta(0) = [\beta/2, 1 - \beta/2] \). Writing \( \xi_\beta(t) \) as the vector with components \( y_\beta(t), \overline{y}_\beta(t) \), the ordinary initial value problem is obtained as

\[
\begin{align*}
\xi'_\beta(t) &= \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \xi_\beta(t), \\
\xi_\beta(0) &= \begin{pmatrix} \beta/2 \\ 1 - \beta/2 \end{pmatrix} \quad \text{for} \quad 0 \leq \beta \leq 1.
\end{align*}
\]

It is easy to see that

\[
\xi_\beta(t) = \frac{1}{2} e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta - 1}{2} e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

that is,

\[
y_\beta(t) = \left[ \frac{e^{-2t}}{2} + (\beta - 1)\frac{e^{2t}}{2}, \frac{e^{-2t}}{2} + (1 - \beta)\frac{e^{2t}}{2} \right].
\]
Hence the solution of (1.3.3) is \( y(t) = \frac{e^{-2t}}{2} + \left( \frac{e^{-2t}}{2} ; \frac{e^{2t}}{2} \right)_s \). Thus (1.3.3) has an unstable solution, in stark contrast to the behavior of the associated ordinary problem

\[
\frac{dz}{dt} = -2z(t), \quad z(0) = \frac{1}{2} \tag{1.3.4}
\]

which has the solution \( z(t) = \frac{e^{-2t}}{2} \). Thus introducing of some uncertainty, in the initial condition totally changes the qualitative behavior of the solution. Indeed, an arbitrary small fuzzification \( y(0) = \left( \frac{1}{2} - \epsilon; \frac{1}{2} - \epsilon \right)_s \) has the same effect, the solution being \( y(t) = \frac{e^{-2t}}{2} + (-\epsilon e^{2t}, \epsilon e^{2t})_s \), although as \( \epsilon \to 0^+ \), the ordinary solution is the limit.

**Example 1.3.3** Consider the fuzzy initial value problem

\[
\frac{dy}{dt} = c_1 y^2 + c_2, \quad y(0) = 0, \tag{1.3.5}
\]

where \( c_i > 0 \), for \( i = 1, 2 \) are triangular fuzzy numbers and

\[
[c_1]^\alpha = [\xi_1^\alpha, \xi_2^\alpha], \quad [c_2]^\alpha = [\xi_2^\alpha, \xi_2^\alpha].
\]

(1.3.5) is reduced to the following system of ordinary differential equations

\[
\frac{d\tilde{y}_1^\alpha}{dt} = \xi_1^\alpha \{\tilde{y}_1^\alpha\}^2 + \xi_2^\alpha,
\]

\[
\frac{d\tilde{y}_2^\alpha}{dt} = \xi_2^\alpha \{\tilde{y}_2^\alpha\}^2 + \xi_2^\alpha.
\]

The exact solution is given by

\[
\tilde{y}_1^\alpha(t) = l^\alpha \tan(w^\alpha),
\]

\[
\tilde{y}_2^\alpha(t) = \bar{l}^\alpha \tan(\bar{w}^\alpha),
\]

where

\[
l^\alpha = \sqrt{\frac{\xi_2^\alpha}{\xi_1^\alpha}}, \quad \bar{l}^\alpha = \sqrt{\frac{\xi_2^\alpha}{\xi_1^\alpha}},
\]

\[
w^\alpha = \sqrt{\frac{\xi_1^\alpha \xi_2^\alpha}{\xi_1^\alpha \xi_2^\alpha}}, \quad \bar{w}^\alpha = \sqrt{\frac{\xi_1^\alpha \xi_2^\alpha}{\xi_1^\alpha \xi_2^\alpha}}.
\]
1.4 The Hybrid Fuzzy Differential System

Hybrid systems are used in modelling, design and validation of interactive systems of computer programs and continuous systems, that is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics. The differential systems involving fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems [59].

Consider the hybrid fuzzy differential system

\[
\begin{align*}
\dot{\tilde{y}}(t) &= \tilde{f}(t, y(t), \lambda_k(y_k)), \quad t \in [t_k, t_{k+1}], \\
\tilde{y}(t_k) &= \tilde{y}_k,
\end{align*}
\]

where \(0 \leq t_0 < t_1 < \cdots < t_k < \cdots, t_k \to \infty, \tilde{f} \in C[\mathbb{R}^+ \times E \times E, E], \lambda_k \in C[E, E]\).

Here we assume that the existence and uniqueness of solutions of the hybrid system hold on each \([t_k, t_{k+1}]\). To be specific, the system would look like

\[
\dot{\tilde{y}}(t) = \begin{cases} \\
\tilde{y}_0(t) = \tilde{f}(t, y_0(t), \lambda_0(y_0)), & \tilde{y}_0(t_0) = \tilde{y}_0, \quad t_0 \leq t \leq t_1, \\
\tilde{y}_1(t) = \tilde{f}(t, y_1(t), \lambda_1(y_1)), & \tilde{y}_1(t_1) = \tilde{y}_1, \quad t_1 \leq t \leq t_2, \\
\vdots & \\
\tilde{y}_k(t) = \tilde{f}(t, y_k(t), \lambda_k(y_k)), & \tilde{y}_k(t_k) = \tilde{y}_k, \quad t_k \leq t \leq t_{k+1}, \\
\end{cases}
\]

By the solution of (1.4.1) we mean the following function

\[
\tilde{y}(t) = \tilde{y}(t, t_0, \tilde{y}_0) = \begin{cases} \\
\tilde{y}_0(t), & t_0 \leq t \leq t_1, \\
\tilde{y}_1(t), & t_1 \leq t \leq t_2, \\
\vdots & \\
\tilde{y}_k(t), & t_k \leq t \leq t_{k+1}, \\
\end{cases}
\]

(1.4.2)

We note that the solutions of (1.4.1) are piecewise differentiable in each interval for \(t \in [t_k, t_{k+1}]\) for a fixed \(\tilde{y}_k \in E, \quad k = 0, 1, 2, \ldots \).
1.5 Numerical Methods

Numerical analysis helps to solve higher mathematical problems using a computer, offering a technique widely used by scientists and engineers to solve their problems. An advantage of numerical analysis is that a numerical solution can be obtained even when a problem has no "analytical" solution. Analytical methods usually yield a result in terms of mathematical functions that can then be evaluated for specific instances. Thus there is an advantage over the analytical results, in that the behavior and properties of the function are often apparent; this is not the case for purely numerical results. However, numerical results can be plotted to show some behavior of the solution. Another important distinction is that though the result from numerical analysis is approximate, it can be made as accurate as desired. To achieve high accuracy, different kinds of methods are available [44].

1.5.1 Single-Step and Multi-Step Methods

The methods for the solution of the initial value problem

\[ y'(t) = f(t, y(t)), \quad t \in [a, b], \]
\[ y(a) = y_0, \quad (1.5.1) \]

can be classified mainly into two types. They are

(i) Single-step methods.

(ii) Multi-step methods.

**Single-Step Methods:** In single-step methods, the solution at any point is obtained using the solution at the previous point only. Thus, a general single step method can be written as

\[ y_{j+1} = y_j + h\phi(t_{j+1}, t_j, y_{j+1}, y_j, h), \quad (1.5.2) \]
where $\phi$ is a function of the arguments $t_j, t_{j+1}, y_j, y_{j+1}, h$ and also depends on $f$. If $y_{j+1}$ can be obtained simply by evaluating the right hand side of (1.5.2), then the method is called an explicit method. If the right hand side of (1.5.2) depends on $y_{j+1}$ also, then it is called an implicit method. Some of the single-step methods which are used to find the solution of the initial value problem are

(a) Taylor series method

(b) Euler method

(c) Runge-Kutta methods.

A method is said to be **consistent** if it is atleast of order 1. If the effect of the total error including the roundoff error remains bounded as $j \to \infty$ with fixed step size, then the difference method is said to be **stable**; otherwise **unstable**.

**Multi-Step Methods:** In multi-step methods, the solution at any point is obtained using the values of $y(t)$ and $y'(t)$ at $m + 1$ successive mesh points $t_{j+1}, t_j, t_{j-1}, \ldots, t_{j-m+1}$, namely the values $y_{j+1}, y_j, y_{j-1}, \ldots, y_{j-m+1}$ and $y'_{j+1}, y'_j, y'_{j-1}, \ldots, y'_{j-m+1}$. Some of the multi-step methods are

(a) Adams-Bashforth method

(b) Adams-Moulton method

(c) Predictor-corrector method

(d) Nystrom method

**Definition 1.5.1.** An *m-step method* for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point $t_{i+1}$ can be represented by the following equation
\[ y(t_{i+1}) = a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + \cdots + a_0y(t_{i+1-m}) \\
\quad + h[b_m f(t_{i+1}, y_{i+1}) + b_{m-1} f(t_i, y_i) \\
\quad + \cdots + b_0 f(t_{i+1-m}, y_{i+1-m})], \] (1.5.3)

for \( i = m-1, m, \ldots, N-1 \), such that \( a = t_0 \leq t_1 \leq \cdots \leq t_N = b \), \( h = \frac{(b-a)}{N} = t_{i+1} - t_i \)
and \( a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_m \) are constants with the starting values
\[ y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \cdots, \quad y_{m-1} = \alpha_{m-1}. \]

When \( b_m = 0 \), the method is known as explicit, since (1.5.3) gives \( y_{i+1} \) explicit in terms of previously determined values. When \( b_m \neq 0 \), the method is known as implicit, since \( y_{i+1} \) occurs on both sides of (1.5.3) and is specified only implicitly.

**Definition 1.5.2.** Associated with the difference equation
\[ y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \cdots + a_0y_{i+1-m} + hF(t_i, y_i, y_{i+1}, \cdots, y_{i+1-m}), \]
\[ y_0 = \alpha, \quad y_1 = \alpha_1, \cdots, \quad y_{m-1} = \alpha_{m-1}, \] (1.5.4)

the characteristic polynomial of the method is defined as
\[ p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0. \]

If \( |\lambda_i| \leq 1 \) for each \( i = 1, 2, \cdots, m \) and all the roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

**Theorem 1.5.1.** A multi-step method of the form (1.5.4) is stable if and only if it satisfies the root condition.

### 1.5.2 Interpolation of Fuzzy Number

The problem of interpolation for fuzzy sets is as follows [53, 61, 100]: Suppose that at various time instants \( t \), the information \( f(t) \) is presented as a fuzzy set. The aim is to approximate the function \( f(t) \), for all \( t \), in the domain of \( f \). Let \( t_0 < t_1 < \cdots < t_n \)
be \( n + 1 \) distinct points in \( \mathbb{R} \) and let \( \tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_n \) be \( n + 1 \) fuzzy sets in \( E \). A fuzzy polynomial interpolation of the data is a fuzzy-valued continuous function \( f : \mathbb{R} \to E \) satisfying

- \( f(t_i) = \tilde{u}_i, \quad i = 1, \ldots, n. \)
- If the data are crisp, then the interpolation \( f \) is called a crisp polynomial.

A function \( f \) which fulfills these conditions may be constructed as follows:

Let \( C^i_\alpha = [\tilde{u}_i]^\alpha \) for any \( \alpha \in [0, 1], \quad i = 0, 1, \ldots, n. \) For each \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \), the unique polynomial of degree \( \leq n \), denoted by \( P_X \), such that

\[
P_X(t_i) = x_i, \quad i = 0, 1, \ldots, n,
\]

\[
P_X(t) = \sum_{i=0}^{n} x_i \left( \prod_{i \neq j} \frac{t - t_j}{t_i - t_j} \right).
\]

Finally, for each \( t \in \mathbb{R} \) and all \( \xi \in \mathbb{R} \), define \( f(t) \in E \) by

\[
(f(t))(\xi) = \sup \{ \alpha \in [0, 1] : \exists X \in C^0_\alpha \times \cdots \times C^n_\alpha \text{ such that } P_X(t) = \xi \}.
\]

The interpolation polynomial can be written level set wise as

\[
[f(t)]^\alpha = \{ y \in \mathbb{R} : y = P_X(t), \quad x \in [\tilde{u}_i]^\alpha, i = 1, 2, \ldots, n \} \text{ for } 0 \leq \alpha \leq 1.
\]

When the data \( \tilde{u}_i \) are presented as triangular fuzzy numbers, the values of the interpolation polynomial are also triangular fuzzy numbers. Then \( f(t) \) has a simple form that is well suited to computation.

**Theorem 1.5.2.** Let \((t_i, \tilde{u}_i), \quad i = 0, 1, 2, \ldots, n, \) be the observed data and suppose that each of the \( \tilde{u}_i = (u^l_i, u^c_i, u^r_i) \) is an element of \( E \). Then, for each \( t \in [t_0, t_n], \) \( \tilde{f}(t) = (f^l(t), f^c(t), f^r(t)) \in E, \) where

\[
f^l(t) = \sum_{l_i(t) \geq 0} l_i(t)u^l_i + \sum_{l_i(t) < 0} l_i(t)u^r_i,
\]

\[
f^c(t) = \sum_{i=0}^{n} l_i(t)u^c_i.
\]
\[ f^r(t) = \sum_{l_i(t) \geq 0} l_i(t) u_i^r + \sum_{l_i(t) < 0} l_i(t) u_i^l, \]

such that \( l_i(t) = \prod_{j \neq i} \frac{t - t_j}{t_i - t_j} \).

The most important problem, examined so far, is concerned with the existence of solutions of considered equations. The present study deals mainly with the numerical solution of fuzzy differential equations by using various methods. It is solved mostly by using the predictor-corrector method, Nystrom method and RKGL method. Finally we prove the existence of solutions of fuzzy Volterra integral equations with infinite delay by using the successive approximation method.

### 1.6 Author’s Contributions

In the light of the above description of the problem and methodology, the author has obtained some significant results on the following topics:

1. Numerical solution of hybrid fuzzy differential equations by predictor-corrector method
2. Numerical solution of hybrid fuzzy differential equations by the Nystrom method
3. Hybrid numerical methods for fuzzy exponential models of growth
4. Numerical solution of fuzzy differential equations by the RKGL method
5. Fuzzy Volterra integral equations with infinite delay.

The rest of the thesis presents the various results established by the author on the above topics.