Chapter VI

Fuzzy Volterra Integral Equations with Infinite Delay
Chapter 6

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6.1 Introduction

The initial value problem for fuzzy differential and integral equations has been studied by several authors [22, 42, 43, 62, 74, 77, 95] on the metric space \((E^n, D)\) of normal fuzzy convex sets with the distance \(D\) given by the maximum of the Hausdorff distances between the corresponding level sets. Subrahmanyam and Sudarsanam [97, 98, 99] proved existence theorems for fuzzy valued mappings using the concept of a fuzzy integral due to Puri and Ralescu [86]. Diamond [32] introduced a theory to study the fuzzy Volterra integral equations. Song et al. [93] discussed the existence and comparison theorems for fuzzy Volterra integral equations in \((E^n, D)\). Balachandran and Prakash [16] studied the existence of solutions of nonlinear fuzzy Volterra-Fredholm integral equations by using the successive approximation method. Agarwal et al. [5] have given a very general formulation of the stacking theorem approach for fuzzy Volterra integral equations. In this chapter, we study the problem of existence of solutions of fuzzy Volterra integral equations with infinite delay by using the method of successive approximation.
6.2 Preliminaries

Consider the fuzzy Volterra integral equation with infinite delay of the form

\[ y'(t) = h(t, y(t)) + \int_{-\infty}^{t} q(t, s, y(s))ds, \quad t \in T = (-\infty, \infty), \quad (6.2.1) \]

where \( h : T \times E^n \to E^n \) and \( q : T \times T \times E^n \to E^n \) are levelwise continuous and satisfy the generalized Lipschitz conditions.

**Basic Assumption:** For each \( t_0 \in T \), there exists a nonempty convex subset \( B(t_0) \) of the space of continuous functions \( \phi : T_1 = (-\infty, t_0] \to E^n \) such that \( \phi \in B(t_0) \) implies

\[ \int_{-\infty}^{t_0} q(t, s, \phi(s))ds = Q(t, t_0, \phi) \]

is continuous on \( T_2 = [t_0, \infty) \). For a given \( t_0 \in T_1 \) and a continuous initial function \( \phi : T_1 \to E^n \), we seek a continuous solution \( y(t, t_0, \phi) \) satisfying (6.2.1) for \( t \in [t_0, t_0 + \beta) \) for some \( \beta > 0 \) with \( y(t, t_0, \phi) = \phi(t) \) for \( t \leq t_0 \).

Assume that \( h : T_0 \times E^n \to E^n \) and \( q : T_0 \times T_0 \times E^n \to E^n \) are levelwise continuous, where \( T_0 = \{ t \in T : t_0 \leq t < t_0 + \beta \} \). Consider the fuzzy Volterra integral equation (6.2.1) where \( \phi(t_0) \in E^n \). We let \( J = T_0 \times B(\phi(t_0), b) \) and \( J_0 = T_0 \times T_0 \times B(\phi(t_0), b) \) where \( a > 0, \ b > 0, \ \phi(t_0) \in E^n \), and

\[ B(\phi(t_0), b) = \{ y \in E^n : D(y, \phi(t_0)) \leq b \}. \]

**Definition 6.2.1.** A mapping \( y : T_0 \to E^n \) is a solution to the problem (6.2.1) if it is levelwise continuous and satisfies the integral equation

\[ y(t) = \phi(t_0) + h(s, y(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, y(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi)du, \]

for all \( t \in T_0 \).
To establish the results, we shall assume the following conditions:

(A) $h : J \rightarrow E^n$ is levelwise continuous and for any pair $(t, x), (t, y) \in J$ and $\alpha \in [0, 1]$, we have

$$d([h(t, x)]^\alpha, [h(t, y)]^\alpha) \leq k_h d([x]^\alpha, [y]^\alpha),$$

where $k_h$ is a given constant.

(B) $q : J_0 \rightarrow E^n$ is levelwise continuous and for any pair $(t, s, x_1), (t, s, x_2) \in J_0$, $-b \leq s \leq t \leq b$ and $\alpha \in [0, 1]$, we have

$$d([q(t, s, x_1)]^\alpha, [q(t, s, x_2)]^\alpha) \leq k_q d([x_1]^\alpha, [x_2]^\alpha),$$

where $k_q$ is a given constant.

(C) Let $K = \max\{k_h, k_q\}$ be such that $0 < K < 1$.

### 6.3 Main Result

**Theorem 6.3.1.** If the conditions (A)-(C) hold, then there exists a unique solution $y = y(t)$ of (6.2.1) defined on the interval $t_0 \leq t < t_0 + \beta$.

**Proof.** Let $0 < L < \beta$ be given. Therefore $t_0 \leq t \leq t_0 + L$.

Let

$$\delta = \min \left\{ L, \sqrt{\left( \frac{M + M_2}{M_1} \right)^2 + \frac{2b}{M_1} - \left( \frac{M + M_2}{M_1} \right)} \right\},$$

where $M = D(h(t, y), \hat{0}), \hat{0} \in E^n$, such that $\hat{0}(t) = 1$ for $t = 0$ and 0 otherwise, and for any $(t, y) \in J$ and $M_1 = D(q(u, s, \phi(t_0)), \hat{0})$ for any $(u, s, \phi(t_0)) \in J_0$ and $M_2 = D(Q(u, t_0, \phi(t_0)), \hat{0})$ for any $(u, t_0, \phi(t_0)) \in J_0$. 

We show that the sequence of functions is defined inductively on \([t_0, t_0 + L]\) by

\[
y_0(t) \equiv \phi(t_0), \quad t \in T_0,
\]
\[
y_n(t) = \phi(t_0) + \int_{t_0}^{t} h(s, y_{n-1}(s)) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, y_{n-1}(s)) \, ds \, du
\]
\[
+ \int_{t_0}^{t} Q(u, t_0, \phi) \, du, \quad n = 1, 2, 3, \ldots.
\]  \hfill (6.3.1)

From (6.3.1), it follows that, for \(n = 1\),

\[
y_1(t) = \phi(t_0) + \int_{t_0}^{t} h(s, \phi(t_0)) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, \phi(t_0)) \, ds \, du
\]
\[
+ \int_{t_0}^{t} Q(u, t_0, \phi(t_0)) \, du,
\]  \hfill (6.3.2)

which proves that \(y_1(t)\) is levelwise continuous on \(|t - t_0| \leq L\) and hence on \(|t - t_0| \leq \delta\). Moreover, for any \(\alpha \in [0, 1]\), we have

\[
d([y_1(t)]^{\alpha}, [y_0(t)]^{\alpha}) = d\left(\left[\phi(t_0) + \int_{t_0}^{t} h(s, \phi(t_0)) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, \phi(t_0)) \, ds \, du
\right]^{\alpha}, [\phi(t_0)]^{\alpha}\right)
\]
\[
\leq \int_{t_0}^{t} d([h(s, \phi(t_0))]^{\alpha}, \hat{0}) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} d([q(u, s, \phi(t_0))]^{\alpha}, \hat{0}) \, ds \, du
\]
\[
+ \int_{t_0}^{t} d([Q(u, t_0, \phi(t_0))]^{\alpha}, \hat{0}) \, du,
\]

and by the definition of \(D\), we get

\[
D(y_1(t), y_0(t)) \leq \int_{t_0}^{t} D(h(s, \phi(t_0)), \hat{0}) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} D(q(u, s, \phi(t_0)), \hat{0}) \, ds \, du
\]
\[
+ \int_{t_0}^{t} D(Q(u, t_0, \phi(t_0)), \hat{0}) \, du
\]
\[
\leq (M + M_2)|t - t_0| + M_1 \frac{|t - t_0|^2}{2!}
\]
\[
\leq (M + M_2)\delta + M_1 \frac{\delta^2}{2!}
\]
\[
\leq b.
\]  \hfill (6.3.3)
Now assume that \( y_{n-1}(t) \) is levelwise continuous on \( |t - t_0| \leq \delta \) and that

\[
D(y_{n-1}(t), y_0(t)) \leq b.
\]

From (6.3.1), we deduce that \( y_n(t) \) is levelwise continuous on \( |t - t_0| \leq \delta \) and that

\[
D(y_n(t), y_0(t)) \leq b.
\]

Consequently, we conclude that \( y_n(t) \) consists of levelwise continuous mappings on \( |t - t_0| \leq \delta \) and that

\[
(t, y_n(t)) \in J \text{ and } (t, s, y_n(t)) \in J_0, \quad |t - t_0| \leq \delta, \quad n = 1, 2, \ldots
\]

Let us prove that there exists a fuzzy set-valued mapping \( y : [t_0, t_0 + L] \to E^n \) such that \( D(y_n(t), y(t)) \to 0 \) uniformly on \( |t - t_0| \leq \delta \) as \( n \to \infty \).

For \( n = 2 \), from (6.3.1),

\[
y_2(t) = \phi(t_0) + \int_{t_0}^{t} h(s, y_1(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, y_1(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du.
\]

(6.3.4)

From (6.3.2) and (6.3.4), we have

\[
d([y_2(t)]^\alpha, [y_1(t)]^\alpha) = d\left(\left[\int_{t_0}^{t} h(s, y_1(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, y_1(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du\right]^\alpha, \right.
\]

\[
\left.\left[\int_{t_0}^{t} h(s, \phi(t_0))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, \phi(t_0))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du\right]^\alpha\right) \leq k_h \int_{t_0}^{t} d([y_1(s)]^\alpha, [\phi(t_0)]^\alpha)ds + k_q \int_{t_0}^{t} \int_{t_0}^{u} d([y_1(s)]^\alpha, [\phi(t_0)]^\alpha)dsdu.
\]

So, by the definition of \( D \), we have

\[
D(y_2(t), y_1(t)) \leq k_h \int_{t_0}^{t} D(y_1(s), \phi(t_0))ds
\]

\[
+ k_q \int_{t_0}^{t} \int_{t_0}^{u} D(y_1(s), \phi(t_0))dsdu.
\]

(6.3.5)
Now we apply the first inequality (6.3.3) in the right-hand side of (6.3.5) to get
\[
D(y_2(t), y_1(t)) \leq (M + M_2) K \frac{|t - t_0|^2}{2!} + M_1 K \frac{|t - t_0|^3}{3!} + (M + M_2) K \frac{|t - t_0|^3}{3!} + M_1 K \frac{|t - t_0|^4}{4!} \\
\leq K \left[ (M + M_2) \frac{\delta^2}{2!} + (M + M_1 + M_2) \frac{\delta^3}{3!} + M_1 \frac{\delta^4}{4!} \right]. \tag{6.3.6}
\]

From (6.3.3) and (6.3.6), we get
\[
D(y_n(t), y_{n-1}(t)) \leq K^{n-1} \left[ \sum_{i=0}^{n-1} (n-1) C_0 (M + M_2) \frac{\delta^n}{n!} \right. \\
\quad + \left. \sum_{i=0}^{n-1} (n-1) C_1 (M + M_2) + (n-1) C_0 M_1 \frac{\delta^{n+1}}{(n+1)!} + \cdots \right] \\
\quad + \left[ \sum_{i=0}^{n-1} (n-1) C_{n-1} (M + M_2) + (n-1) C_{n-2} M_1 \frac{\delta^{2n-1}}{(2n-1)!} \right] \\
\quad + M_1 \frac{\delta^{2n}}{2n!}, \tag{6.3.7}
\]

and we prove that such an inequality holds for \(D(y_{n+1}(t), y_n(t))\).

Indeed, from (6.3.1) and the assumptions, it follows that
\[
d([y_{n+1}(t)]^\alpha, [y_n(t)]^\alpha) =
\left[ \phi(t_0) + \int_{t_0}^t h(s, y_n(s)) ds + \int_{t_0}^t \int_{t_0}^u q(u, s, y_n(s)) ds du + \int_{t_0}^t Q(u, t_0, \phi) du \right]^\alpha \\
\left[ \phi(t_0) + \int_{t_0}^t h(s, y_{n-1}(s)) ds + \int_{t_0}^t \int_{t_0}^u q(u, s, y_{n-1}(s)) ds du + \int_{t_0}^t Q(u, t_0, \phi) du \right]^\alpha \\
\leq k_n \int_{t_0}^t d([y_n(s)]^\alpha, [y_{n-1}(s)]^\alpha) ds + k_q \int_{t_0}^t \int_{t_0}^u d([y_n(s)]^\alpha, [y_{n-1}(s)]^\alpha) ds du,
\]

for any \(\alpha \in [0, 1]\) and from the condition on \(D\), we have
\[
D(y_{n+1}(t), y_n(t)) \leq k_n \int_{t_0}^t D(y_n(s), y_{n-1}(s)) ds \\
+ k_q \int_{t_0}^t \int_{t_0}^u D(y_n(s), y_{n-1}(s)) ds du.
\]
By (6.3.7), we get
\[ D(y_{n+1}(t), y_n(t)) \leq K^n \left[ nC_0(M + M_2) \frac{\delta^{n+1}}{(n+1)!} \right. \]
\[ + \left[ nC_1(M + M_2) + nC_0M_1 \right] \frac{\delta^{n+2}}{(n+2)!} + \cdots \]
\[ + \left[ nC_n(M + M_2) + nC_{n-1}M_1 \right] \frac{\delta^{2n+1}}{(2n+1)!} \]
\[ + M_1 \frac{\delta^{2n+2}}{(2n+2)!} \right]. \]

Consequently inequality (6.3.7) holds for \( n = 1, 2, \ldots \). We also write
\[ D(y_n(t), y_{n-1}(t)) \leq \frac{K^n}{K} \left[ n^{-1}C_0(M + M_2) \frac{\delta^n}{n!} \right. \]
\[ + \left[ (n-1)C_1(M + M_2) + (n-1)C_0M_1 \right] \frac{\delta^{n+1}}{(n+1)!} + \cdots \]
\[ + \left[ (n-1)C_{n-1}(M + M_2) + (n-1)C_{n-2}M_1 \right] \frac{\delta^{2n-1}}{(2n-1)!} \]
\[ + M_1 \frac{\delta^{2n}}{(2n)!} \right], \] for \( n = 1, 2, \ldots \), and \( |t - t_0| \leq \delta \). (6.3.8)

Let us mention that
\[ y_n(t) = y_0(t) + [y_1(t) - y_0(t)] + \cdots + [y_n(t) - y_{n-1}(t)], \]
which implies that the sequence \( \{x_n(t)\} \) and the series
\[ y_0(t) + \sum_{n=1}^{\infty} [y_n(t) - y_{n-1}(t)] \]
have the same convergence properties.

From (6.3.8), it follows that \( D(y_n(t), y_{n-1}(t)) \to 0 \) uniformly on \( |t - t_0| \leq \delta \) as \( n \to \infty \). Hence there exists a fuzzy set-valued mapping \( y : [t_0, t_0 + L] \to E^n \) such that \( D(y_n(t), y(t)) \to 0 \) uniformly on \( |t - t_0| \leq \delta \) as \( n \to \infty \). From the assumptions, we get
\[ d([h(t, y_n(t))]^\alpha, [h(t, y(t))]^\alpha) \leq k_\alpha d([y_n(t)]^\alpha, [y(t)]^\alpha) \]
for any \( \alpha \in [0, 1] \) and so
\[ D(h(t, y_n(t)), h(t, y(t))) \leq k_h D(y_n(t), y(t)) \rightarrow 0 \] (6.3.9)

uniformly on \(|t - t_0| \leq \delta\) as \(n \rightarrow \infty\). Furthermore

\[ d([q(t, s, y_n(s))]^\alpha, [q(t, s, y(s))]^\alpha) \leq k_q d([x_n(s)]^\alpha, [x(s)]^\alpha) \]

for any \(\alpha \in [0, 1]\) and

\[ D(q(t, s, y_n(s)), q(t, s, y(s))) \leq k_q D(y_n(s), y(s)) \rightarrow 0 \] (6.3.10)

uniformly on \(|t - t_0| \leq \delta\) as \(n \rightarrow \infty\).

Taking (6.3.9) and (6.3.10) into account, from (6.3.1), we obtain

\[
y(t) = \phi(t_0) + \int_{t_0}^{t} h(s, y(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, y(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du \quad \text{for} \quad n \rightarrow \infty.
\]

Consequently there is at least one levelwise continuous solution of (6.2.1).

We prove now that this solution is unique, that is, from

\[
z(t) = \phi(t_0) + \int_{t_0}^{t} h(s, z(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, z(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du \] (6.3.11)

on \(|t - t_0| \leq \delta\), we show that \(D(y(t), z(t)) \equiv 0\). Indeed, from (6.3.1) and (6.3.11), we have

\[
d([z(t)]^\alpha, [y_n(t)]^\alpha) = 
\begin{align*}
d\left([\phi(t_0) + \int_{t_0}^{t} h(s, z(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, z(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi)du]^\alpha, \\
[\phi(t_0) + \int_{t_0}^{t} h(s, y_{n-1}(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, y_{n-1}(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi)du]^\alpha\right)
\end{align*}
\leq k_h \int_{t_0}^{t} d([z(s)]^\alpha, [y_{n-1}(s)]^\alpha)ds + k_q \int_{t_0}^{t} \int_{t_0}^{u} d([z(s)]^\alpha, [y_{n-1}(s)]^\alpha)dsdu
\]

where \(k_h, k_q\) are constants related to the functions \(h, q\).
for any $\alpha \in [0, 1]$, $n = 1, 2, \ldots$.

By the definition of $D$, we have

$$D(z(t), y_n(t)) \leq K \int_{t_0}^{t} D(z(s), y_{n-1}(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{u} D(z(s), y_{n-1}(s)) ds du.$$

But $D(z(t), y_0(t)) \leq b$ on $|t - t_0| \leq \delta$, $z(t)$ being a solution of (6.3.11). It, follows from (6.3.12), that

$$D(z(t), y_1(t)) \leq K \int_{t_0}^{t} D(z(s), y_0(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{u} D(z(s), y_0(s)) ds du$$

$$\leq K b \left[ |t - t_0| + \frac{|t - t_0|^2}{2!} \right] \text{ on } |t - t_0| \leq \delta.$$ 

Also

$$D(z(t), y_2(t)) \leq K \int_{t_0}^{t} D(z(s), y_1(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{u} D(z(s), y_1(s)) ds du$$

$$\leq K^2 b \left[ \frac{|t - t_0|^2}{2!} + 2 \frac{|t - t_0|^3}{3!} + \frac{|t - t_0|^4}{4!} \right]$$

on $|t - t_0| \leq \delta$. Now assume that

$$D(z(t), y_n(t)) \leq K^n b \left[ (n-1) C_0 \frac{|t - t_0|^n}{n!} + (n-1) C_1 \frac{|t - t_0|^{n+1}}{(n+1)!} \right. $$

$$\left. + \cdots + (n-1) C_n \frac{|t - t_0|^{2n}}{(2n)!} \right]$$

on the interval $|t - t_0| \leq \delta$. From

$$D(z(t), y_{n+1}(t)) \leq K \int_{t_0}^{t} D(z(s), y_n(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{u} D(z(s), y_n(s)) ds du$$

and (6.3.13), we get

$$D(z(t), y_{n+1}(t)) \leq K^{n+1} b \left[ n C_0 \frac{|t - t_0|^{n+1}}{(n+1)!} + n C_1 \frac{|t - t_0|^{n+2}}{(n+2)!} \right.$$

$$\left. + \cdots + n C_n \frac{|t - t_0|^{2n+1}}{(2n+1)!} \right].$$
Consequently (6.3.13) holds for any $n$ leading to the conclusion

$$D(z(t), y_n(t)) = D(x(t), x_n(t)) \to 0$$

on the interval $|t - t_0| \leq \delta$ as $n \to \infty$. Thus there exists a unique solution on $[t_0, t_0 + L]$ since $L$ is arbitrary in $(0, \beta)$. Therefore there exists a unique solution on $[t_0, t_0 + \beta)$. □