Chapter 1

Nonlinear dynamics in condensed matter systems
1.1 Birth of soliton

The study of natural systems driven by powerful nonlinear force provides a fruitful opening to the new field called nonlinear dynamics and is governed by nonlinear evolution equations which admit nonlinear excitations in the form of solitons. It started more than 100 years ago in the field of hydrodynamics, where the mutual influence of the experimental investigation of the nonlinear propagation of water waves and the theoretical effort to relate the results to the solutions of the Korteweg-de Vries equation which turned out to be very fruitful.

We begin our discussion of solitary waves or solitons with a historical account from the person who first observed this phenomena, J. Scott Russell [1]. In 1834, J. Scott Russell observed on the Edinburgh-Glasgow canal a great wave of translation. Riding alongside a canal, he observed a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the canal apparently without change of form or diminution of speed and which was preserving its original figure for a very long distance. Thus, the solitary wave is a single hump of water which propagates without deformation. Years later after this first observation he built a tank in his own garden and started to experiment with shallow water waves to generate solitary waves by dropping a weight at one end of a water channel. He discovered that their shape can be described by a \( \text{sech}^2 \) function and deduced an empirical relation for the volume of water in the wave is equal to the volume of water displaced and further that the speed \( c \) of the solitary wave is obtained from the relation \( c^2 = g(h + a) \), where \( a \) is the amplitude of the wave, \( h \) and \( g \) represent the undisturbed depth of water and the acceleration due to gravity respectively. It is found that their peak amplitude is proportional to the velocity of the wave. However, at that time there was no equation describing such water waves and possessing solitary solutions.

It took more than sixty years until Korteweg and de Vries, a Dutch physicists, in 1895 derived a nonlinear wave equation which describes the evolution of wave in a shallow one-dimensional water channel [2]. As a confirmation of
Russel's experimental investigations they have shown theoretically that such a system admits solitary wave solutions. In 1955, Fermi, Pasta and Ulam performed numerical experiment in Los Alamos laboratory and they investigated how the equilibrium state is approached in a one-dimensional nonlinear lattice [3]. It was expected that the nonlinear interactions among the normal modes of the linear system would lead to the energy of the system being evenly distributed throughout all the modes, that is, the system would be ergodic. The results of numerical analysis contradicted this idea. The energy is not distributed equally into all the modes, but they found that only a very small number of modes were actually participating in the system dynamics and the system returns to the initial state after some period (the recurrence phenomena). In 1965 Zabusky and Kruskal, while studying the Fermi-Pasta-Ulam problem, rederived the Korteweg-de Vries (KdV) equation as a continuum approximation [4]. Further, they have numerically solved this equation for periodic boundary conditions and they found an unexpected property of the KdV equation. From a smooth initial waveform, waves with sharp peaks emerge. Those pulse-waves move almost independently with constant speed and pass through each other after collision. A detailed analysis confirmed that each pulse is a solitary wave of $\text{sech}^2$-type and thus the soliton was discovered. The remarkable quality of these solitary waves was that they could collide with each other and preserves their shape and speed after the collision and the solitary waves behave like stable particles. In modern physics, a suffix-on is used to indicate the particle property, for example, phonon and photon. Zabusky and Kruskal named a solitary wave with the particle property as a soliton. During the past fifteen years a rather complete mathematical description of solitons has been developed. The amount of information on nonlinear wave phenomena obtained through the fruitful collaboration of mathematicians and physicists using this description make the soliton concept one of the most significant developments in modern mathematical physics. The non-dispersive nature of the soliton solutions to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$  \hspace{1cm} (1.1)
which arises because of the robust balance between the dispersion and nonlinearity. The presence of both the dispersion and nonlinearity can be appreciated by considering simplified versions of the KdV equation.

The absence of nonlinear term $uu_x$ in the KdV equation yields $u_t + u_{xxx}$ and its associated dispersion relation $\omega = k^3$ with the phase velocity and group velocity $v_p = \omega/k = k^2$ and $v_g = \partial \omega/\partial k = 3k^2$ respectively. The phase velocity measures how fast a point of constant phase is moving, while the group velocity measures how fast the energy of the wave moves. The waves described by $u_t + u_{xxx} = 0$ are said to be dispersive because a wave with large $k$ will have larger phase and group velocities than a wave with small $k$. Therefore, a wave composed of a superposition of elementary components with different wave numbers will disperse or change its form as it propagates, whereas in the absence of dispersion term in the KdV equation, the corresponding wave solution breaks, that is, portions of the wave undergoing greater displacements move faster than those undergoing smaller displacements. This multi-valuedness is a result of the nonlinearity and leads to a change in shape as the wave propagates. A remarkable property of the KdV equation is that dispersion and nonlinearity balance each other and allow wave solutions that propagate without changing its form as depicted in Fig. (1.1). The manifestation of balance between dispersion and nonlinearity can be quite different from system to system. In 1967 Gardner, Greene, Kruskal and Miura while exploring the initial value problem for the Korteweg-de Vries equation, they discovered a new method of mathematical physics namely direct and inverse scattering transformation [5]. One year later Lax generalized these ideas and in 1971 Zakharov and Shabat proved that this method also can be applied to another physically significant nonlinear evolution equation, namely the nonlinear Schrödinger (NLS) equation which have soliton solutions that are distinct from the bell-shaped solitons of the KdV equation [6,7]. This equation is generic to all conservative systems that are weakly nonlinear but strongly dispersive. In the same way as a single hump, a wave packet involving some carrier wave modulated by an envelope can be stabilized by the combined effect of dispersion and nonlinearity. Three
years later Ablowitz, Kaup, Newell and Segur showed that this method is ana-
log to the Fourier transform for nonlinear problems. They called this procedure
as the inverse scattering transform [8]. Although the concept of the soliton
which emerged from these studies proved to be useful in many different fields
of physics such as optics, nuclear and particle physics and plasma physics [9-
16]. In solid state physics, soliton like excitations have been of much interest
particularly since Krumhansl and Schrieffer described how to deal with them in
thermal equilibrium and concluded that they should be considered as new type
of elementary excitations [15]. After this seminal work, soliton like excitations
were investigated in many branches of condensed matter physics for example
dislocations, Josephson transmission lines, lattice dynamics, conducting poly-
mers, charge-density waves, superfluid $^3$He and magnetic spin chains [17-29].
Historically, particular interest has been devoted to nonlinear solutions to the
equations of motions of integrable models, that is, models which can be shown
to allow a complete solution for any given initial condition. Among these models
are Toda lattice, continuum sine-Gordon equation, KdV equations and nonlin-
ear Schrödinger equation and so on. The general solutions to these models are usually composed of several types of mode such as the kink and the breathers [30-35].

The field of soliton like excitations in many respects is equivalent to the field of magnetic domain walls which have been investigated in considerable details. Domain walls were introduced by Bloch in 1932 and Döring in 1948 was the first to consider moving Bloch walls in slightly anisotropic three dimensional Heisenberg magnets quantitatively [36,37]. Somewhat later Enz in 1964 gave an explicit discussion of the analogous problem for a magnet with easy axis (Ising) symmetry and related the domain wall to the soliton solutions of the sine-Gordon equation [38]. This early work, although closely related to the developments since the mid 1970s dealt with regular three dimensional magnets and mainly emphasized the static macroscopic nature of the nonlinear configurations.

The modern era of nonlinear one-dimensional magnetism began when it was realized that the dynamics of nonlinear excitations in classical magnetic chains are particularly interesting and that localized large amplitude waves propagating with a permanent profile are possible. Akhiezer and Brodovik have given solutions of this type for easy-plane and easy axis ferromagnets and antiferromagnets [39]; Nakamura and Sasada [40] and Tjon and Wright [41] gave an analysis of soliton solutions in an isotropic Heisenberg spin chain. It has been realized that one-dimensional magnets have quite interesting equations of motion, which may serve as model systems to exhibit many types of nonlinear behaviour. Somewhat later, some of these systems were shown to be completely integrable and thus in the same class as the KdV and the sine-Gordon equation. In the next section, we will present the historical developments and its perspectives of the magnetism and magnetic materials since the nonlinear spin dynamics play a vital role in the modern technology.
1.2 History of magnetism

Magnetic phenomena have been recognized since ancient times, but understanding of magnetism began only with developments in physics in the 19th century. Magnetism is the most familiar phenomenon through the compass, indicating the direction of the earth magnetic poles, also through small permanent magnets used in household fixtures and as children’s toys. Permanent magnet is a piece of iron or certain metal alloy with strong attraction or repulsion from one another.

In technology, magnetism is employed on a wide-ranging scale, from enormous electromagnets used and lift massive loads and small electromagnets used for the recording of sound and television pictures on magnetic tape. The use of applied magnetic field for the rapid reversal of the direction of magnetization of magnetic oxides or thin metallic films of iron alloy is being exploited in computer memories. Thus the magnetic materials are vital for today information technology industry. The use of high magnetic field with specially shaped configuration has been explored as a method of containing the plasma of charged ions at the temperature of about $10 \times 10^6 \, ^\circ C$ required to produce nuclear power from controlled fusion reactions.

The mineral magnetite, a magnetic oxide of iron, was the first permanent magnetic materials appear in Greek writings from as early as 800 BC. The mineral called magnetite, which even in its natural state has a strong attraction of iron, was mined in the Greek province of Magnesia in Thessaly. The name lodestone, by which the mineral was generally known in English, signifies leading stone and refers to the first application of magnetism, the compass. Some writers believe that the compass was in use in China as early as the 26th century BC; others that it was introduced to China only in the 13th century AD, its invention being of Italian or Arab origin. The first known experimental investigation of the lodestone is described in Latin treatise dated 1269 by Petrus Peregrinus de Maricourt in which by mapping the direction of a small rectangular piece of iron on the surface of a spherical lodestone, he obtained lines that circled the lodestone and intersected at two points in the same way
as lines of longitude on the earth intersect at the poles. By analogy, these two points called the pole of the magnet. In 1600, William Gilbert, a English medical doctor and scientist, who repeated and extended such experiments and summarized in his famous treatise De Magnete is that the Earth itself is a magnet and the lodestone loses its magnetic properties when heated it to bright red heat but then regains them on cooling.

In 1785 the French physicist Charles Coulomb using torsion balance who established the inverse square law of force between both electric charges and magnetic poles. In each case, like poles repel one another whereas unlike poles attract one another. From these results, the theory of magnetostatics, study of steady state magnetic field, was developed using the concept of a magnetic potential analogous to the electrical potential. The origin of magnetic properties remained a mystery but a major step forward occurred in 1820, when Danish physicists, Hans Christian Orsted, observed that an electric current flowing in a wire affected a nearby magnet. The same discovery had been made and reported by Gian Dominico Romagnosi, an Italian jurist, in Gazetta di Trentino, August 3, 1802, but was ignored. In France, a physicist, Francois Arago, showed that a current acts like an ordinary magnet, in its ability both to attract iron fillings and to induce permanent magnetism in iron needles. A year later the French scientists Jean-Baptiste Biot and Felix Savart derived the magnetic field around a current carrying wire and during 1820-1825 Andre Marie Ampere considered the forces between current carrying wires. This led to the famous laws named after the discoverers.

Classical electromagnetism peaked with the work of two of the greatest physicists of the 19th century, the experimentalist Michael Faraday and the theorist James Clerk Maxwell [42]. In 1831 Faraday discovered electromagnetic induction and in 1845 he discovered a direct connection between magnetism and light [43]. He also found that the plane of vibration of polarized light is rotated when the light passes through a medium in a direction parallel to a magnetic field (the Faraday effect). Faraday's discoveries of electric motors, generators and transformers have become the foundation of the in-
dustrialized society. Maxwell used Faraday's notion of a connection between electricity and magnetism on a firm mathematical footing, developed in his book Treatise on Electricity and Magnetism. This constituted the birth of electromagnetism and the electromagnetic field. In 1855 W.E. Weber had derived the numerical value for $1/\sqrt{\mu_0\varepsilon_0} = 3.1074 \times 10^8 \text{ m/s}$ however he could not understand why this value was close to the speed of light. This connection was made by Maxwell whose studies on the equations describing electric and magnetic fields was led to the value $c = 1/\sqrt{\mu_0\varepsilon_0}$. Maxwell concluded that light is a form of electromagnetic wave. So thus the connection between magnetism and light has been established. Maxwell's theories and their experimental verification by H. Hertz in Germany, who discovered radio waves in 1888, today are the basis for global communications at the speed of light. The 19th century development of magnetism concluded with Zeeman's discovery in 1896. The century was crowned by the discovery of the electron by J.J. Thomson in 1897 and independently around the same time by E. Wiechert. The understanding of magnetic phenomena in the 20th century largely concentrated on the development of an atom-based picture.

1.3 Ordered magnetic systems

The magnetic moment of solids originates from the partly filled inner electron shells, of particular importance are the iron-series transition-metal elements (3d elements) Fe, Co, Ni, and the rare-earth or 4f elements, such as Nd, Sm, Gd and Dy. There are two sources of the atomic magnetic moment $m$: current associated with the orbital motion of the electron around the nucleus and an intrinsic magnetic moment which is associated with spin angular momentum and the orbital moments of these electrons do not make an important contribution for strong magnetism [44,45]. In 1907 Weiss introduced a theory of ferromagnetism due to mutual interaction between the atomic magnets based on a molecular field concept [46]. Therefore the macroscopic magnetic moment appears due to the spin ordering of the electrons in the partially filled shells under certain conditions and interaction takes place between them. Based on
this assumption the quantum theory of strong magnetism is formulated by Heisenberg [47] for which he was awarded Nobel prize in 1932. Since magnetic moments of iron, cobalt and nickel are caused mostly by spin motion of the electrons, which refer to the atomic magnetic moment simply as the spin. The origin of diamagnetism is mainly due to the orbital motion of electrons about the nuclei induced electromagnetically by the application of an external field according to Lenz’s law. In paramagnetic materials, the magnetic atoms or ions whose spins are isolated from their magnetic environment and can more or less freely change their directions. At finite temperatures, the spins are thermally agitated and take random orientations. By the application of a magnetic field, the average orientations of spins are slightly changed so that it produces a weak induced magnetization parallel to the applied magnetic field. In the case of ferromagnetism, the spins are aligned parallel to one another as a result of strong positive interaction acting between the neighbouring spins when the temperature falls below a certain critical temperature called Curie temperature [48,49].
For this reason ferromagnets have spontaneous magnetic moments even in the absence of an external magnetic field. In spite of the presence of spontaneous magnetization, the ferromagnetic substance is usually not spontaneously magnetized but exists rather in a demagnetized state. This is because, the interior of the ferromagnetic single crystal is divided into many magnetic domains, each of which is spontaneously magnetized. The dimensions of domains, their shapes and mutual positions are governed by the conditions for the existence of the minimum thermodynamic potential of the system [50-57]. In antiferromagnetic materials, these include carbonates, anhydrous sulphates, oxides and fluorides of transition metals Mn, Ni, Co and Fe, the spins are aligned antiparallel to each other and the mean atomic magnetic moments compensate each other and hence no net magnetization is observed below the transition temperature called Neel temperature [58,59]. Finally, another type of magnetically ordered crystal known commonly as ferrites in which all the spins are aligned antiparallel with unequal magnetic moments and hence exhibiting a nonzero magnetization. Examples of this type are compounds of transition metals such as the salts $MnO.Fe_2O_3$ and $3Y_2O_3.5Fe_2O_3$. This different types of
magnetic materials are depicted in Fig. (1.2).

Another important ordered magnetic system is the spin ladder system with two ferro or antiferro magnetic legs coupled ferro or antiferromagnetically which provides a novel and potential mechanism for the discovery of high temperature superconducting materials [60-70]. Spin ladders having an even number of chains have been predicted to exhibit interesting dynamics including superconductivity. The spin ladders are those in which the exchange coupling along the rungs is very similar to the exchange coupling along the chains. To illustrate the spin ladders we can quote the following examples, the two-leg ladder $SrCu_2O_3$ as depicted in Figs. (1.3), the three-leg ladder $Sr_2Cu_3O_5$ and the zigzag chain $SrCuO_2$ in which the ladder structure are formed by the spins on the copper sites coupled through the oxygen atoms. It is well established that the Heisenberg spin-1/2 ladders with an even number of legs are found to have finite energy gap while those with an odd number of legs show gapless spin excitations [71].

1.4 Origin of nonlinearity

Ordered ferromagnetic systems are inherently nonlinear because of the nature of the exchange interaction between the ferromagnetic spins and the relationship between magnetic field strength $H$ and magnetic flux density $B$ is nonlinear which is evident from the hysteresis loop. When an external magnetic field is applied to a ferromagnet, the atomic dipoles align themselves with the external field. Even when the external field is removed, part of the alignment will be retained: the material has become magnetized. The relationship between magnetic field strength $H$ and magnetic flux density $B$ is not linear in such materials. If the relationship between the two is plotted for increasing levels of field strength, it will follow a curve up to a point where further increase in magnetic field strength will result in no further change in flux density. This condition is called magnetic saturation. This situation leads to the inherent nonlinearity in ferromagnetic systems. If the magnetic field is now reduced linearly, the plotted relationship will follow a different curve back towards zero.
field strength the point at which it will be offset from the original curve by an amount called the remanent flux density or remanence. If this relationship is plotted for all strengths of applied magnetic field the result is a sort of S-shaped loop as depicted in Fig. (1.4). The thickness of the middle bit of the S describes the amount of hysteresis, related to the coercivity of the material. Thus the output of the system is no longer proportional to the input and therefore the ordered ferromagnetic systems are inherently nonlinear because of the nature of exchange interactions between the ferromagnetic spins.

![Figure 1.4: Hysteresis loop.](image)

### 1.5 Magnetic interactions

In this section, we consider different types of physically significant magnetic interactions such as exchange interaction, anisotropic interaction, super exchange interaction, Dzyaloshinskii-Moriya interaction, octupole-dipole interaction and interaction with external magnetic field.

#### 1.5.1 Heisenberg exchange interaction

This interaction should be analyzed by means of quantum theory which is of electrostatic in origin. More specifically, on a scale in the order of the atomic scale, the exchange interaction tends to align neighboring spins. In
view of a continuum average analysis in terms of magnetization vector field, we expect that the exchange interaction tends to produce small uniformly magnetized regions, indeed observed experimentally and called magnetic domains. In this respect, the existence of domains [46] was postulated by Weiss in the early 1900s to explain the inverse temperature dependence of susceptibility for ferromagnetic materials investigated by Curie. This theory was partially validated by the work of Barkhausen (1915), in which the emergence of irreversible jumps in magnetization reversal was connected to the Weiss domains. Successively, experimental observations [52] based on Faraday and Kerr effect measurements, definitely stated the existence of magnetic domains. However, in 1928 Heisenberg [47] described ferromagnetic systems in terms of exchange interactions, justifying the Weiss theory on molecular field. The exchange interaction defines the spin systems in magnetic materials and is responsible for the existence of parallel and antiparallel spin alignments called ferromagnet and antiferromagnet respectively. Hence Heisenberg addressed microscopic origin of exchange interaction, after his work on the $^4\text{He}$ atom in 1926 and the Heitler-London calculation for $H_2$ in 1927 and hence the theory of ferromagnetism originates. He-spectrum holds the key for the quantum mechanical understanding of magnetism. The origin of the singlet-triplet splitting of the excited states of electrons is due to the exchange interaction between the two electrons. The exchange interaction arises from the coulomb interaction between two electrons and symmetrization postulate. The important requirement of symmetrization postulate is the total wave function should be antisymmetric with respect to the simultaneous interchange of the coordinates and of the spin variables of electrons. This means that the symmetric spatial part of total wave function will be associated with antisymmetric spin function which gives singlet excited state and conversely, an antisymmetric space function will be associated with symmetric spin function which leads to the triplet excited state. The spin function will be symmetric if the resultant spin $S$ of the two electrons is equal to unity ($S = 1$) and antisymmetric if $S = 0$. Therefore the space wave function will be antisymmetric for $S = 1$ and symmetric for $S = 0$. 
If the total spatial wave function is antisymmetric, both electrons occupies the different orbital with the same spin states (symmetric spin part) leading to the triplet excited states with parallel spin configurations, whereas in the singlet state, both electron occupies different orbital with the different spin states (antisymmetric spin state) leading to the antiparallel spin configurations. Thus, the symmetrization postulate leads to either parallel or antiparallel alignment of spins. Heisenberg formulated exchange model that formally builds on the Heitler-London model which now known as Heisenberg Hamiltonian. The effective Hamiltonian for a many electron system is usually written as

$$H = - \sum_{<i,j>} J_{ij} \vec{S}_i \cdot \vec{S}_j,$$

where $J_{ij}$ represents the exchange integral because it reflects the energy associated with a change of quantum states between the two electrons. In Eq. (1.2) the double sum runs over all the magnetic atoms and it is sufficient to consider only the interaction of nearest neighbours and is represented by the brackets $<i,j>$. In the isotropic case, $J_{i,j}$ can be replaced by the constant $J$ and according to the positive and negative sign of $J$, the tendency is for nearest neighbours to align parallel and antiparallel respectively. The difference in energy between the parallel and antiparallel spin configuration is the exchange energy which depends on their relative orientation ($\Delta E = 2J$, corresponding to the singlet-triplet splitting). The electrostatic Coulomb repulsion force regulates the spin configuration in atoms, molecules and solids via Pauli principle and therefore the effect of the electrostatic repulsion is greater in the singlet state.

### 1.5.2 Anisotropic interaction

The energy of a magnet depends on the orientation of the magnetization with respect to the crystal axes, which is known as magnetic anisotropy. Permanent magnets need a high magnetic anisotropy in order to keep the magnetization in a desired direction. Soft magnets are characterized by a very low anisotropy, whereas materials with intermediate anisotropies are used as magnetic recording media. In the iron group ions, the 3d shell which is responsible for strong
magnetism experiences an intense inhomogeneous electric field produced by neighbouring ions. This inhomogeneous electric field is known in the literature as crystal field effect [48]. Due to the competition between electrostatic crystal field interaction and the spin-orbit coupling in a ferromagnetic crystal the magnetization is directed along certain crystallographic axis or plane known to be direction of easy magnetization. The second-order anisotropy-energy expression for a single ion is

\[ H_{\text{ani}} = \sum_i K_1 (S_i^z)^2. \] (1.3)

This anisotropy is known as uniaxial anisotropy and \( K_1 \) is the uniaxial anisotropy constant. It is noted that without the existence of magnetic anisotropy, 2D objects such as thin films could not order magnetically and even in 3D samples, the magnetization would twist itself into interwoven curls and be hardly observable [72]. This happens because the exchange interaction is a short range and the magnetization can thus be rotated over a distance called the magnetic coherence length into a new direction at a very low expense of energy.

1.5.3 Super exchange interaction

A number of ionic solids, including some oxides and fluorides, have magnetic ground states. For example, \( MnO \) and \( MnF_2 \) are both antiferromagnets, though this observation appears at first sight rather surprising because there is no direct overlap between the electrons on \( Mn^{2+} \) ions in each system. The exchange interaction is normally very short-ranged so that the longer-ranged interaction that is operating in this system must be in some sense super. The exchange mechanism which is operative here is known as superexchange that was first pointed out by Kramers in 1934 [73] and the theory was more formally developed by Anderson in 1950 [74]. It can be defined as an indirect exchange interaction between non-neighbouring magnetic ions which is mediated by a non-magnetic ion placed in between the magnetic ions. It arises because of the advantage of kinetic energy that present in the antiferromagnet which can be understood through Fig. (1.5) in which two transition metal ions separated by
an oxygen ion. For simplicity we will assume that the magnetic moment on the
transition metal ion is due to a single unpaired electron. Hence if this system
were perfectly ionic, each metal ion would have a single unpaired electron in
a d-orbital and the oxygen would have two p-electrons in its outermost occu-
pied states. By hypothesis there is no direct overlap between the two d-wave
functions and also there is a finite probability that an electron will move from
\( O^- \) ion to the s-state or d-state of one of the magnetic ions and thus the oxy-
gen is paramagnetic and can enter into magnetic interaction which is called
as superexchange interaction. The figure demonstrates that antiferromagnetic
coupling lowers the energy of the system by allowing these electrons to become
delocalized over the whole structure, thus lowering the kinetic energy. In Fig.
(1.5), the arrows show the spins of the four electrons and how they are dis-
tributed over the transition metal (M) and oxygen (O) atoms. If the moments
on the transition metal atoms are coupled antiferromagnetically (a, b, c), the
ground state is (a) and this can mix with excited configurations like (b) and
(c). The magnetic electrons can thus be delocalized over the M-O-M unit, thus
lowering the kinetic energy. In some circumstances, superexchange can ac-
tually be ferromagnetic. For example, imagine a situation in which there is a
coupling, through an oxygen ion, between an occupied \( e_g \) orbital on one mag-
netic ion and an unoccupied \( e_g \) orbital on another magnetic ion. There is an
energetic advantage to the \( e_g \) electron hopping onto the unoccupied orbital, if
when it arrives its spin which is aligned with the spin of the \( t_{2g} \) electrons be-
cause of the Hund's rule coupling. If the moments on the metal (M) atoms are
coupled ferromagnetically (d, e, f) the ground state (d) cannot mix with excited
configurations like (c) and (f) because these configurations are prevented by
the exclusion principle. The ferromagnetic configuration therefore costs more
energy. Thus the superexchange could be ferromagnetic in this case, but this
is weaker interaction and less common than the usual antiferromagnetic su-
perexchange [75].
Figure 1.5: Super exchange interaction in a magnetic oxide.

1.5.4 Dzyaloshinskii-Moriya (D-M) interaction

This is an extension of the superexchange theory to include the effect of spin-orbit coupling. The spin-orbit interaction plays an important role in a similar manner to that of the oxygen atom in superexchange. Here the excited state is not connected with oxygen but it is produced by the spin-orbit interaction in one of the magnetic ions. Thus, there is an exchange interaction between the excited state of one ion and the ground state of the other ion. This is known in the literature as the anisotropic exchange interaction, or also as the Dzyaloshinskii-Moriya interaction. Antisymmetric coupling between two localized magnetic moments was first suggested by Dzyaloshinskii [76] in 1957 solely on grounds of symmetry and later derived theoretically by Moriya [77] in 1960, to explain the mechanism of weak ferromagnetism in some antiferromagnetic crystals such as $\alpha - Fe_2O_3$, $MnCO_3$ and $CoCO_3$. The D-M interaction arises due to interplay between superexchange and spin-orbit coupling, which play an important role to describe certain class of insulators as well as in the study of spin glasses and the electron paramagnetic resonance line width observed in
Figure 1.6: The presence of the Dzyaloshinskii-Moriya interaction ($\vec{D} = 0$) that leads to a slight canting of the magnetic moments and a resulting net magnetization $M \neq 0$.

some one-dimensional antiferromagnet [78-82]. The D-M interaction has also been utilized for describing the emergence of weak ferromagnetism observed in the low temperature superconducting materials such as $La_2CuO_4$ and Mitra and Halperin have considered a time dependent D-M Hamiltonian to explain infrared transmission measurement on the $S = 1$ one-dimensional antiferromagnet [83]. The antisymmetric microscopic coupling between two localized magnetic moments $\vec{S}_i$ and $\vec{S}_j$ can be written as

$$\hat{H}_{DM} = \sum_{i<j} \vec{D} \cdot \vec{S}_i \times \vec{S}_j.$$  \hfill (1.4)

The vector $\vec{D}$ vanishes when the crystal field has an inversion symmetry with respect to the center between the two magnetic ions. However, in general, $\vec{D}$ may not vanish and then will lie parallel or perpendicular to the line connecting the two spins depending on the symmetry. The presence of D-M interaction in the antiferromagnetic spin chain leads to small canting (a small deviation from an overall collinear magnetic configurations) between the interacting moments which is the mechanism of weak ferromagnetism in antiferromagnetic crystal whereas in ferromagnetic spin chain the D-M vector tries to force $\vec{S}_i$ and $\vec{S}_j$ to be at right angles in a plane perpendicular to the vector $\vec{D}$ in such an orientation as to ensure that the energy is negative. This produces a small ferromagnetic
component of the moments in a direction perpendicular to the spin-axis of the antiferromagnet as depicted in Fig. (1.6). The effect is known as weak ferromagnetism.

1.5.5 Octupole-Dipole interaction

The existence of a third-order spin-spin exchange interaction was proposed by Toru Moriya [85] on the extension of theory of superexchange interaction to include spin-orbit coupling in spin Hamiltonian in addition to biquadratic exchange interaction. Physically, this interaction was interpreted by Bleany [85] as octupole-dipole interaction which describes the hyperfine structure of \( \Gamma_3 \) ions in cubic symmetry in the magnetic compound [86-88] and it is acting as an interesting nonlinear dynamical models. The Hamiltonian for octupole-dipole interaction can be written as

\[
H_{OD} = -J_0 \sum_{<ij>} (\vec{S}_i \cdot \vec{S}_{i+1})(\vec{S}_{i+1} \cdot \hat{k})^2,
\]

where \( J_0 \) represents the octupole-dipole parameter and \( \hat{k} = (0, 0, 1) \) is a constant vector directed along z-axis.

1.5.6 Biquadratic interaction

For the past few decades, the existence of the biquadratic exchange coupling has been pointed out by many authors. Anderson [89] has pointed out that for a high-spin system \( (S > 1) \) the biquadratic exchange interaction should be considered. This biquadratic exchange effects are of importance in such magnets as \( Fe, EuO \) (ferromagnet), \( MnO, MnF_2 \) (antiferromagnet), \( Fe_3O_4 \) (ferrimagnet), as well as rare earth vanadates, arsenates and phosphates. Many research articles have been focused to highlight the effect of biquadratic exchange interaction [90,91] which can illustrate some experimental observations where the usual Heisenberg model is unable to explain. Up to date, much interest is shown in the magnetic properties, such as the Curie temperature, magnetization and magnetic susceptibility, using the Green function technique with
different decoupling approximations [92-94]. Relatively less attention has been paid to the magnetic excitations.

The theoretical explanations for the origin of the biquadratic exchange interaction were given by Anderson [89] and Huang and Orbach [95] in the theory of the superexchange interaction and by Kittel [96] in the theory of the magnetoelastic effect. The biquadratic term may arise for various reasons, but the most transparent one is due to the spin-lattice coupling. The other possible origins of the biquadratic exchange interaction were pointed out by Allan and Betts [89, 95, 96, 86] in the calculation with the use of the permutation operator and by Iwashita and Uryu [97,98] in the theory of perturbation expansion. For the ferromagnetic metallic layers, the origin of this interaction was discussed by Edwards et al., [99] and Slonczewski [100] in the theory associated with higher harmonics in the oscillatory exchange coupling. The effect of this interaction on the magnetic properties of the Heisenberg ferromagnet have been investigated by Chen and Levy [101,102], Allan and Betts [86], Brown [90,103], Biegala [104], Micnas [105], Adler et al., [106], and Iwashita and Uryu [107,108]. It was pointed out that the higher-order exchange interactions are smaller than the bilinear ones for the 3d group ions [86,97], and comparable with the bilinear ones in the rare-earth compounds [96,109]. The Heisenberg Hamiltonian for the biquadratic exchange interaction can be written as

$$H_{biq} = -K \sum_{<ij>} [\vec{S}_i \cdot \vec{S}_j]^2,$$

where $K$ is the biquadratic exchange parameter.

### 1.5.7 Interaction with external magnetic field

In the absence of a magnetic field the magnetization naturally breaks up into domains with different magnetization directions that cancel on a macroscopic scale. When a strong external magnetic field is applied in a ferromagnetic crystal, all the spins gain energy and align themselves parallel to the applied magnetic field. The application of a magnetic field is therefore required to induce a well-defined magnetic direction which makes the magnetic material useful.
In the early days of magnetism, for example, the Zeeman interaction was used to align iron compass needles by the magnetic field of lodestones. Today, we all use the Zeeman effect daily as we save information in our computers. The Hamiltonian representing the interaction of the applied external magnetic field with the spontaneous magnetization is in the form of Zeeman energy and can be written as

$$H_{ex} = -g\mu_B\vec{H} \cdot \sum_i \vec{S}_i,$$

(1.7)

where $g$ is the gyromagnetic ratio, $\mu_B$ is the Bohr magneton and $\vec{H}$ is the externally applied magnetic field.

1.5.8 Site-dependent exchange interaction

Most of the work on magnetic chains has been based on the homogeneous Heisenberg Hamiltonian, where the nearest neighbor exchange integral is merely a single constant $J$ or two constants as in the case of uniaxial anisotropy. The inhomogeneous or site-dependent Heisenberg Hamiltonian [110] can written as

$$H = -J \sum_{<i,j>} f_i \vec{S}_i \cdot \vec{S}_j,$$

(1.8)

where the exchange integral $J$ is bond-dependent. The function $f_i$ characterize the variation in the bilinear exchange interaction. Normally, the bond dependency of exchange integral in magnetic compound may occur because of the following two factors (a) if the distance between neighbouring atoms varies along the chain, thereby altering the overlap of electronic wave functions (assumed to be identical at all sites). Thus the interaction between the spins depends upon the site in the crystal lattice, which is known as site-dependent interaction e.g. charge transfer complexes $TCNQ$, $Ni(\text{CN})_4$, organo-metallic insulators, TTF-bisdithiolenes and $Ni(\text{Co})_4$ [111]. (b) If the atomic wave functions itself varies from site to site although the atoms themselves may be equally spaced. This type of inhomogeneity occurs when magnetic insulators placed in a weak, static and inhomogeneous electric field. It can also be simulated
by the deliberate introduction of imperfections such as impurities or organic complexes in the vicinity of a bond so as to alter the electronic wave functions without causing appreciable lattice distortion. By gradually changing the concentration of impurities in a specified manner as we move along the chain, it may be possible to achieve a controlled inhomogeneity.

1.6 Exactly solvable quantum spin models

Bethe ansatz models are extremely useful in solving quantum spin chain models and to obtain exact results, including ground state, excited states and thermodynamics [112-120]. The isotropic Heisenberg spin chain with spin-1/2 is the original quantum integrable system for which Bethe first set down his famous ansatz [112]. In the case of Ising model, the spins have a simple coupling in which only one component plays a role. All quantum states that are eigenstates of the relevant spin components are stationary states corresponding to the Ising model and it has no dynamics [121]. In the case of XY model for spin one-half, it turned to be possible to diagonalize the Hamiltonian completely. In this quantum-mechanical model, the interaction between neighbouring spins is confined to a plane, which condition results in a complete exact solution atleast for well chosen boundary conditions. The generalization to fully anisotropic coupling, so called XYZ model, has an amazingly rich mathematical structure being essentially equivalent to the Yang-Baxter [122] eight vortex model of classical statistical mechanics [115]. For \( S = 1 \) systems, it has been found that there exists a gap between the excited states and the lowest existing state [123,124]. Apart from the physical dimension, the spin value is an important property, which determines strongly the collective behaviour of large systems.

As the value of the spin \( S \) plays a crucial role, in the case of spin systems having high spin value, the quantum mechanical description corresponding to the spin Hamiltonian is extremely complicated because one cannot indicate the particular state of individual spin. Even in the ground state this is impossible. For large spin values, the corresponding size of the spin matrix will also be
large. One real problem with the above analysis is that, although all the energy levels of the Bethe ansatz system can be found and the wave functions written down in principle, it is very difficult to diagonalize the large spin matrix and hence connect the known excitation spectrum with what is observed in.

One-dimensional quantum spin chains are interesting objects to study for a number of reasons. On the one hand, experimental systems are generally very well described by simple yet nontrivial Hamiltonians involving very few unknown parameters, a considerable amount of research has been done in understanding the dynamics of quantum spin systems which sometimes leads to exactly solvable models [121, 125-128]. Most of the progress in theoretical and experimental investigations on critical phenomena has resulted from the introduction of large variety of magnetic models. In that way theoretical physicists have been able to obtain exact or approximate solutions for the elementary excitations, due in most cases to the insurmountable mathematical problems associated with cooperative phenomena. The original aim of theoretical physicists in devising these various models was to get a better understanding of the corresponding phenomena.

1.7 Spin wave theory

For real magnetic systems which are commonly having high spin values the quantum fluctuation $\frac{1}{s(s+1)}$ ceases and also the quantum mechanical description is much complicated. Hence in order to deal with real system with high spin values, we should seek for limiting procedures. The more renounced limiting procedures are classical and semiclassical owing to their highlighting characteristics. The strong correlation between the directions of atomic spins in magnetically ordered crystals leads to the existence of particular type of collective modes in such crystals. To understand the origin of these modes to begin with let us consider a ferromagnet at $T = 0$. All the spins then point along the same direction which corresponds to minimum energy of the ferromagnet and let us now deflect the magnetic moment of a particular spin and let it go. This disturbance will not remain localized in the original spin, but owing
to the presence of the exchange interaction, it will be propagated through the
crystal in the form of a mode of wave motion, which is nothing but a form of an
excitation of much lower energy if all the spins share the reversal. Such waves
are commonly known to be ‘spin waves’ first proposed by Bloch in 1930 [129,
36]. Spin waves are oscillations in the relative orientations of spins on a lattice.
Spin waves are characterized by a definite dependence of the frequency on the
wave vector. It can also be regarded as oscillations in the magnetic moment
density, propagating through a magnetically ordered crystal, provided the tem-
perature is sufficiently low. Now it is clear that if the magnetic moment of any
given atom in the ferromagnet is pushed from the equilibrium orientation and
then left alone, a spin wave will propagate through the crystal. The spin wave
energy must be equal to the excitation energy of the crystal required to cause
the change in the orientation of the spin. The energy of an elementary excita-
tion connected with the spin wave can be obtained by multiplying its frequency
$\omega(k)$ by $\hbar$, where $k$ is the wave vector of the spin wave. Thus, the quantized form
of a spin wave is called magnons [130-134]. The energy of a magnon is equal to
the spin wave frequency multiplied by $\hbar$, just as the energy of a phonon is equal
to the frequency of the sound wave multiplied by $\hbar$. Though, in the linear case,
spin wave theory is an useful approximation, in order to make the dynamics
concrete we need to consider nonlinear mode which is inherently present and
play a leading role in the dynamics.

1.8 Semiclassical approach

In the semiclassical picture, the spin vectors precess around the quan-
tization axis $z$ with random phases in the ground state. Hence addition of
the spins of two distinct electrons to its mechanical momentum and the cou-
pling of two or more angular momenta of various origins require mathematical
techniques beyond simple formulation of models. In semiclassical theories of
magnetism it is common to approximate the unwieldy spin operators or ma-
trices by harmonic oscillator operators which resemble in many respects the
matrix structure of harmonic oscillator operators. The following are some of
the machineries established to handle the spin operators: (i) Schwinger-Boson representation [135] (ii) Holstein-Primakoff transformation [136] (iii) Dyson-Maleev transformation [137, 138] (iv) Jordon-Weigner transformation [139]. All the above representations obey the spin commutation relation

\[ [S_i^k, S_j^{k'}] = 0 \quad \text{for} \quad i \neq j, \]  

(1.9)

for all components \( k, k' = x, y, z \) of the spin vectors. Among them the Holstein-Primakoff (H-P) transformation is the well known and most useful [136, 140, 141]. In the Schwinger-Boson (S-B) representation, the spin operators are represented by the coupled-harmonic oscillator operators i.e., bosons [142, 143]. The Dyson-Maleev transformation consists of rational operators but breaks the conjugate relation of the spin operators. The Jordan-Weigner (J-W) transformation is the exact analogue of coupled-boson picture, but replaces bosons by fermions. The first formal justification of the Bloch theory of spin waves may be found in the work of Holstein and Primakoff. Bloch had naturally assumed that spin waves obey Bose-Einstein statistics, but Holstein and Primakoff showed how spin operators could be expressed in terms of true Bose fields. The H-P representation is best understood as a special case of the Schwinger coupled boson representation, that is, as an irreducible representation of the later in a subspace of field. Holstein and Primakoff considered the behaviour of a three dimensional ferromagnetic array of spins in an external magnetic field. They succeeded in defining a set of co-ordinates which describe accurately the quantum state of the system and which have the appearance of “spin wave amplitude”. In terms of these co-ordinates the Hamiltonian of the system splits into two parts, one quadratic in its amplitude and the another one is of higher order. The quadratic part alone would give a theory of noninteracting spin-waves, identical with the linear approximation of Bloch. The nonquadratic part represents the effects of interaction between spin waves. It therefore appeared that a consistent treatment of spin-wave interactions would be possible, taking the quadratic part of perturbation theory. The H-P transformation is thus essentially nonlinear and hence not amenable to exact analysis.
1.8.1 Holstein-Primakoff transformation

The Holstein-Primakoff transformation in quantum mechanics is a mapping of angular momentum operators to the Boson creation and annihilation operators. For any set of right-handed orthogonal axes we can define the components of this vector operator as $S_x$, $S_y$ and $S_z$, which are mutually noncommuting, $[S_x, S_y] = i\hbar S_z$ and cyclic permutations. In order to uniquely specify the states of a spin, we can diagonalize any set of commuting operators. Normally we use the $SU(2)$ Casimir operators $S^2$ and $S_z$, which leads to states with the quantum numbers $|S, m_S\rangle$:

\[ S^2 |S, m_S\rangle = \hbar^2 S(S + 1)|S, m_S\rangle, \]  
\[ S_z |S, m_S\rangle = \hbar m_S |S, m_S\rangle. \]

Now take the state with minimal projection $|S, m_S = -S\rangle$, the extremal weight state as a vacuum for a set of boson operators, and each subsequent state with higher projection quantum number as a boson excitation of the previous one,

\[ |S, S + n\rangle \rightarrow \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle_B. \]

Each added boson then corresponds to a decrease of $\hbar$ in the spin projection. The spin raising and lowering operators $\hat{S}^+ = \hat{S}_x + i\hat{S}_y$ and $\hat{S}^- = \hat{S}_x - i\hat{S}_y$, therefore correspond to the bosonic annihilation and creation operators. The precise relations between the operators must be chosen to ensure the correct commutation relations for the angular momentum operators. The Holstein-Primakoff transformation can be written as:

\[ \hat{S}^+_n = \sqrt{2S} \left[ 1 - \frac{a^\dagger_n a_n}{2S} \right]^{1/2} a_n, \]
\[ \hat{S}^-_n = \sqrt{2S} a^\dagger_n \left[ 1 - \frac{a^\dagger_n a_n}{2S} \right]^{1/2}, \]
\[ \hat{S}^z_n = [S - a^\dagger_n a_n], \]

by which a quantum spin can be represented by a single boson. The transformation is particularly useful in the case where $S$ is large, when the square
roots can be expanded as Taylor series to give an expansion in decreasing powers of $S$. The disadvantage of this representation is the Bose operators act in an infinite-dimensional Hilbert space, while the spin operators act only in a $(2S + 1)$ dimensional space, corresponding to the $(2S + 1)$ different possibilities for the spin orientations [144]. But the factor $\sqrt{2S - a^\dagger a}$ in $\hat{S}^+$ annihilates the state of maximal polarization, $|m = +S\rangle$, and thus for any Hamiltonian which can be written in terms of the spin algebra operators, the infinite-dimensional boson Hilbert space is effectively divided into two parts. The physical states all have $0 \leq n_h \leq 2S$, and there are no matrix elements connecting this subspace to the unphysical one where $n_h > 2S$. Many approaches were proposed in order to remove the effect of the extra unphysical magnon states [21,145-148]. If one relaxes the requirement that $\hat{S}_i^+$ and $\hat{S}_i^-$ be Hermitian conjugate operators, we can obtain Dyson-Maleev transformation.

### 1.8.2 Equation of motion through H-P transformation

By introducing dimensionless spins $\hat{S}_i = \frac{\hat{S}_i}{\hbar}$ and $\hat{S}_i^\pm = \hat{S}_i^\mp \pm i\hat{S}_i^\eta$ in the given Hamiltonian which leads to the dimensionless form, where $\hat{S} = \lim_{\hbar \to 0,S \to \infty}(\hbar S)$. The new dimensionless spin operators $\hat{S}_i^+, \hat{S}_i^-$ and $\hat{S}_i^\eta$ satisfy the commutation relations

$$[\hat{S}_i^\pm, \hat{S}_j^\mp] = \mp \hat{S}_i^\eta \delta_{ij},$$

$$[\hat{S}_i^+, \hat{S}_j^-] = 2\hat{S}_i^\eta \delta_{ij},$$

with $\hat{S}_i^+.\hat{S}_i^- = S(S+1)$. After introducing the H-P transformation for the spin operators we can write the dimensionless spin operators in terms of boson operators which are standard magnon operators. In the low temperatures $a_i^\dagger a_i << 2S$ and hence we can use the following semiclassical expansions in Eqs. (1.13-1.14).

$$\frac{\hat{S}_i^+}{S} = \sqrt{2}\left[1 - \frac{\varepsilon^2}{4}a_i^\dagger a_i - \frac{\varepsilon^4}{32}a_i^\dagger a_i a_i^\dagger a_i + O(\varepsilon^6)\right] \varepsilon a_i,$$

$$\frac{\hat{S}_i^-}{S} = \sqrt{2}\varepsilon a_i^\dagger\left[1 - \frac{\varepsilon^2}{4}a_i^\dagger a_i - \frac{\varepsilon^4}{32}a_i^\dagger a_i a_i^\dagger a_i + O(\varepsilon^6)\right],$$

where $a_i^\dagger a_i$ are the magnon operators and $\varepsilon$ is related to the quasi-momentum of the magnon.
where $\epsilon = \frac{1}{\sqrt{s}}$ is a small dimensionless parameter in this approximation. Then, the dimensionless Hamiltonian can be written as a power series in $\epsilon$. The Heisenberg equation of motion for the Boson operator is then written as

$$i\hbar \frac{da_j}{dt} = [a_j, H].$$  \hspace{1cm} (1.20)

Specific form of the equation of motion can be derived by substituting the dimensionless Hamiltonian for the particular spin model in Eq. (1.20) and evaluating the commutations involved.

### 1.8.3 Glauber's coherent state representation

We then introduce Glauber's coherent state representation for Bose operators [149]

$$|\alpha\rangle = \Pi_j |\alpha_j\rangle,$$  \hspace{1cm} (1.21)

where

$$|\alpha_j\rangle = \exp \left( -\frac{1}{2} |\alpha_j|^2 \right) \sum_{m=0}^{\infty} \left[ \frac{\alpha_j^m}{\sqrt{m!}} \right] |m\rangle,$$  \hspace{1cm} (1.22)

with $<\alpha | \alpha > = 1$. The semiclassical approach allows us to consider the projection of spins which can be continuously distributed along the $z$-axis. The states represented in Eq. (1.21) are the eigen states of the operators $a_i$ with eigen value $\alpha_i$.

$$a_i^\dagger |\alpha\rangle = \alpha_i^* |\alpha\rangle,$$  \hspace{1cm} (1.23)

$$a_i |\alpha\rangle = \alpha_i |\alpha\rangle.$$  \hspace{1cm} (1.24)

For the system in the state $|\alpha\rangle$, one finds the equation for the average $<\alpha | a_j | \alpha >$. Thus working in Glauber's coherent state representation and making semiclassical approximation leads to highly nontrivial nonlinear differential-difference equations in general. However, in the long wavelength and low temperature limits, one can go to the continuum limit [150-155] by making Taylor expansion of the variables which leads to nonlinear evolution equations. However, in
the spin-coherent state representation [156], one can work directly with spin operators, without making approximations to the Hamiltonian [150]. The other coherent state treatments [157-165] use a severely truncated H-P transformation for $S_i^\pm$ and further approximate H by a Hamiltonian which is biquadratic in boson operators.

1.9 Classical equation of motion

The nature of nonlinear spin excitations in a Heisenberg ferromagnet can be understood through the dynamical equation derived from the associated model Hamiltonian. The Heisenberg equation of motion for the quantum spin operator $\vec{S}_i$ can be written as

$$i\hbar \frac{d\vec{S}_i}{dt} = [\vec{S}_i, H], \quad i = 1, 2, ..., N. \quad (1.25)$$

The square bracket in the right hand side of Eq. (1.25) represents the quantum mechanical commutator. In the classical limit $(\vec{S}_i)_{\text{quantum}} \rightarrow n \rightarrow 0, s \rightarrow \infty (\vec{S}_i)_{\text{classical}}$ and the quantum mechanical commutator is replaced by the classical Poisson bracket.

$$[\vec{S}_i, H] \sim i\hbar \{\vec{S}_i, H\}_{PB}. \quad (1.26)$$

The dynamics of spins can also be considered from a classical point of view, by developing a Hamiltonian formalism from the outset. Here the spin is considered as a basic dynamical or canonical variable and suitable canonical equations can be obtained in analogy with a spinless non-relativistic particle. The classical Poisson bracket is thus generalized to include spin [166-171]. Hence, in the classical limit, the Heisenberg equation of motion (2.25) becomes

$$\frac{d\vec{S}_i}{dt} = \{\vec{S}_i, H\}_{PB}. \quad (1.27)$$

The above equation implies that the length of each spin vector does not change with time and that the equation of motion corresponds to the assumption of rigidity of the spins. Hence we assume that all the spins to have unit length and the spin angular momentum is represented by three component unit vector...
\tilde{S}_i = (S_i^x, S_i^y, S_i^z)$, and \(\tilde{S}_i^2 = 1\). The spin Poisson brackets in Eq. (1.27) for any two arbitrary functions \(f(\tilde{S}_i)\) and \(g(\tilde{S}_i)\) of spins can be defined in terms of the spin components as

\[
\{f, g\}_{PB} = \sum_{i=1}^{N} \sum_{\alpha, \beta, \gamma} \varepsilon_{\alpha \beta \gamma} \frac{\partial f}{\partial S_i^\alpha} \frac{\partial g}{\partial S_i^\beta} S_i^\gamma,
\]

(1.28)

where \(\varepsilon_{\alpha \beta \gamma}\) is the complete antisymmetric Levi-Civita tensor which can be defined as the permutation tensor of rank three in the three dimensional space [172]. The generalized Poisson brackets satisfy the same algebraic relations as the usual Poisson brackets [173, 174]. The classical equation of motion describing spin dynamics in a Heisenberg ferromagnet can be derived by substituting the relevant Hamiltonian in Eq. (1.27). As an example, we write below the equation of motion for a simple case, namely the Hamiltonian for the isotropic bilinear ferromagnet given by Eq. (1.2)

\[
\frac{d\tilde{S}_n}{dt} = J\tilde{S}_n \times (\tilde{S}_{n+1} + \tilde{S}_{n-1}), \quad \tilde{S}_n^2 = 1.
\]

(1.29)

where \(J_{ij} = J\). While writing Eq. (1.29), the spin-spin interaction is assumed to act only between nearest neighbours. Thus, Eq. (1.29) describes the dynamics of spins in the lattice model of a classical isotropic Heisenberg ferromagnetic spin system. However, Eq. (1.29) can not be understood analytically due to its nonlinear nature. Hence in the long wavelength and low temperature limit it is reasonable to rewrite the equation under continuum approximation. This is also a valid approximation when we assume that the distance between two lattice points is very small. Hence \(\tilde{S}_n(t)\) is replaced by \(\tilde{S}(r, t)\) where \(r = (r_1, r_2, ..., r_N)\) such that \(r = na\) in a simple N-dimensional lattice and expand \(\tilde{S}_{n \pm 1}\) in a Taylor series as

\[
\tilde{S}_{n \pm 1} = \tilde{S}(r, t) \pm a \nabla \tilde{S} + \frac{a^2}{2!} (\nabla^2 \tilde{S}) \pm \frac{a^3}{3!} (\nabla^3 \tilde{S}) + ...
\]

(1.30)

Using the above expansion the discrete equation of motion when restricted upto \(O(a^2)\) can be written as

\[
\frac{\partial \tilde{S}}{\partial t} = J(\tilde{S} \times \nabla^2 \tilde{S}).
\]

(1.31)
This equation was originally derived by Landau and Lifshitz [175,176] from phenomenological arguments to represent magnetic excitations in ferromagnets which can be written in general as

\[
\frac{\partial \vec{S}}{\partial t} = J(\vec{S} \times \vec{F}_{eff}),
\]

(1.32)

where \( \vec{F}_{eff} \) is the effective field contribution due to magnetic interaction. In real magnets, dissipative effects also play an important role and hence Landau-Lifshitz (LL) equation should be generalized further by including the Gilbert damping which takes into account the interaction between magnetic and conduction electrons. The LL equation with Gilbert damping is written as [177,178],

\[
\frac{\partial \vec{S}}{\partial t} = J(\vec{S} \times \vec{F}_{eff}) + \lambda(\vec{S} \times [\vec{S} \times \vec{F}_{eff}]),
\]

(1.33)

where \( \lambda \) is the Gilbert damping parameter. The solution of the damped isotropic ferromagnetic spin chain represented by Eq. (1.33) was made equivalent to that of the undamped one by a simple complex rescaling of time in terms of the damping parameter [178]. Also, Eq. (1.33) was mapped to a perturbed NLS equation through Lakshmanan’s space curve mapping procedure which shows soliton damping [177]. Further, Eq. (1.33) possess universal attractor [179,180]. The one-dimensional version of LL equation is given by

\[
\vec{S}_i = J(\vec{S} \times \vec{S}_{xx}),
\]

(1.34)

where the subscript denotes partial derivative. Pioneering work by Lakshmanan [35] showed that Eq. (1.34) is gauge equivalent to the nonlinear Schrödinger equation. Eq. (1.34) was solved by Takhtajan [181] using inverse scattering transform method and obtained N-soliton solution. Thus the elementary spin excitations were identified in terms of soliton in addition to linear magnons.

Suppose that the anisotropic energy density \( H_{ani} \) varies as the square of the components of the magnetization vector \( \vec{S} \), the Hamiltonian for an anisotropic Heisenberg ferromagnet,

\[
H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + K_1 \sum_i (S_i^z)^2,
\]

(1.35)
where the z-axis coincides with the anisotropic axis. The one-dimensional version of associated LL equation in the classical continuum limit can be derived through Poisson bracket \( \frac{d\vec{S}}{dt} = (\vec{S}, H) \) which is given by

\[
\frac{d\vec{S}}{dt} = \vec{S} \times \left[ J\vec{S}_{xx} - 2K_1\vec{S}^z\hat{k} \right],
\]

(1.36)

where \( \hat{k} = (0, 0, 1) \). The integrability of the above model was established in Refs. [182,183] and multi-soliton solutions were first obtained in Refs. [184,185] by the Hirota method. Later, in Refs. [186-189] multi-soliton and even more complicated solutions were obtained through various methods.

The nonlinear spin dynamics associated with one-dimensional anisotropy Heisenberg ferromagnet involving nearest neighbor spin-spin exchange interaction and D-M interaction is given by

\[
\vec{S}_i = \vec{S} \times \left[ J\vec{S}_{xx} - \vec{D} \times \vec{S}_x - 2K_1\vec{S}^z\hat{k} \right].
\]

(1.37)

The dynamics of the above spin system described in Eq. (1.37) is studied separately in the isotropic and anisotropic cases by choosing \( \vec{D} = D\vec{m} \) where \( \vec{m} = (1,1,1) \). In the isotropic case, i.e., when \( K_1 = 0 \), Eq. (1.37) is found to be integrable and the elementary spin excitations are governed by solitons when the effective field due to weak interaction is considered within a small angle cone. In an anisotropic case, the system is not integrable however the system is found to be integrable under the long wavelength approximation and for small angle variation of spins since the weak anisotropic axis lies parallel to the easy axis of magnetization and the elementary spin excitations are governed by solitons [190]. The corresponding Hamiltonian is written as

\[
H = -\sum_{<ij>} \left[ J\vec{S}_i \cdot \vec{S}_j + \vec{D} \cdot \vec{S}_i \times \vec{S}_j \right] + K_1 \sum_i (S_i^z)^2,
\]

(1.38)

where \( K_1 \) represents an anisotropic parameter. The classical LL equation associated with Heisenberg Hamiltonian involving bilinear and biquadratic exchange interaction is given by

\[
\frac{d\vec{S}}{dt} = \vec{S} \times [\vec{S}_{xx} + \gamma_1\vec{S}_{xxxx} + \gamma_2(\vec{S} \cdot \vec{S}_{xx})\vec{S}_{xx} + \frac{2}{3}\vec{S} \cdot \vec{S}_{xxx} \vec{S}_x],
\]

(1.39)
where $\gamma_1 = \frac{g^2}{12}$, $\gamma_2 = \frac{K a^2}{(1+2K)}$ and $a$ is the lattice parameter. Solitary waves in one-dimensional ferromagnets have been reported due to magnon-magnon interactions [158, 191] and is connected with the lattice deformation. The existence of solitary waves in an anisotropic Heisenberg spin chain with biquadratic exchange interaction under the Dyson-Maleev representation has been reported in [192]. It is found that such nonlinear excitations appear as a solution of NLS equation with high degree of nonlinearity that vanishes in the isotropic spin chain. In Ref. [193], it was shown that the isotropic ferromagnetic spin chain with biquadratic exchange interaction exhibits nonlinear spin excitations in the form of soliton under the Holstein-Primakoff transformation in the semi-classical limit.

When the one-dimensional isotropic ferromagnet is placed in an external magnetic field, the Hamiltonian takes the form

$$H = - \sum_{<i,j>} J \tilde{S}_i \cdot \tilde{S}_j - g \mu_B \tilde{H} \cdot \sum_i \tilde{S}_i. \quad (1.40)$$

The corresponding LL equation is given by

$$\frac{d\tilde{S}}{dt} = \tilde{S} \times \left[ J \tilde{S}_{xx} + g \mu_B \tilde{H} \right]. \quad (1.41)$$

When the magnetic field is constant or time dependent the term in the LL equation corresponding to the Zeeman energy can be transformed away by simple transformation [194]. The resultant LL equation is reduced to that of isotropy spin chain and admits soliton solution. But when the external magnetic field is the component of EM field, it can not be transformed away by the transformation. Because it depends on both space and time and the variation of external magnetic field is governed by Maxwell equations. Therefore the spin dynamics in this case can be understood by solving the relevant LL equation combined with Maxwell equations. From the Refs. [195-198], it is found that the magnetic field component of EM field excites the magnetization of the medium in the form of solitons and hence the EM wave has been modulated in the form of solitons. The equation of motion for inhomogeneous Heisenberg exchange
interaction Eq. (1.8) is given by

$$\frac{d\vec{S}_i}{dt} = J f_i(\vec{S}_i \times \vec{S}_{i+1}) + J f_{i-1}(\vec{S}_i \times \vec{S}_{i-1}).$$

(1.42)

A continuum description in which $\vec{S}_i \rightarrow \vec{S}(x, t)$, $f_i \rightarrow f(x, t)$ is suitable when $\vec{S}_i$, $f_i$ vary slowly over the lattice separation $a$. Inserting Taylor expansions of $\vec{S}(x+a, t)$ and $f(x-a, t)$ in Eq. (1.42), we get the following inhomogeneous LL equation as

$$\vec{S}_i = f(\vec{S} \times \vec{S}_{xx}) + f_x(\vec{S} \times \vec{S}_x).$$

(1.43)

Eq. (1.43) is proved by Lakshmanan to be equivalent to the conventional non-linear Schrödinger equation [35] through Painlevé singularity structure analysis which is integrable only when the function $f(x)$ is linear function of $x$ and the underlying spin excitations are governed by solitons.

The magneto-elastic coupling which exists in real physical models [199] was taken into account only on very few occasions [200-202]. The nonlinear spin-phonon excitations in a biquadratic Heisenberg spin chain under the influence of inhomogeneity and compressibility coupled with bilinear exchange interaction at the harmonic and anharmonic levels have been studied through gauge transformation and space curve mapping [203]. In the harmonic limit of lattice vibrations the system is made gauge equivalent to a perturbed NLS equation which is non-integrable in general but shows integrability under quasi-static limit and the associated elementary excitations are governed by solitons.

Through singularity structure analysis, it is found that the n-dimensional radially symmetric isotropic ferromagnet is integrable only when $n = 2$ and the elementary excitations are governed by solitons [204]. Also, explicit solutions to n-dimensional radially symmetric LL equation in the presence of external magnetic field were obtained [205]. Among the two explicit solutions obtained, the first type of solutions blows up in time and the second one is smooth globally in time. Moreover, it was proved that the radially symmetric inhomogeneous ferromagnetic system is integrable and is governed by soliton spin excitations only when suitable inhomogeneity is added to the system [206].

The study of higher dimensional spin chains is also of great interest not only from the physical point of view but also of great interest in applied mathemat-
ics [207-212]. A (2+1) dimensional integrable Heisenberg model of ferromagnet as a mathematical generalization was done by Ishimori [212]. Similar to the one-dimensional case, the dynamical equation governing the spins is related to a higher dimensional generalization of the NLS equation, namely the Davey-Stewartson equation [213]. Moreover, in Ref. [214] the localized coherent structures for Ishimori equations have been analyzed for time dependent boundaries and different solutions such as rationally localized soliton, exponentially localized soliton and rationally-exponentially localized soliton have been constructed using inverse scattering transform method.

Recently, there was some numerical study on soliton dynamics in (3+1) dimensional ferromagnets also. Ioannidou and Sutcliffe [215] studied the dynamics of solitons in a LL equation describing the magnetization of a three dimensional ferromagnet with an easy axis anisotropy. They numerically computed the energy dispersion relations and the structure of moving solitons using a minimization algorithm. They also studied the interaction of two solitons where it is found that the interaction depends on the relative phases of the solitons and the collision of solitons leads to the formation of unstable magnon loops.