Chapter 4

Controlled nanoscale magnetization switching through soliton
4.1 Introduction

Magnetization switching in magnetic material is one of the fundamental issues in spin dynamical studies. This problem is of considerable importance in high density magnetic recording, data storage, fast programmable logic and in high frequency devices for telecommunication. In magnetic memory device, a logical bit is stored in a unit cell of magnetic materials under the influence of external magnetic field by setting the orientation of the magnetization vector inside the cell in one of the two possible directions. But the speed and reliability with which the bit can be set is still limited. Conventionally, the bit has been changed by applying external magnetic field which causes the nucleation of the magnetic domains in the unit cell followed by expansion of the magnetic domains. The speed of both the processes are inherently limited. But it was realized that the speed of the magnetization reversal process has been increased to certain limit through precession of magnetization vector by the application of instantaneous magnetic field [327,328]. Moreover, the magnetization reversal through the domain wall motion and spin lattice relaxation in a nanosecond regime was studied under quasi-equilibrium conditions [329]. The process of magnetization reversal in a ferromagnet can occur via a nucleation of a domain with a reversed magnetization, motion of domain walls and destruction of the domain structure [330,331].

The defects are the active centers for the formation of nuclei of a magnetization that have been investigated experimentally [332]. The dynamics of large-amplitude magnetic inhomogeneities are the next stage of the formation of a reversed domain with allowance for magnetostatic interaction which can be studied in terms of a one-dimensional soliton models [333-338]. However, due to the enormous increase in data transfer rates, the analysis of dynamics of magnetization switching become a crucial issue. The interactions of solitons with the imperfections in the ferromagnetic medium is an important aspect of nonlinear physics. In recent years, the investigation of the nonlinear dynamics of inhomogeneous systems has assumed much significance because these systems are considered to be realistic. Also, the importance becomes particularly
apparent when solitons are used to model the nonlinear wave propagation in dispersive and inhomogeneous media such as radio waves in the ionosphere, waves in the ocean, optical pulses in glass fibers, laser radiations in a plasma and impurities or defects in magnetic systems [339]. Most of the studies on magnetic spin chains have been based on the homogeneous Heisenberg Hamiltonian when the exchange integral associated with the coupling interaction between nearest-neighbour pair of spins is a single constant $J$. But due to the presence of various magnetic defects such as imperfect grain boundaries, impurities as well as elastic defects such as misfitting precipitates and dislocation, there arises an inhomogeneity in the magnetic lattice that makes the dynamics more complex. Defects are local deviations from a given ordered structure and have attracted large interest in many areas like biology [340], liquid crystals [341], soap bubbles [342], alloys [343] and soft matter [344] as their formation and evolution are one of the fundamental problems to understand the complexity in these systems. The numerical solution of Landau-Lifshitz-Gilbert equation provides the theoretical background for the switching process of ferromagnetic structures [345]. The magnetization switching is mainly depends on inhomogeneity present in the spin chain and switching time entirely depends on the damping parameter and size of the magnetic particles. In Ref. [346], Braun has proposed that with increasing the length of the ferromagnetic Heisenberg spin chain, a different reversal mechanism becomes energetically favorable through soliton-antisoliton nucleation process. In this chapter, we investigate the magnetization reversal process through soliton flip in a site-dependent biquadratic ferromagnetic spin chain with crystal field anisotropy in semi-classical limit by making use of inherent properties present in the spin chain and the integrability aspect of the above system is also investigated through the Painlevé's singularity analysis. Also, we show that an inhomogeneous ferromagnetic spin chain with nonlinear inhomogeneity is a good candidate for inducing the magnetization reversal process in a nano-scale regime.
4.2 Model Hamiltonian and spin dynamics

The Heisenberg model of ferromagnet with different magnetic interactions in the semi-classical continuum limit have been identified as an interesting class of nonlinear dynamical systems because of their connection to the nonlinear Schrödinger family of equations and soliton spin excitations [151, 347-349]. The Heisenberg Hamiltonian for a crystal field anisotropic (including higher order anisotropy) ferromagnetic spin system with site-dependent bilinear and biquadratic exchange interactions are given by

\[
\tilde{H} = - \sum_i \left[ J_{e} f_i (\vec{S}_i \cdot \vec{S}_{i+1}) + J_{b} g_i (\vec{S}_i \cdot \vec{S}_{i+1})^2 + A(S^z_i)^2 + A_1 (S^z_i)^4 \right],
\]

where \( \vec{S}_i = (S^x_i, S^y_i, S^z_i) \) is three component spin vector and \( J_e \) and \( J_b \) are the bilinear and biquadratic exchange interaction coefficients respectively. The functions \( f_i \) and \( g_i \) represent their site-dependent character respectively and \( A \) and \( A_1 \) are the lower and higher order anisotropy parameters respectively. For our analysis, we define \( \hat{S}_i^\pm = \hat{S}_i^x \pm i\hat{S}_i^y \), where \( \hat{S}_i \equiv \vec{S}_i/\hbar \) and rewrite Eq. (4.1) in the dimensionless form

\[
H = \frac{1}{2S^2} \sum_i \left[ J_{e} f_i \left[ \hat{S}_i^+ \hat{S}_{i+1}^+ + \hat{S}_i^- \hat{S}_{i+1}^- + 2\hat{S}_i^z \hat{S}_{i+1}^z \right] + \frac{J_{b} g_i}{S^2} \left[ \hat{S}_i^+ \hat{S}_{i+1}^- \hat{S}_i^+ \hat{S}_{i+1}^- + \hat{S}_i^- \hat{S}_{i+1}^+ \hat{S}_i^- \hat{S}_{i+1}^+ + 2\hat{S}_i^z \hat{S}_{i+1}^z \hat{S}_i^+ \hat{S}_{i+1}^- + 4\hat{S}_i^z \hat{S}_{i+1}^z \hat{S}_i^- \hat{S}_{i+1}^+ \right] + 4A(S^z_i)^2 + \frac{4A_1}{S^2} (\hat{S}_i^z)^4 \right],
\]

where \( H = \frac{\hbar}{\hbar^2 S^2} J_b \), \( J_b = \hbar^2 S^2 J_b \) and \( A_1 = \hbar^2 S^2 A_1 \). The dimensionless spin operator \( \hat{S}_i \) satisfies the commutation relations \([\hat{S}_n^+, \hat{S}_m^-] = 2\delta_{mn} \hat{S}_n^z, [\hat{S}_n^+, \hat{S}_m^+] = \mp \delta_{mn} \hat{S}_n^+ \) with \( \hat{S}_n \cdot \hat{S}_n = S(S+1) \). In order to treat the dynamics of our spin system in the semi-classical limit, we introduce the Holstein-Primakoff transformation [136] for the spin operators in terms of boson operators,

\[
\hat{S}_n^+ = (2S)^{1/2} \left[ 1 - \frac{a_n^+ a_n}{2S} \right]^{1/2} a_n,
\]

\[
\hat{S}_n^- = (2S)^{1/2} a_n^\dagger \left[ 1 - \frac{a_n^+ a_n}{2S} \right]^{1/2},
\]

\[
\hat{S}_n^z = \frac{2S}{(2S)^{1/2}} \left[ \frac{2S}{(2S)^{1/2}} \right]^{1/2} a_n^\dagger a_n + \frac{1}{2} S(S+1).
\]
\[ \hat{S}_n^z = \left[ S - a_n^\dagger a_n \right]^{1/2}, \quad (4.5) \]

with boson operators \( a_n^\dagger \) and \( a_n \) satisfying the usual commutation relations. For large spins \( < a_n^\dagger a_n >> 2S \) and therefore we use the semi-classical expansions for \( \hat{S}_n^+ \) and \( \hat{S}_n^- \) in the following form

\[ \frac{\hat{S}_n^+}{S} = \sqrt{2} \left[ 1 - \frac{\epsilon^2}{4} a_n^\dagger a_n - \frac{\epsilon^4}{32} a_n^\dagger a_n^\dagger a_n^\dagger a_n - \frac{\epsilon^6}{128} a_n^\dagger a_n^\dagger a_n^\dagger a_n^\dagger a_n^\dagger a_n - O(\epsilon^8) \right] \epsilon a_n, \quad (4.6) \]

\[ \frac{\hat{S}_n^-}{S} = \sqrt{2} \epsilon a_n^\dagger \left[ 1 - \frac{\epsilon^2}{4} a_n^\dagger a_n - \frac{\epsilon^4}{32} a_n^\dagger a_n^\dagger a_n^\dagger a_n - \frac{\epsilon^6}{128} a_n^\dagger a_n^\dagger a_n^\dagger a_n^\dagger a_n^\dagger a_n - O(\epsilon^8) \right], \quad (4.7) \]

where \( \epsilon = 1/\sqrt{S} \). Using this formulation, our Hamiltonian Eq. (4.2) can be written as a series in \( \epsilon \). Knowing the Hamiltonian for a crystal field anisotropic ferromagnetic spin system, the dynamics of the spins can be expressed in terms of the Heisenberg equation of motion for the Bose operator as

\[ i\hbar \frac{\partial a_n}{\partial t} = [a_n, H] = F(a_n^\dagger, a_n, a_{n+1}^\dagger a_{n+1}). \quad (4.8) \]

After introducing Glauber’s coherent-state representation [149] for the Bose operators \( a_n^\dagger |u >= u_n^\dagger |u >, a_n |u >= u_n |u >, |u >= \Pi |u_n > \) with \( < u |u > = 1 \) where \( |u(n) > \) is the coherent-state eigen vector for the operator \( a_n \) and \( u_n \) represents the coherent amplitude for the system in the state \( |u > \). Then, we write down equation of motion for the average \( < u |a_j |u > \) using Eq. (4.8).
\begin{align}
-2f_j|u_j|^2|u_{j+1}^*|^2 - & f_{j-1}|u_{j-1}|^4u_j - 3f_{j-1}|u_j|^4u_{j-1} + J'_b\{-32g_j|u_{j+1}|^2|u_j|^2u_j^* - 32g_{j-1}|u_{j-1}|^2|u_j|^2u_j^* - 48g_j|u_{j+1}|^2u_j^2 - 48g_{j-1}|u_{j-1}|^2u_j^2u_j^* + 24g_j|u_{j+1}|^2u_j^2u_{j+1}^* + 24g_{j-1}|u_{j-1}|^2u_j^2u_{j-1}^* - 32g_{j-1}|u_{j-1}|^2u_j^2u_j^* + 36g_j|u_{j+1}|^2u_j^2u_j^* - 36g_{j-1}|u_{j-1}|^2u_j^2u_j^* + 12g_j|u_{j+1}|^2u_j^2u_{j+1}^* + 12g_{j-1}|u_{j-1}|^2u_j^2u_{j-1}^* + 36g_j|u_{j+1}|^2u_j^2u_{j+1}^* - 36g_{j-1}|u_{j-1}|^2u_j^2u_{j-1}^* - 384\lambda_1|u_j|^4u_j^*\}.
\end{align}

Eq. (4.9) represents the equation of motion which is very difficult to solve in its normal form because of its high nonlinearity and discreteness. Therefore, it is necessary to make continuum approximation in the low temperature and long wave length limit by assuming that the lattice constant is very small compared to the length of the lattice and restrict to one-dimension by expanding

\[ u_{j\pm 1} = u(x, t) \pm \gamma u_x + \frac{\gamma^2}{2!} u_{xx} \pm \frac{\gamma^3}{3!} u_{xxx} \ldots \pm O(\gamma^5), \]

where \( x = j\gamma \), \( \gamma \) is the lattice parameter and suffix \( x \) represents partial derivative with respect to \( x \). Also, we introduce the expansions for \( f_{j-1} \) and \( g_{j-1} \) in the same way as in Eq. (4.10)

\begin{align}
 f_{n+1} &= f(x, t) \pm \gamma f_x + \frac{\gamma^2}{2!} f_{xx} \pm \frac{\gamma^3}{3!} f_{xxx} \ldots \pm O(\gamma^5), \quad (4.11a) \\
 g_{n+1} &= g(x, t) \pm \gamma g_x + \frac{\gamma^2}{2!} g_{xx} \pm \frac{\gamma^3}{3!} g_{xxx} \ldots \pm O(\gamma^5). \quad (4.11b)
\end{align}

The final form of equation of motion at \( O(\gamma^m\epsilon^n) \) with \( m + n = 6 \) when \( \epsilon = \gamma \) can be written as

\[ iu_t + K_1u_{xx} + K_2|u|^2u + K_3[u_{xxx} + 8|u|^2u_{xx} + 6u^*u_x^2 + 4|u_x|^2u + 2u^2u_{xx}] + K_4|u|^4u + K_5u_{xxxx} = 0, \]

(4.12)

where \( K_1 = 8\kappa + \gamma^2\kappa_{xx} \), \( K_2 = -[8J'_b g + \frac{A}{3}\gamma + \gamma^2\kappa_{xx}] \), \( K_3 = \gamma^2\kappa \), \( K_4 = -\gamma^2(3J'_b g - A_1) \), \( K_5 = \frac{2}{3}\gamma\kappa \) and \( \kappa = J_c f + 2J'_b g \). Thus, the dynamics of one-dimensional anisotropic (crystal field) continuum Heisenberg ferromagnetic spin chain with varying bilinear and biquadratic exchange interactions in the semi-classical limit is gov-
4.3 Search for integrable models

Magnetic solitons have been identified as one of the interesting class of nonlinear excitations representing nonlinear spin dynamics in one-dimensional classical continuum Heisenberg ferromagnetic spin systems. Therefore, we now try to see whether the nonlinear spin excitations of our ferromagnetic model can be expressed in terms of magnetic solitons by solving Eq. (4.12). However, before actually solving the evolution equation (4.12) we try to check for integrability of it by carrying out Painlevé singularity structure analysis [353].

4.3.1 Painlevé singularity structure analysis

The Painlevé (P-) singularity structure analysis [353] is a well recognized systematic and algorithmic procedure to verify whether a given nonlinear partial differential equation is integrable and to analyze the integrability properties. This approach aims to decide the presence or absence of movable noncharacteristic critical manifolds. The absence of critical manifolds indicate that Painlevé property holds suggesting the evolution equation's integrability. Otherwise, the system is nonintegrable in general.

The P-analysis aims to isolate those cases for which the system is free from
movable critical manifolds so that the general solution of Eq. (4.12) will be single valued around a noncharacteristic singular manifold \( \phi(x, t) = 0 \). For this we express the solution of the nonlinear evolution equation in the form of a Laurent series. The P-analysis involves the following major steps: (i) the determination of leading orders of the Laurent series (ii) identification of the powers called resonance at which the arbitrary functions can enter into the Laurent series (iii) verification at the resonance values to check sufficient number of arbitrary functions that exist without the introduction of movable critical manifolds and finally as a consequence (iv) construction of the Lax pair, soliton solutions and other integrability properties. Thus, the given nonlinear evolution equation satisfies the P-test, as commonly applied, if it passes the following four hurdles: (i) each possible leading term must have an integer power (with no logarithmic power) (ii) for each such leading term, the so called resonance powers must all be integers (iii) the terms successive to the leading terms generated recursively must have powers increasing by integer steps (and no logarithms) and (iv) at every resonance, the resonance condition must be satisfied assuming that no logarithmic term is generated. In order to check whether Eq. (4.12) is completely integrable in general so that the elementary spin excitations can be expressed in term of solitons if not for specific choice of \( K_i, i = 1, 2, 3, 4 \), we perform Painleve singularity structure analysis [354] on Eq. (4.12). Expressing \( u \) and \( u^* \) by \( E(x, t) \) and \( G(x, t) \) respectively, Eq. (4.12) and its conjugate equation can be written as

\[
iE_t + K_1 E_{xx} + K_2 E^2 G + K_3 [E_{xxxx} + 8EGE_{xx} + 2E^2G_{xx} + 4EE_xG_x \
+ 6E_x^2G] + K_4 E^3 G^2 + K_5 E_{xxxx} = 0,
\]

(4.13)

\[
iG_t + K_1 G_{xx} + K_2 EG^2 + K_3 [G_{xxxx} + 8EGG_{xx} + 2G^2E_{xx} + 4GE_xG_x \
+ 6G_x^2E] + K_4 E^2 G^3 + K_5 E_{xxxx} = 0.
\]

(4.14)

We expand the functions \( E(x, t) \) and \( G(x, t) \) locally in the form of Laurent series
about a nonlinear characteristic singular manifold \( \phi(x, t) = 0 \),

\[
E = E_0(x, t)\phi(x, t)^p + \sum_{j=1} E_j(x, t)\phi(x, t)^{p+j}, \tag{4.15}
\]

\[
G = G_0(x, t)\phi^q(x, t) + \sum_{j=1} G_j(x, t)\phi(x, t)^{q+j}. \tag{4.16}
\]

Using the leading order solutions i.e., \( E \sim E_0(x, t)\phi^p(x, t) \) and \( G \sim G_0(x, t)\phi^q(x, t) \) in Eqs. (4.13-4.14) and balancing the dominant terms, we obtain two branches of leading order results given by

\[
p = q = -1, \quad E_0G_0 = \frac{1}{K_4}[-15K_3 \pm \sqrt{225K_3^2 - 24K_3K_4}]. \tag{4.17}
\]

Now, to find the resonances, i.e., the powers at which free coefficients enter into the generalized Laurent expansion, we expand

\[
E = E_0\phi^p + \ldots + \delta_1\phi^{p+r}, \tag{4.18a}
\]

\[
G = G_0\phi^q + \ldots + \delta_1\phi^{q+r}, \tag{4.18b}
\]

where \( \delta_1 \) and \( \delta_2 \) are nonzero coefficients. Using these expansions for \( E \) and \( G \) in Eqs. (4.13-4.14) containing the leading order terms alone, we get the resultant resonance equations as

\[
\begin{pmatrix}
R_1 & -E_0^2R_2 \\
-G_0^2R_2 & R_1
\end{pmatrix}
\begin{pmatrix}
\delta_1 \\
\delta_2
\end{pmatrix} = 0, \tag{4.19}
\]

where

\[
R_1 = K_3[(r - 1)(r - 2)(r - 3)(r - 4) - E_0G_0(8r^2 - 40r + 60)] + 3K_4E_0^2G_0^2,
\]

\[
R_2 = 2K_3(r^2 - 5r + 15) + 2K_4E_0G_0.
\]

Making use of the leading order results in Eqs. (4.19), we find that integer resonances are possible only when \( A_1' = 3J_\epsilon f + \frac{15}{2}J'_\epsilon g \). For this choice, \( E_0G_0 \) reduce to the values \(-1\) and \(-4\) respectively and resonances corresponding to
the branches are written as

**Branch(i)**

\[ E_0 G_0 = -1, \ r = -1, 0, 1, 2, 3, 4, 5, 6. \]  \hspace{1cm} (4.20)

The resonances \(-1\) and \(0\) correspond to the arbitrariness of the singular manifold and the coefficients \(E_0\) or \(G_0\) respectively. In order to verify the existence of sufficient number of arbitrary functions at other resonance values, we substitute the complete Laurent expansion in the full Eqs. (4.13-4.14), collect the terms proportional to different power of \(\phi\) and solve them. For the branch (i) at \((\phi^{-4}, \phi^{-4})\), we find that \(E_1\) or \(G_1\) is arbitrary only when \(\kappa_x = 0\) implying that \(J_e f_x + 2J_bg_x = 0\). In the next order i.e., at \((\phi^{-3}, \phi^{-3})\), \(E_2\) or \(G_2\) becomes arbitrary when the anisotropy function \(A\) is expressed in terms of \(f\) and \(g\) as

\[ A = -24\gamma^2(2J_e f + 5J_b g) \]  Proceeding further we find that \(E_3\) or \(G_3\), \(E_4\) or \(G_4\), \(E_5\) or \(G_5\) and \(E_6\) are arbitrary.

**Branch(ii)**

\[ E_0 G_0 = -4, \ r = -3, -2, -1, 0, 5, 6, 7, 8. \]  \hspace{1cm} (4.21)

We also find that without introducing movable critical manifolds \(E_5\) or \(G_5\), \(E_6\) or \(G_6\), \(E_7\) or \(G_7\) and \(E_8\) or \(G_8\) are arbitrary at orders \((\phi^{-2}, \phi^{-2})\), \((\phi^{-1}, \phi^{-1})\), \((\phi^0, \phi^0)\) and \((\phi^1, \phi^1)\) respectively. Though, for both the cases we verified that the system passes Painlevé test without introducing movable critical manifolds for the (ii) because of the presence of negative resonances \(r = -3, -2\), we can write only a lesser parameter solution in this case and hence we pay our attention to the branch (i) only. Having found that the inhomogeneous biquadratic spin

<table>
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<td>(i)</td>
<td>( A'_1 = 3J_e f ) [+\frac{15}{2}J'_bg ]</td>
<td>( p = q = -1, \ E_0 G_0 = -1 )</td>
<td>( r = -1, 0, 1, 2, 3, 4, 5, 6 )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( A = -24\gamma^2(2J_e f ) [+5J'_bg ]</td>
<td>( p = q = -1, \ E_0 G_0 = -4 )</td>
<td>( r = -3, -2, -1, 0, 5, 6, 7, 8 )</td>
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Table 4.1: Leading order results of the singularity structure analysis.

chain is completely integrable for the choices \(J_e f_x + 2J'_bg_x = 0, A = -24\gamma^2(2J_e f + \)
\(5J'_g(g)\) and \(A'_1 = (3J_c f + \frac{15}{2} J'_c g)\) and the elementary spin excitations are governed by solitons. It may also be noted that after suitable rescaling Eq. (4.12) is completely integrable when the functions \(f(x) = g(x) = \text{constant}, A'_1 = 0\) and for the choice \(J_c = -\frac{5}{2} J'_c\),

\[
iu_t + uu_{xx} + 2|u|^2 u + \lambda [u_{xxxx} + 8|u|^2 u_{xx} + 2u^2 u_{xx} + 4|u_x|^2 u + 6u^* u_x^2 + 6|u|^4 u] = 0, \tag{4.22}\]

where \(\lambda = \frac{c^2}{12}\). The corresponding integrable spin chain can be written as

\[
\vec{S}_t = \vec{S} \wedge \left\{ \vec{S}_{xx} + \lambda \left[ \vec{S}_{xxxx} - \frac{5}{2} (\vec{S} \cdot \vec{S}_{xx}) \vec{S}_{xx} - \frac{5}{3} (\vec{S} \cdot \vec{S}_{xxx}) \vec{S}_x \right] \right\}. \tag{4.23}\]

Eq. (4.22) takes place in other contexts as well. For example, the molecular excitations along the hydrogen bonding spine in an alpha helical protein with higher order molecular interactions under specific parametric choices is also governed by the completely integrable higher order NLS equation. The linear eigenvalue problem for Eq. (4.22) which is obtained through the AKNS formalism [355] as

\[
V_x = U_1 V, \quad V_t = U_2 V, \quad V = (V_1, V_2)^T, \tag{4.24}\]

where

\[
U_1 = M - i\lambda \sigma_3, \tag{4.25}\]

\[
U_2 = \begin{bmatrix}
3i\gamma |q|^4 + i|q|^2 + i\gamma (q^* q_{xx} + q q_{xx}^* - |q_x|^2) + 8i\gamma \lambda^4 + 2\lambda \gamma (qq^* - q q^*) \\
-2i\lambda^2 (2\gamma |q|^2 + 1) \sigma_3 - 8\gamma \lambda^3 M - 4i\gamma \lambda^2 \sigma_3 M_x + 6i\gamma M^2 M_x \sigma_3 + i\sigma_3 M_x \\
+ i\gamma \sigma_3 M_{xxx} + 2\lambda (M + \gamma M_{xx} - 2\gamma M^3)
\end{bmatrix}, \tag{4.26}\]

here

\[
M = \begin{pmatrix}
0 & q \\
-q^* & 0
\end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}. \tag{4.27}\]

Also, the linear eigenvalue problem for the corresponding spin chain Eq. (4.23) is obtained using a suitable gauge transformation [356,357,318], which takes
the form
\[ \hat{U}_1 = -i \lambda S, \]  
(4.28)
\[ \hat{U}_2 = (8i \gamma \lambda^4 - 2i \lambda^2) \tilde{S} + \lambda \left[ \tilde{S} \tilde{S}_x + \gamma (\tilde{S} \tilde{S}_{xxx} + \frac{5}{2} \tilde{S} \tilde{S}_x^3 + \tilde{S}_x \tilde{S}_{xx} + 2 \tilde{S}_{xx} \tilde{S}_x) \right] - i \gamma \lambda^2 \left( 2 \tilde{S}_{xx} \tilde{S}_x^2 - 4i \lambda \tilde{S}_x \tilde{S}_x^2 \right), \]  
(4.29)
where \( \tilde{S} = \tilde{S} \cdot \tilde{\sigma} = g^{-1} \sigma_3 g, \) \( \tilde{\sigma} = (\sigma_2, \sigma_2, \sigma_3), \) and \( g \) is an arbitrary \( 2 \times 2 \) matrix. Further, using the Bäcklund transformation [358-362] the one soliton solution of Eq. (4.22) is found to be
\[ q_1 = -2n_1 \text{sech}\xi_1 e^{-2i f_1}, \]  
(4.30)
where
\[ \xi_1 = 2 \left[ \eta_1 \delta_1 + \eta_1 x - 32 \gamma (\mu_1 - \eta_1)^2 \mu_1 \eta_1 t + 4 \mu_1 \eta_1 t \right], \]  
(4.31)
\[ f_1 = \mu_1 \delta_2 + \mu_1 x + 2 (\mu_1 - \eta_1^2) t - 8 \gamma (\mu_1^4 + \eta_1^4) t + 48 \gamma \mu_1^2 \eta_1^2 t, \]  
(4.32)
and \( \delta_1 \) and \( \delta_2 \) are phase constants. The corresponding spin components can be written as [299,363]
\[ S^x = - \frac{2n_1}{(\mu_1^2 + \eta_1^2)} \left[ \mu_1 \cos (\mu_1/\eta_1) \xi_1 - \eta_1 \tanh \xi_1 \sin (\mu_1/\eta_1) \xi_1 \right] \text{sech}\xi_1, \]  
(4.33)
\[ S^y = - \frac{2n_1}{(\mu_1^2 + \eta_1^2)} \left[ \eta_1 \tanh \xi_1 \cos (\mu_1/\eta_1) \xi_1 + \mu_1 \sin (\mu_1/\eta_1) \xi_1 \right] \text{sech}\xi_1, \]  
(4.34)
\[ S^z = 1 - \frac{2n_1^2}{(\mu_1^2 + \eta_1^2)} \text{sech}^2\xi_1. \]  
(4.35)

### 4.4 Nonlinear inhomogeneity as a perturbation

Now, we try to understand the dynamics of spins in the more general case (other than the above choices) by carrying out a multiple scale perturbation analysis [301]. After suitable rescaling, we rewrite Eq. (4.12) as
\[ iu_t + u_{xx} + 2|u|^2 u + \lambda [u_{xxxx} + 8|u|^2 u_{xx} + 2u^2 u_{xx}^* + 4|u_x|^2 u + 6u^* u_x^2 \]  
\[ + e|u|^4 u + \frac{2}{3} hu_{xxx}] = 0, \]  
(4.36)
where \( e = \frac{2A_f - \frac{3}{2} f_g}{(f_f + 2f_g)} \) and \( h = \frac{2f_f + 2f_g}{2f_f + 2f_g} \). When \( \lambda = 0 \), Eq. (4.36) reduces to the completely integrable cubic NLS equation which admits N-soliton solutions [7]. The envelope one soliton solution of the cubic NLS can be written as

\[
u = \eta \text{sech}_\eta(\theta - \theta_0)\exp[i(\xi(\theta - \theta_0) + i(\sigma - \sigma_0))] \tag{4.37}
\]

where \( \theta = 2\xi, \theta_0 = 1, \sigma = \eta^2 + \xi^2, \sigma_0 = 0 \), and \( \eta \) and \( \xi \) are related to the scattering parameter of the inverse scattering transform analysis. Now, we write \( \eta, \xi, \theta, \theta_0 \) and \( \sigma_0 \) as functions of a new time (slow time variable) scale \( T = \lambda t \) and so that envelope soliton solutions of Eq. (4.36) is given by

\[
u = \hat{u}(\theta, T; \lambda)\exp[i(\xi(\theta - \theta_0) + i(\sigma - \sigma_0))] \tag{4.38}
\]

Under the assumption of quasi-stationary, we then expand \( \hat{u} \) in terms of \( \lambda \) as

\[
\hat{u}(\theta, T; \lambda) = \hat{u}_0(\theta, T) + \lambda \hat{u}_1(\theta, T) + \ldots \]

where \( \hat{u}_0 = \eta \text{sech}_\eta(\theta - \theta_0) \) and making use of Eq. (4.37) in Eq. (4.36), at \( O(\lambda) \) one can obtain

\[
-\eta^2 \hat{u}_1 + \hat{u}_{1\theta\theta} + 2\hat{u}_0^2 \hat{u}_1 + 4\hat{u}_0^3 \hat{u}_1 = F_1(\hat{u}_0), \tag{4.39}
\]

where

\[
F_1(\hat{u}_0) = \xi_\tau(\theta - \theta_0)\hat{u}_0 - \hat{u}_{0\theta\theta\theta} + 6\xi^2\hat{u}_{0\theta\theta} - \xi^4\hat{u}_0 - 10|\hat{u}_0|^2\hat{u}_{0\theta\theta} - 10\hat{u}_0\hat{u}_{0\theta}^2
+ 20\xi^2|\hat{u}_0|^2\hat{u}_0 - e|\hat{u}_0|^4\hat{u}_0 - \frac{2}{3}\hat{u}_{0\theta\theta\theta} + 2h\xi^2\hat{u}_{0\theta} + i[4\xi^3\hat{u}_0
- 4\xi\hat{u}_{0\theta\theta} - 24\xi|\hat{u}_0|^2\hat{u}_{0\theta} - \hat{u}_{0\tau} - 2h\xi\hat{u}_{0\theta\theta} + \frac{2}{3}h\xi^3\hat{u}_0]. \tag{4.40}
\]

Substituting \( \hat{u}_1 = \phi_1 + \imath \psi_1 \), where \( \phi_1 \) and \( \psi_1 \) are real functions, in Eqs. (4.39), we get

\[
L_1\phi_1 = -\eta^2\phi_1 + \phi_{1\theta\theta} + 6\hat{u}_0^2\phi_1 = \Re F_1(\hat{u}_0), \tag{4.41}
\]

\[
L_2\psi_1 = -\eta^2\psi_1 + \psi_{1\theta\theta} + 2\hat{u}_0^2\psi_1 = \Im F_1(\hat{u}_0), \tag{4.42}
\]

where \( L_1 \) and \( L_2 \) are self-adjoint operators, and \( \Re F_1(\hat{u}_0) \) and \( \Im F_1(\hat{u}_0) \) are the real and imaginary parts of \( F_1(\hat{u}_0) \) respectively. It should be noted that \( \hat{u}_{0\theta} \) and \( \hat{u}_0 \) are the solutions of the homogeneous part of Eq. (4.41) and Eq. (4.42) respectively. Hence, the secularity conditions give

\[
\int_{-\infty}^{\infty} \hat{u}_{0\theta} \Re F_1 d\theta = 0, \tag{4.43}
\]
and
\[ \int_{-\infty}^{\infty} \dot{u}_0 \Re F_1 d\theta = 0. \] (4.44)

For evaluating these integrals explicitly, we need to substitute the explicit form of \( h(x) = \frac{\Delta x}{\kappa} \) in \( \Re F_1 \) and \( \Im F_1 \). For our convenience we assume that the functions \( f(x) = g(x) = \exp(\frac{\beta}{\gamma} x) \) and substituting it in \( h(x) \), finally we get \( h(x) = \frac{L}{f} = \frac{\beta}{\gamma} \).

Making use of it in the expressions \( \Re F_1(\dot{u}_0) \) and \( \Im F_1(\dot{u}_0) \) and evaluating the above integrals, one can obtain the evolution equation for soliton parameters in terms of \( \xi \) and \( \eta \) as
\[ \xi_T = \frac{\beta}{\gamma^2} \left( \frac{28}{45} \eta^4 + \frac{4}{3} \xi^2 \eta^2 \right), \] (4.45)
\[ \eta_T = \frac{4\beta}{3\gamma^2} (\eta^2 + \xi^2) \xi \eta. \] (4.46)

where \( \xi, \eta \) represent the velocity and amplitude of the soliton respectively. We employ the modified extended tangent hyperbolic function method [364,300] to solve the above evolution equations for soliton parameter and we find
\[ \xi = a_0 - \frac{\Gamma a_0 (3s_0^2 + a_0^2)}{3\sqrt{3} s_0} \tanh \left( \frac{4\beta a_0 \sqrt{(3a_0^2 s_0^2 + a_0^4 + 2s_0^4)(-6s_0^2 + a_0^2)}}{3\sqrt{3} \gamma^2 s_0} \right), \] (4.47)
\[ \eta = s_0 - \frac{1}{\Gamma} \coth \left( \frac{4\beta a_0 \sqrt{(3a_0^2 s_0^2 + a_0^4 + 2s_0^4)(-6s_0^2 + a_0^2)}}{3\sqrt{3} \gamma^2 s_0} \right), \] (4.48)

where \( \Gamma = \sqrt{\frac{(-6s_0^2 + a_0^2)}{(3a_0^2 s_0^2 + a_0^4 + 2s_0^4)}} \), \( s_0, a_0 \) are arbitrary constants and \( \beta, \gamma \) are the constant coefficients of exponential inhomogeneity.

### 4.5 Optimization of switching frequency

Exploring the physical limit to the speed at which the magnetic moments can be switched is complicated, partly because switching the magnetization from one direction to the other can occur in multiple ways and along different paths. For example, magnetic and electric fields, electric current and laser pulses can all stimulate magnetic switching and the trajectory of the magnetization vector from its initial to its final state will vary with each of these
Figure 4.1: Magnetization switching through soliton flip in terms of (i) amplitude and (ii) velocity of soliton when $\beta = 0.2$.

Figure 4.2: Magnetization switching through soliton flip in terms of (i) amplitude and (ii) velocity of soliton when $\beta = 0.4$.

Figure 4.3: Magnetization switching through soliton flip in terms of (i) amplitude and (ii) velocity of soliton when $\beta = 0.6$.  

$\phi_{1} = \frac{1}{2} \phi_{2}$ (4.40)
switching mechanisms. Passing through a state of magnetization in the solitonic state seems to be the key to obtain the controlled speed and to optimize the switching frequency ultimately. Once the spin system in the presence of higher order magnetic interactions between the spins is excited in the form of solitonic evolution it leads to controlled magnetization switching by tuning the values of parameters \( a_0 = 0.7 \), \( s_0 = 0.3 \) and \( \gamma = 0.1 \), when a lattice deformity in the form of exponential inhomogeneity is introduced. More especially the presence of nonlinear inhomogeneity in the form of \( f = e^{\beta x} \) in the spin chain makes the solitonic magnetization state to switch without losing its coherent nature. Also, it is found that the reversal time is optimized notably as depicted in Figs. (4.1 and 4.2)). From the plots, it is inferred that the reversal time decreases with increase of the values of \( \beta \) which drives the inhomogeneity. When the value of \( \beta \) is 0.2, the soliton parameters take 6 ns to flip or to reverse from one state to other. While increasing the value of \( \beta \) to 0.4, the flipping time is reduced by 3 ns. Further increase in the value of \( \beta \) above 0.6, we observe no appreciable change in the reversal time and corresponding reversal time is 0.2 ns. From the plots, its is clearly inferred that the reversal time is successfully reduced and optimized to 0.2 ns by tuning the value of parameter \( \beta \) in the range of 0.2 – 0.73. Under the influence of the nonlinear exponential inhomogeneity, the amplitude and velocity of the soliton increase with time [see Figs. (4.2) and (4.3)] and when they reach some critical values, the soliton suddenly reverses its magnetization and the switching process repeats periodically.

### 4.5.1 Perturbed soliton

We have also construct the perturbed soliton solutions for Eq. (4.36) by following the Kodama and Ablowitz nonlinear perturbation technique. The homogeneous part of Eq. (4.41) admits the following two particular solutions,

\[
\phi_{11} = \text{sech} \eta (\theta - \theta_0) \tanh \eta (\theta - \theta_0),
\]

(4.49)

and

\[
\phi_{12} = -\frac{1}{\eta} \left[ \text{sech} \eta (\theta - \theta_0) - \frac{3}{2} \eta (\theta - \theta_0) \phi_{11} - \frac{1}{2} \tanh \eta (\theta - \theta_0) \sinh \eta (\theta - \theta_0) \right].
\]

(4.50)
Figure 4.4: Real part of perturbed soliton and its contour plots with \( s_0 = 1.23, \quad \alpha_0 = 0.2, \quad \beta = 0.04, \quad \delta = 0.025, \quad \sigma = 0.03 \quad \epsilon = 0.075 \) and \( h = 0.01 \).

Figure 4.5: Imaginary part of perturbed soliton and its contour plots with \( s_0 = 0.37, \quad \alpha_0 = 0.25, \quad \beta = 0.13, \quad \delta = 1 \) and \( h = 0.25 \).
Knowing the two particular solutions, the general solution can then be written using the formula,

\[ \phi_1 = \alpha_1 \phi_{11} + \alpha_2 \phi_{12} - \phi_{11} \int_{-\infty}^{\theta} \phi_{12} R F_1 d\theta' + \phi_{12} \int_{-\infty}^{\theta} \phi_{11} R F_1 d\theta', \]  

(4.51)

with \( \alpha_1 \) and \( \alpha_2 \) being arbitrary constants. By substituting the values of \( \phi_{11}, \phi_{12} \) and \( R F_1 \) in Eq. (4.51) and evaluating the integrals, we obtain the general solution for \( \phi_1 \) as

\[
\phi_1 = \left[ \alpha_1 + \frac{3}{2} \alpha_2 (\theta - \theta_0) - \frac{4}{45} h \eta^2 + \frac{2}{3} h \xi^2 - \frac{1}{4} h^2 \xi T - \frac{4}{5} h \eta^2 \ln \cosh(\eta(\theta - \theta_0)) \right. \\
+ \left( \frac{3}{2 h \eta T} - 3 \xi^2 \eta^2 \right) \tanh(\eta(\theta - \theta_0)) \tanh(\eta(\theta - \theta_0)) + \left[ \frac{5}{4} (6 \xi^2 \eta^2 - \xi^4 - \eta^4)(\theta - \theta_0) \right. \\
+ \frac{71}{45} h \eta^2 + \frac{2}{3} h \xi^2 \left. \right] \sech^3(\eta(\theta - \theta_0)) \tanh(\eta(\theta - \theta_0)) + \frac{13}{10} h \eta^2 \sech^5(\eta(\theta - \theta_0)) \\
\times \tanh(\eta(\theta - \theta_0)) + \left[ \frac{\xi T}{2 \eta} + \frac{32}{5} h \eta^3 \right] (\theta - \theta_0) + \frac{21 h}{4} \xi^2 \eta^2 - \frac{\xi^4}{2 \eta} - \frac{\eta^3}{2 \eta} - \frac{\alpha_2}{\eta} \\
\times \sech(\eta(\theta - \theta_0)) + \left[ \frac{16}{5} h \eta^3 + \frac{1}{2 h \eta} \xi T \right] (\theta - \theta_0) - \frac{1}{2 h} \xi^4 - \frac{1}{2 h \eta} \xi^2 \eta^2 + \frac{123}{4} \xi^2 \eta^2 \\
- \frac{1}{2} (6 - e) \eta^3 \] \sech^3(\eta(\theta - \theta_0)) - \left[ \frac{32}{5} h \eta^3 (\theta - \theta_0) - \frac{9}{2} \xi^2 \eta^2 + \frac{23}{5} \xi^2 \eta \right. \\
\times \sech^5(\eta(\theta - \theta_0)) - \frac{1}{6} (6 - e) \eta^3 \sech^7(\eta(\theta - \theta_0)) + \frac{1}{4 h \eta^2 \xi T} - \frac{1}{2 \eta} \xi^2 \eta^3 \\
\left. + \frac{2}{3} h \eta \xi^3 \right] + \frac{2}{3} h \eta^2 + \frac{1}{2 \eta} \xi T + \frac{8}{5} h \eta^3 - \frac{4}{3} h \eta^2 \xi^2 + \alpha_2 \\
\times \tanh(\eta(\theta - \theta_0)) \left( \tanh(\eta(\theta - \theta_0)) \right) \left( \eta(\theta - \theta_0) \right). \]  

(4.52)

However, the solution \( \phi_1 \) contains secular term which make the solution unbounded and we remove the secular terms by choosing \( \alpha_2 = 0 \). Further using the boundary conditions \( \phi_1(0)|_{\theta_0 = 0} = 0 \); \( \phi_{1\theta}(0)|_{\theta_0 = 0} = 0 \), we get \( \alpha_1 = \frac{17}{10} h (\xi^2 - \xi^2) \). Using \( \alpha_1 \) and \( \alpha_2 \), the final form of \( \phi_1 \) can be explicitly constructed. In this way, \( \psi \) can also be evaluated using the two particular solutions corresponding to the homogeneous parts of Eq. (4.42) which is of the form,

\[
\psi_{11} = \sech(\eta(\theta - \theta_0)), \]  

(4.53)

\[
\psi_{12} = \frac{1}{2 \eta} \left[ \eta(\theta - \theta_0) \sech(\eta(\theta - \theta_0)) + \sinh(\eta(\theta - \theta_0)) \right]. \]  

(4.54)
Making use of above two particular solutions, the general solution can be written using the formula,

$$
\psi_1 = \alpha_3 \psi_{11} + \alpha_4 \psi_{12} - \psi_{11} \int_{-\infty}^{\theta} \psi_{12} \mathcal{F}_1 d\theta' + \psi_{12} \int_{-\infty}^{\theta} \psi_{11} \mathcal{F}_1 d\theta',
$$

(4.55)

where $\alpha_3$ and $\alpha_4$ are the arbitrary constants. Using the $\psi_{11}$, $\psi_{12}$ and $\mathcal{F}_1$ in Eq. (4.55) and after evaluating the integrals, we obtain the solution for $\psi_1$ as

$$
\psi_1 = [\alpha_3 + \frac{1}{2} \alpha_4 (\theta - \theta_0) - 2 (\xi^3 - \xi \eta^2) - (2 \xi \eta^3 - 2 \eta \xi^3 + \frac{1}{2} \eta (\theta - \theta_0) T
+ \frac{1}{4} \eta T (\theta - \theta_0) (\theta - \theta_0) \sech(\eta (\theta - \theta_0)) - \frac{2}{3} h (\xi \eta^2 + \xi^3) (\theta - \theta_0)
\times \sech(\eta (\theta - \theta_0)) \tanh(\eta (\theta - \theta_0)) + \frac{4}{3} h \xi \eta \sech(\eta (\theta - \theta_0))
\times \ln \cosh(\eta (\theta - \theta_0)) + \frac{\alpha_4}{2 \eta} + \frac{1}{\eta} \left( \frac{2}{3} h \xi \eta^2 - \frac{1}{2} \eta T + \frac{2}{3} h \xi \eta^2 \right)
+ \frac{1}{\eta} \left( \frac{2}{3} h \xi \eta^2 - \frac{1}{2} \eta T + \frac{2}{3} h \xi \eta^2 \right) \tanh(\eta (\theta - \theta_0)) \right] \sinh(\eta (\theta - \theta_0)).
$$

(4.56)

We remove the secular terms which make the solution unbounded by choosing $\alpha_4 = 0$. By using the boundary conditions $\psi_1(0)|_{\theta_0=0} = 0; \psi_1(0)|_{\theta_0=0} = 0$, we obtain $\alpha_3 = 2 (\xi^3 - \xi \eta^2 - \frac{1}{3} h \xi \eta)$. Using $\alpha_3$ and $\alpha_4$, the final form of $\psi_1$ can be explicitly constructed and making use of $\phi_1$ and $\psi_1$, the first order perturbed soliton solution $\hat{u}_1$ can now be explicitly written as

$$
\hat{u}_1 = \left[ \left( \frac{302}{25} h \eta^3 + \frac{3}{3} h \xi \eta^2 \right) (\theta - \theta_0) - \frac{21}{4} \xi^2 \eta - \frac{\xi^4}{2 \eta} - \frac{\eta^3}{2} \right] \sech(\eta (\theta - \theta_0))
- \left[ \left( \frac{151}{5} h \eta^3 + \frac{2}{3} h \xi \eta^2 \right) (\theta - \theta_0) - \frac{1}{2} \xi^4 - \frac{1}{2} \eta^3 + \frac{123}{4} \xi^2 \eta - \frac{1}{2} (6 - e) \eta^3 \right]
\times \sech^3(\eta (\theta - \theta_0)) - \left[ \left( \frac{32}{5} h \eta^5 (\theta - \theta_0) - \frac{9}{2} \xi^2 \eta + \frac{1}{3} (6 - e) \eta^3 \right) \sech^5(\eta (\theta - \theta_0))
- \frac{1}{6} (6 - e) \eta^3 \sech^7(\eta (\theta - \theta_0)) + \left[ \left( \frac{165}{8} \xi^2 \eta^2 - \frac{1}{4} \xi T (\theta - \theta_0) + \frac{1}{5} \eta^4 + \frac{1}{2} \xi^4 \right) (\theta - \theta_0)
+ \frac{29}{90} h \eta^2 - \frac{2}{3} h \xi^2 - \frac{4}{5} h \eta \ln \cosh(\eta (\theta - \theta_0)) \right] \sech(\eta (\theta - \theta_0)) \tanh(\eta (\theta - \theta_0))
+ \frac{5}{4} (6 \xi^2 \eta^2 - \xi^4 - \eta^4) (\theta - \theta_0) + \frac{71}{45} h \eta^2 + \frac{2}{3} h \xi \eta \right] \sech^3(\eta (\theta - \theta_0)) \tanh(\eta (\theta - \theta_0)) + \frac{13}{10} h \eta^2 \sech^5(\eta (\theta - \theta_0)) \tanh(\eta (\theta - \theta_0)) + i \left[ 2 \xi^3 - 2 \xi \eta^2 + (\xi \eta^3 - \xi^3 \eta
$$

(4.59)
\[ \frac{1}{2} \eta(\theta - \theta_0)\tau - \frac{1}{4} \eta \tau (\theta - \theta_0) ((\theta - \theta_0)) \left[ \text{sech}(\eta(\theta - \theta_0)) \right] - \frac{2}{3} \hbar (\xi^2 + \xi^3)(\theta - \theta_0) \]

\[ \times \text{sech}(\eta(\theta - \theta_0)) \tanh(\eta(\theta - \theta_0)) + \frac{4}{3} \hbar \xi \eta \text{sech}(\eta(\theta - \theta_0)) \text{tncosh}(\eta(\theta - \theta_0)) \]

(4.57)

We have plotted the real and imaginary part of perturbed soliton by choosing appropriate values to the arbitrary constants as shown in Figs. (4.4) and Figs. (4.5). In the Fig. (4.4a) the plot corresponds to the real part of perturbed soliton in which the soliton exhibits complete magnetization reversal behavior through flipping. Fig. (4.5a) shows imaginary part of perturbed soliton in which soliton moves along the spin chain that exhibits the magnetization reversal behavior. As time passes the magnetization reversal process is controlled, then soliton parameters evolves with constant speed without loss of energy along the spin chain as depicted in Fig. (4.5).

### 4.6 Deformed isotropic biquadratic spin chain

In the previous section, we have discussed the effect of inhomogeneity on the switching frequency of flipping soliton in the semiclassical continuum approximation. Now we would like to extend the analysis by mapping the governing dynamical equation using the differential geometry procedure for the isotropic case of the deformed biquadratic spin chain in the classical continuum limiting approximation. For this, we consider the isotropic case of site-dependent bilinear and biquadratic exchange interactions described by the Hamiltonian for the pure isotropic case (4.1) with \( A = A' = 0, \)

\[ H = -J_e \sum_{i=1}^{N} f_i (\vec{S}_i \cdot \vec{S}_{i+1}) + \sum_{i=1}^{N} J_b g_i (\vec{S}_i \cdot \vec{S}_{i+1})^2. \]

(4.58)

The time evolution of the magnetization follows the Landau-Lifshitz equation [175] of motion corresponding to Eq. (4.58), which can be obtained from the relation \( \frac{d\vec{S}_i}{dt} = \{\vec{S}_i, H\}_{PB} \) as

\[ \frac{d\vec{S}_i}{dt} = \vec{S}_i \wedge \{ J_e (f_i \vec{S}_{i+1} + f_{i-1} \vec{S}_{i-1}) + J_b (g_i (\vec{S}_i \cdot \vec{S}_{i+1}) \vec{S}_{i+1} + g_{i-1} (\vec{S}_i \cdot \vec{S}_{i-1}) \vec{S}_{i-1}) \}. \]

(4.59)
A classical continuum description in which $\vec{S}_i \rightarrow \vec{S}(x, t)$, $f_i \rightarrow f(x)$ and $g_i \rightarrow g(x)$ is suitable when $\vec{S}_i$, $f_i$ and $g_i$ vary slowly over different lattice separations $a$ and $b$ respectively. In the long-wavelength and low temperature limit, we introduce the series expansions as in Eqs. (4.10-4.11) in the equation of motion for the discrete lattice. It may be noted that in the above expansions, the spins have been expanded to $O(a^4)$ and the exchange coefficients only up to $O(b^2)$. In the normal cases $a$ and $b$ are expected to coincide with each other. But, nevertheless there may be situations where $b \neq a$. The purpose of introducing two different lattice spacing for the spin and exchange interactions respectively, is to identify the possible inhomogeneous integrable spin models or their perturbations which are generated for unequal values of $a$ and $b$. Then after suitable rescaling, we obtain the following continuous equation of motion written upto $O(b^2)$ and $O(a^4 b^m)$ such that $l + m = 4$:

$$
\vec{S}_i(x, t) = \vec{S} \times \left[ \left( a b A_x - \frac{a^2 b}{2} A_{xx} + \frac{J_b g a^4}{3} (\vec{S} \cdot \vec{S}_{xx}) \right) \vec{S}_x + \left( a^2 A - \frac{a^2 b}{2} A_x \right) \vec{S}_y + \frac{a^2 b^2}{4} A_{xx} + \frac{J_b a^4 g}{4} (\vec{S} \cdot \vec{S}_{xx}) \right] \vec{S}_{xx} + \frac{a^3 b A_x}{6} \vec{S}_{xxx} + \frac{a^4 A}{12} \vec{S}_{xxxx},
$$

where $A = J_e f(x) + J_b g(x)$. As the LL Eq. (4.60) is a nontrivial vector nonlinear partial differential equation, it is very difficult to solve in its natural form to understand the underlying nonlinear spin dynamics. Hence, we try to map the spin vector $\vec{S}(x, t)$ onto a moving helical space curve [35,297] to one of the well known nonlinear evolution equation that admits soliton solutions.

### 4.6.1 Space curve mapping

In this procedure, a local coordinate system $\vec{e}_i$ ($i = 1, 2, 3$) is formed on the space curve by identifying the unit spin vector $\vec{S}(x, t)$ with the unit tangent vector $\vec{e}_1(x, t)$ of the space curve and defining the unit principal and binormal vectors $\vec{e}_2(x, t)$, $\vec{e}_3(x, t)$ respectively in the usual way. The change in orientation of the trihedral $\vec{e}_i$ ($i = 1, 2, 3$) which defines the space curve uniquely within rigid motions is determined by the Serret-Frenet (S-F) equation [295]
\[
\begin{bmatrix}
\tilde{e}_{1x} \\
\tilde{e}_{2x} \\
\tilde{e}_{3x}
\end{bmatrix} = \begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix} \begin{bmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3
\end{bmatrix},
\]
(4.61)
and the time evolution of \( \tilde{e}_i, i = 1, 2, 3 \) can be evaluated by using the SF equations. After lengthy algebraical calculations, \( \tilde{e}_{it} \) can be written as,
\[
\begin{bmatrix}
\tilde{e}_{1t} \\
\tilde{e}_{2t} \\
\tilde{e}_{3t}
\end{bmatrix} = \begin{bmatrix}
0 & \omega_3 & -\omega_2 \\
-\omega_3 & 0 & \omega_1 \\
\omega_2 & -\omega_1 & 0
\end{bmatrix} \begin{bmatrix}
\tilde{e}_1 \\
\tilde{e}_2 \\
\tilde{e}_3
\end{bmatrix},
\]
(4.62)
where
\[
\omega_1 = \frac{1}{\kappa} \left\{ \tilde{h}_x (\kappa_x - \kappa \tau^2) + (\tilde{h}_x \kappa_x + \frac{a^2}{2} J_b g (\kappa^3 \tau^2 - 6\kappa \kappa_x^2 - 3\kappa^2 \kappa_x) \\
+ \frac{a^2}{12} A (\kappa_{xxx} - 12\kappa \kappa_x^2 - 6\kappa_x \kappa_{xx} - 6\kappa_x \tau^2 - 12\kappa \tau_x \tau_x - 3\kappa \tau_x^2) \\
- 4\kappa \tau_x \tau_x + \kappa^3 \tau^2 + \kappa \tau^4) + \frac{ab}{4} A_x (\kappa_{xxx} - 4\kappa^2 \kappa_x - 3\kappa \tau_x^2 - 3\kappa \tau_x) \\
+ \frac{ab}{12} A_x (\kappa_{xxx} - \kappa^3 - \kappa \tau^2) - ab J_b g \kappa^2 \kappa_x \right\},
\]
(4.63)
\[
\omega_2 = (\tilde{h}_x \kappa_x + \frac{a^2}{12} A (\kappa_{xxx} - 6\kappa^2 \kappa_x - 3\kappa \tau^2 - 3\kappa \tau_x) + \frac{ab}{6} A_x (\kappa_{xx} - \kappa^3 - \kappa \tau^2) \\
- \frac{3}{2} a^2 J_b g \kappa^2 \kappa_x,
\]
(4.64)
\[
\omega_3 = -\tilde{h}_x \tau - \frac{a^2}{12} A (3\kappa \tau_x \tau + 3\kappa \tau_x - \kappa^3 \tau + \kappa \tau_x - \kappa \tau_x) - \frac{ab}{6} A_x (2\kappa \tau_x + \kappa \tau_x) \\
+ \frac{a^2}{2} J_b g \kappa^3 \tau.
\]
(4.65)
The compatibility of Eq. (4.61) and Eq. (4.62) leads to the following evolution equations for the curvature and torsion of the space curve
\[
\kappa_t = - \tilde{h}_x (2\kappa_x \tau + \kappa \tau_x) - 2 \tilde{h}_x \kappa \tau - \frac{a^2}{12} A (4\kappa_{xxx} \tau + 6\kappa_x \tau_x - 4\kappa \tau_x^3 - 6\kappa \tau_x^2 \tau_x \\
+ 4\kappa_x \tau_x + \kappa_{xxx} - 9\kappa^2 \kappa_x \tau - \kappa^3 \tau_x) - \frac{ab}{4} A_x (3\kappa \tau_x + 3\kappa \tau_x - \kappa \tau_x) \\
+ \kappa \tau_x - \kappa \tau_x^3) - \frac{a^2}{2} J_b g (6\kappa^2 \kappa_x \tau + \kappa^3 \tau_x),
\]
(4.66)
In order to identify the set of coupled equations (4.66-4.67) with the more standard nonlinear partial differential equations, we make the following complex transformation

\[ q = \frac{\kappa}{2} \exp \left[ i \int_{-\infty}^{x} \tau(x', t) dx' \right]. \]  

(4.68)

At \( O(\alpha^2) \), assuming that \( O(ab) > O(\alpha^2) \) and we obtain the following inhomogeneous generalized nonlinear Schrödinger (IGNLS) equation,

\[ i q_t + (h q)_{xx} + 2h|q|^2 q + 2q \int_{-\infty}^{x} h_{xx}|q|^2 dx' + a_{2}|q_{xxx}| + K_1|q|^2 q_{xx} + K_2 q^2 q^*_{xx} + K_3 q|q|^2 + K_4 q^2 q^2 + 3K_2 q|q|^4 = 0, \]  

(4.69)

where \( a = \frac{a^2 A}{12} \), \( h(x) = \tilde{h}(x) + \frac{b_2^2}{6} A_{xx}, \tilde{h}(x) = A - \frac{b_1}{2} A_x + \frac{b_2^2}{2} A_{xx}, K_1 = -12(1 + \frac{4J_0 q}{A}), K_2 = -8(1 + \frac{3J_0 q}{A}), K_3 = -3(1 + \frac{8J_0 q}{3A}) \) and \( K_4 = -14(1 + \frac{24J_0 q}{7A}) \). When \( h(x) = Cx + D \), especially when \( C = 0 \), Eq. (4.69) is the homogeneous fourth order NLS equation which governs the dynamics of spins in a classical one-dimensional isotropic biquadratic ferromagnetic spin chain [291]. In order to study the complete nonlinear spin excitations of the above system, we employ the sine-cosine function method to solve the inhomogeneous generalized nonlinear Schrödinger Eq. (4.69).
4.6.2 Exact analytic solution for the dynamical equation

In order to solve Eq. (4.69), we put \( q(x, t) = E(x, t), \quad q^*(x, t) = G(x, t) \) and it admits travelling wave solution as,

\[
E(x, t) = u(x, t)e^{i(kx-\omega t)} \quad \text{and} \quad G(x, t) = u(x, t)e^{-i(kx-\omega t)},
\]

(4.70)

In the above equation \( u(x, t) \) is a real function. By substituting Eq. (4.70) in Eq. (4.69), we get

\[
\omega u + h(u_{xx} - k^2 u) + 2h_x(x)u_x + 2uv + \alpha[u_{xxxx} - 6k^2u_x + (K_1 + K_2)u_{xxxx} - 6k^2u_x + (K_1 + K_2)u_x] = 0,
\]

(4.71)

\[
u_t + 2kh_xu_x + 2kh_xu + \alpha[4k^3u_{xxx} - 4k^3u_x + 2(K_1 - K_2 + K_4)k^2u_x] = 0,
\]

(4.72)

\[
R_e - 2h_xu_t^2 - 2huvu = 0,
\]

(4.73)

where

\[
v(x, t) = R(x, t) = h(x)E(x, t)G(x, t) + \int_{-\infty}^{x} h(x')E(x', t)G(x', t)dx'.
\]

(4.74)

By using the transformation \( \xi = t - k_1x + \delta \), the above Eqs. (4.71-4.73) yield a system of ordinary differential equations,

\[
\omega u + h(k_1^2 u_{\xi\xi} - k^2 u) - 2k_1^2 h_{\xi} u_{\xi} + 2uv + \alpha[k_1^4 u_{\xi\xi\xi\xi} - 6k_1^2 u_{\xi\xi} + k^4 u \\
+ (K_1 + K_2)(k_1^2 u_{\xi\xi} - k^2 u)u_x^2 + (K_3 + K_4)k_1^2 u_x^2u \\
+ (K_3 - K_4)k^2 u^3 + 3k_2 u^5] = 0,
\]

(4.75)

\[
u_{\xi} - 2h k_k_1 u_{\xi} + 2k_1 h_{\xi} u + \alpha[4k^3 k_1 u_{\xi} - 4kk_1^3 u_{\xi\xi\xi} + 2kk_1(K_2 - K_1 - K_4)u_x^2 u_{\xi}] = 0, \quad (4.76)
\]

\[
k_1 v_{\xi} + 2k_1 h_{\xi} u_t^2 - 2hk_1 u u_{\xi} = 0.
\]

(4.77)

Now, we assume that the Eqs. (4.75-4.77) admit the solutions as follows,

\[
u(\xi) = \lambda_1 \sin^{p_1}(\mu\xi),
\]

(4.78)

\[
v(\xi) = \lambda_2 \sin^{p_2}(\mu\xi),
\]

(4.79)
where $\lambda_1$ and $\lambda_1$ are the parameters. In order to find the parameters $\beta_1$ and $\beta_2$ in Eqs. (4.78-4.79), we balance the linear higher order derivative term with the nonlinear term of Eqs. (4.75-4.77) and finally we obtain $\beta_1 = -1$ and $\beta_2 = -2$. Making use of $\beta_1$ and $\beta_2$, we are interested to study the dynamics for various competing nonlinear inhomogeneities as follows.

### 4.6.3 Creation and annihilation of soliton

Now, by solving the IGNLS equation, we show that the soliton can be created and annihilated under the influence of nonlinear inhomogeneities. This can be achieved through sine-cosine function method [365-372]. The theory of soliton creation is of practical interest because of existing experiments [373,374] in which the initial pulse breaks up into a rather large number (~10) of solitons.

**a)** At the outset, let us start with the linear inhomogeneity $f(x) = p_2 x + p_3$, $g(x) = 0$ and upon substituting Eqs. (4.78-4.79) in Eq. (4.75-4.77), we get the system of algebraic equations. On solving the obtained system of algebraic equations using symbolic computation, we obtain,

$$
\mu = \pm i \sqrt{\frac{1 + \frac{i}{3} a^2 k_1 k^3 (p_2 x + p_3) - 2k k_1 (j_0 (p_2 x + p_3) - \frac{i}{2} b p_2)}{\frac{i}{3} a^2 k_1^3 (p_2 x + p_3)},}
$$

(4.80)

and

$$
\lambda_1 = \pm \sqrt{\frac{2}{3} k_1 \mu} \text{ and } \lambda_2 = \frac{j e}{3} k_1^2 \mu^2 (2p_2 x + 2p_3 - bp_2).
$$

(4.81)

We point out that the results of Eqs. (4.80-4.81) hold if we also use the cosine function method. Now, making use of Eqs. (4.80-4.81) in Eqs. (4.78-4.79), we obtain the following the expression

$$
u(x, t) = \pm \sqrt{\frac{2}{3} k_1 \mu} \csc[\mu(t - k_1 x + \delta)],$$

(4.82)

$$
u(x, t) = \frac{j e}{3} k_1^2 \mu^2 (2p_2 x + 2p_3 - bp_2) \csc[\mu(t - k_1 x + \delta)],$$

(4.83)

and

$$
u(x, t) = \pm \sqrt{\frac{2}{3} k_1 \mu} \sec[\mu(t - k_1 x + \delta)],$$

(4.84)

$$
u(x, t) = \frac{j e}{3} k_1^2 \mu^2 (2p_2 x + 2p_3 - bp_2) \sec[\mu(t - k_1 x + \delta)].$$

(4.85)
From the nature of the solutions of Eq. (4.69), it is found that they admit creation and annihilation of soliton for a variety of linear and nonlinear inhomogeneities. For instance, we have plotted Eqs. (4.82-4.83) the evolution of soliton in Fig. (4.6a) for the choices of parameters \( k_1 = 1.7, k = 2.1, \omega = 0.03, J_a = 0.3, J_b = 0, a = 0.8, b = -0.8, p_3 = 0.3 \) and \( \delta = 0.5 \), when \( f(x) = p_2 x + p_3 \), a linear function and \( g(x) = 0 \), for the case of invariant inhomogeneity i.e., \( p_2 = 0 \). From the figure it is evident that for the homogeneous exchange interaction the dynamics is governed by solitons and found to be completely integrable [290]. The corresponding contour plot represents the homogeneous solitonic evolution in the form a bell shape. In the contour plots the brighter region represents the maximum amplitude and the darker region represents the minimum or zero amplitude of the soliton. Physically, it is also inferred that the spin vector \( \hat{S}(x, t) \) precesses with constant velocity about the effective field arising from exchange interaction and the resulting excitations are governed by solitons. From our recent studies, it is also proved that the dynamics of an one-dimensional site-dependent ferromagnetic spin chain with linear inhomogeneity is governed by solitons which exhibit shape changing property [300].

b) When we introduce nonlinear inhomogeneity, \( f(x) = p_1 x^2 + p_2 x + p_3 \) and \( g(x) = 0 \) and inserting Eqs. (4.78-4.79) in Eq. (4.75-4.77), we get the system of algebraic equations, on solving the same using symbolic computation we obtain

\[
\mu = \pm i \frac{1 + j_e \left( \frac{2}{3} k_1 k^3 - 2 k k_1 \right) (p_1 x^2 + p_2 x + p_3)}{-2 j_e k k_1 \left[ \frac{4}{3} b^2 p_1 - \frac{b}{2} (2 p_1 x + p_2) \right]} \frac{j_e}{3} a^2 k k_1^3 (p_1 x^2 + p_2 x + p_3),
\] (4.86)

\[
\lambda_1 = \pm \sqrt{\frac{2}{3}} k_1 \mu,
\] (4.87)

and

\[
\lambda_2 = \frac{j_e}{9} k_1^2 \mu^2 \left[ 6 p_1 x^2 + 6 p_2 x + 6 p_3 - 6 b p_1 x - 3 b p_2 + 8 b^2 p_1 \right].
\] (4.88)

The solutions for \( u(x, t) \) and \( v(x, t) \) become

\[
u(x, t) = \pm \sqrt{\frac{2}{3}} k_1 \mu \csc \left[ \mu (t - k_1 x + \delta) \right],
\] (4.89)
\[ v(x, t) = \frac{j_e k_1^2 \mu^2}{9} \left[ 6p_1 x^2 + 6p_2 x + 6p_3 - 6bp_1 x - 3bp_2 + 8b^2 p_1 \right] \times \text{cosec} \left[ \mu(t - k_1 x + \delta) \right], \]  
(4.90)

and

\[ u(x, t) = \pm \sqrt{\frac{2}{3}} k_1 \mu \sec \left[ \mu(t - k_1 x + \delta) \right], \]  
(4.91)

\[ v(x, t) = \frac{j_e k_1^2 \mu^2}{9} \left[ 6p_1 x^2 + 6p_2 x + 6p_3 - 6bp_1 x - 3bp_2 + 8b^2 p_1 \right] \times \sec \left[ \mu(t - k_1 x + \delta) \right]. \]  
(4.92)

Now, we introduce the nonlinear inhomogeneity of the form \( f(x) = p_1 x^2 + p_2 x + p_3 \), and we have plotted the associated evolution of soliton in Figs. (4.6b-4.6c). While plotting the graphs in order to analyze the effect of quadratic nonlinear inhomogeneity, we keep the values of the linear coefficient and arbitrary constant as \( p_2 = 0.3 \) and \( p_3 = 0.2 \) respectively. We slowly introduce a nonzero value of \( p_1 = 0.09 \), for the choices of parameters, \( k_1 = 1.7, k = 2.1, \omega = 0.03, J_e = 0.3 \) and \( J_1 = 0 \). Further increase in the value of \( p_1 \) develops multiple folded excitations coupled with a higher degree of distortion and thus ends up with a complete onset of annihilation as depicted in Fig. (4.6c). This can be more clearly seen in the contour plots as portrayed in figures (4.6c-4.6e). Physically this implies that the precession of magnetization vector varies with respect to the strength of the inhomogeneous effective field which arises only from the varying bilinear exchange interaction i.e., \( g(x) = 0 \).

c) In this case, we try to analyze the effect of varying biquadratic exchange interaction \( g(x) \), by assuming the inhomogeneous bilinear exchange interaction \( f(x) = (p_1 + p_2)x + p_3 \) and \( g(x) = q_1 x^2 + q_2 x + q_3 \). In a similar way, we obtain

\[ \mu = \pm i \sqrt{\frac{1}{3} a^2 k_1^3 \left( j_e((p_1 + p_2)x + p_3) + j_b(q_1 x^2 + q_2 x + q_3) \right)} \]  
(4.93)

\[ + k k_1 \left( b(j_e((p_1 + p_2)x + p_3)) + j_b(2q_1 x + q_2) - \frac{8}{3} j_b b^2 q_1 \right) \]
\[ \lambda_1 = \pm \frac{2 \left( j_e(p_1 x + p_2 x + p_3) + j_b(5q_1 x^2 + 5q_2 x + 5q_3) \right) \times \left( j_e(p_1 x + p_2 x + p_3) + j_b(q_1 x^2 + q_2 x + q_3) \right)}{\sqrt{3} \left( j_e(3p_1 x + 3p_2 x + 3p_3) + j_b(15q_1 x^2 + 15q_2 x + 15q_3) \right)}^{1/2} \mu k_1, \]  
\[ (4.94) \]

\[ \lambda_2 = \frac{\lambda_1^2}{6} \left[ 3j_e(2p_1 x + 2p_2 x - b p_1 - b p_2 + 2p_3) + j_b(6q_1 x^2 + 6q_2 x + 6q_3 - 6bq_1 x - 3bq_2 + 8q_1 b^2) \right]. \]  
\[ (4.95) \]

The solutions for \( u(x, t) \) and \( v(x, t) \) becomes

\[ u(x, t) = \pm \frac{2 \left( j_e(p_1 x + p_2 x + p_3) + j_b(5q_1 x^2 + 5q_2 x + 5q_3) \right) \times \left( j_e(p_1 x + p_2 x + p_3) + j_b(q_1 x^2 + q_2 x + q_3) \right)}{\sqrt{3} \left( j_e(3p_1 x + 3p_2 x + 3p_3) + j_b(15q_1 x^2 + 15q_2 x + 15q_3) \right)}^{1/2} \mu k_1 \times \text{cosec} \left[ \mu(t - k_1 x + \delta) \right], \]  
\[ (4.96) \]

\[ v(x, t) = \frac{\lambda_1^2}{6} \left[ 3j_e(2x p_1 + 2x p_2 - b p_1 - b p_2 + 2p_3) + j_b(6q_1 x^2 + 6q_2 x + 6q_3 - 6bq_1 x - 3bq_2 + 8b^2 q_1) \right] \text{cosec} \left[ \mu(t - k_1 x + \delta) \right]. \]  
\[ (4.97) \]

and

\[ u(x, t) = \pm \frac{2 \left( j_e(p_1 x + p_2 x + p_3) + j_b(5q_1 x^2 + 5q_2 x + 5q_3) \right) \times \left( j_e(p_1 x + p_2 x + p_3) + j_b(q_1 x^2 + q_2 x + q_3) \right)}{\sqrt{3} \left( j_e(3p_1 x + 3p_2 x + 3p_3) + j_b(15q_1 x^2 + 15q_2 x + 15q_3) \right)}^{1/2} \mu k_1 \times \text{sec} \left[ \mu(t - k_1 x + \delta) \right], \]  
\[ (4.98) \]

\[ v(x, t) = \frac{\lambda_1^2}{6} \left[ 3j_e(2x p_1 + 2x p_2 - b p_1 - b p_2 + 2p_3) + j_b(6q_1 x^2 + 6q_2 x + 6q_3 - 6bq_1 x - 3bq_2 + 8b^2 q_1) \right] \text{sec} \left[ \mu(t - k_1 x + \delta) \right]. \]  
\[ (4.99) \]
A close inspection on the solutions obtained Eqs. (4.96-4.99), reveals that the annihilated soliton can be recreated under the influence of biquadratic exchange interaction.

d) In this case, we assume the functions \( f(x) = p_1 x^2 + \frac{p_3}{x^3} + p_3 \) and \( g(x) = q_1 x + q_2 x^3 \). Similarly, by solving the system of algebraic equations, we get the solutions for \( \mu, \lambda_1 \) and \( \lambda_2 \)

\[
\mu = \pm i \sqrt{\frac{1 - 2kk_1}{2}} \times \left[ j_e(p_1 x^2 + \frac{p_3}{x^3}) + j_b(q_1 x + q_2 x^3) - \frac{k_1}{2} \left[ j_e(2p_1 x - \frac{2p_3}{x^2}) + j_b(q_1 + 3q_2 x^2) \right] \right]^{1/2}
\]

\[
\mu = \pm \frac{\sqrt{2}}{3} \left[ j_e(p_1 x^2 + \frac{p_3}{x^3}) + j_b(q_1 x + q_2 x^3) \right] \cdot \left. \left[ 3j_e(p_1 x^4 + p_2) + 15j_b(q_1 x^3 + q_2 x^5) \right] \right^{1/2}
\]

\( \lambda_1 = \pm \sqrt{2} \left[ \frac{j_e(p_1 x^4 + p_2) + j_b(5q_1 x^3 + 5q_2 x^5)}{3j_e(p_1 x^4 + p_2) + 15j_b(q_1 x^3 + q_2 x^5)} \right] \mu k_1 \)

(4.100)

\( \lambda_2 = \lambda_1^2 \)

\[
\lambda_2 = \lambda_1^2 = \left[ \frac{6j_e(p_1 x^6 + p_2 x^2 - p_1 bx^5 + p_2 bx) + j_b(q_1 x^5 + q_2 x^7)}{6x^4} \right]^{1/2}
\]

(4.101)

In this case, the solutions for \( u(x, t) \) and \( v(x, t) \) read as

\[
u(x, t) = \pm \sqrt{\frac{2}{3}} \left[ \frac{j_e(p_1 x^4 + p_2) + j_b(5q_1 x^3 + 5q_2 x^5)}{3j_e(p_1 x^4 + p_2) + 15j_b(q_1 x^3 + q_2 x^5)} \right]^{1/2} \cdot \mu k_1 \cdot \text{cosec} \left[ \mu(t - k_1 x + \delta) \right],
\]

(4.102)
\[
\lambda_2^2 \left[ \frac{6 \left[ j_e(p_1^2x^6 + p_2x^2 - p_1b_5x^5 + p_2bx) + j_b(q_1x^5 + q_2x^7) \right]}{6x^4} + j_e(8p_1b^2x^4 + 24b^2p_2) + j_b(24q_2b^2x^5 - 3q_1bx^4 - 9q_2bx^6) \right]^{1/2} \times \text{cosec} \left[ \mu(t - k_1x + \delta) \right],
\]

and

\[
\sqrt{\frac{2}{3}} \left[ \frac{j_e(p_1^2x^4 + p_2) + j_b(5q_1x^3 + 5q_2x^5)}{3j_e(p_1^2x^4 + p_2) + 15j_b(q_1x^3 + q_2x^5)} \right]^{1/2} \times \text{sec} \left[ \mu(t - k_1x + \delta) \right],
\]

\[
\lambda_2^2 \left[ \frac{6 \left[ j_e(p_1^2x^6 + p_2x^2 - p_1b_5x^5 + p_2bx) + j_b(q_1x^5 + q_2x^7) \right]}{6x^4} + j_e(8p_1b^2x^4 + 24b^2p_2) + j_b(24q_2b^2x^5 - 3q_1bx^4 - 9q_2bx^6) \right]^{1/2} \times \text{sec} \left[ \mu(t - k_1x + \delta) \right].
\]

We have plotted the solutions (Eqs. (4.104-4.105)) by assuming \( f(x) = p_1x^2 + \frac{p_2}{2} + p_3 \) and \( g(x) = p_1x + p_2x^3 \) for the choices of parameters \( J_e = 0.22, J_b = 0.20, p_1 = 0.8, p_3 = 0, a = 6, b = 3, k_1 = 0.6, k = 0.5, \omega = 1.75 \) and \( \delta = 0.5 \). Figs. (4.7) show the snap shots of creation of soliton under the influence of cubic nonlinear inhomogeneity in the varying biquadratic interaction. From the figures, it is evident that the annihilated solitons as presented in Figs. (4.6) can now again be created by the combined effect of both \( f(x) \) and \( g(x) \). In plotting the figures the coefficient of \( p_2 \) is kept constant and the increase in the value of \( p_1 \) from negative to positive, slowly regenerate the annihilated soliton in the ferromagnetic medium as depicted in the contour plots. It is also should be pointed out that, in the absence of biquadratic varying exchange interaction i.e., when \( g(x) = 0 \), the increase in the value of the coefficient of \( x^2 \) in \( f(x) \) from 0 to 0.5 led to the annihilation of soliton. However, ironically, the similar increase in \( p_1 \) in the presence of \( g(x) \), recreate the annihilated soliton slowly.
Figure 4.6: Annihilation of soliton with nonlinear inhomogeneity for $k_1 = 1.7$, $k = 2.1$, $\omega = 0.03$, $J_e = 0.3$, $J_b = 0$, $f(x) = p_1 x^2 + p_2 x + p_3$, $g(x) = 0$, $a = 0.8$, $b = -0.8$, $p_1 = 0.2$, $\delta = 0.5$ (a) $p_1 = 0$, $p_2 = 0$, (b) $p_1 = 0.09$, $p_2 = 0.3$, (c) $p_1 = 0.15$, $p_2 = 0.3$ (d) $p_1 = 0.33$, $p_2 = 0.3$ and (e) $p_1 = 0.5$, $p_2 = 0.3$. 
Figure 4.7: Snapshots of creation of soliton and associated contour plots under the influence of nonlinear inhomogeneity with $f(x) = p_1 x^2 + (p_2/x^2)$, $g(x) = p_1 x + p_3$, $J_0 = 0.22$, $J_b = 0.20$, $p_2 = 0.8$, $a = 6$, $b = 3$, $k_1 = 0.6$, $k = 0.5$, $\omega = 1.75$, $\delta = 0.5$ (a) $p_1 = -2$, (b) $p_1 = -0.5$, (c) $p_1 = 1.5$, (d) $p_1 = 3$, (e) $p_1 = 6$ and (f) $p_1 = 10$. 
Therefore, one can conclude that the presence of varying biquadratic exchange interaction controls the precessional motion of the magnetization vector about the effective field.

4.7 Conclusions

The nonlinear spin dynamics of site-dependent biquadratic ferromagnetic spin chain with crystal field anisotropy in the semi-classical limit through Holstein-Primakoff transformation were studied. The integrability condition is established through Painlevé test. In order to understand the nonlinear spin excitation under the influence of inhomogeneity, the multiple scale perturbation analysis is performed on the associated inhomogeneous nonlinear Schrödinger equation. The secularity conditions of perturbation analysis lead a system of coupled evolution equations for soliton parameters. The associated dynamical equations obtained from the secularity conditions exhibit the magnetization reversal phenomena through soliton flipping is understood by solving it through tanh-function method. The nature of perturbed soliton solution is also studied and from the real and imaginary part of perturbed solitons, it is clearly observed that solitons undergoes tremendous changes during its evolution. That is, the perturbed solitons exhibit magnetization reversal phenomena in the nanosecond regime and can be tunable or controllable by changing the values of the parameters associated with it.

As a second part of this chapter, we have demonstrated the creation and annihilation of the soliton in a isotropic site-dependent ferromagnet with varying bilinear and biquadratic exchange interactions. The nonlinear spin dynamics has been studied through an understanding of the underlying geometry of the system. This is achieved by mapping the spin chain onto a moving helical space curve in the three dimensional Euclidean space. The two coupled evolution equations for the curvature and torsion of the space curve have been recasted into a higher order inhomogeneous generalized nonlinear Schrödinger equation. We employed the sine-cosine function method to solve the equation of motion with the aid of symbolic computation. In the presence of invariant inho-
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mogeneity, the spin vector precesses with constant velocity about the effective field and the associated spin excitations are governed by solitons. From the results obtained we conclude that the nonlinear inhomogeneous bilinear exchange interaction disturbs the precessional motion of the magnetization vector leading to the annihilation by destroying the robust nature of the soliton in a ferromagnetic medium. On the other hand, the nonlinear inhomogeneity representing the biquadratic exchange interaction revamps the precessional motion and thus recreate the annihilated soliton. Thus one can create or annihilate soliton by tuning the linear or nonlinear inhomogeneity in the ferromagnetic media. The above controllable magnetization reversal phenomenon and creation and annihilation of soliton in site-dependent ferromagnetic medium is expected to have potential applications in construction of magnetic memory and recording devices.