CHAPTER 3

Three-Dimensional Delay Difference Systems

3.1 Introduction

In the previous chapter, we discussed the existence and asymptotic behavior of nonoscillatory solutions of second order nonlinear difference equations. In this chapter, we are concerned with the delay difference system of the form

\[ \Delta x_n = a_n y_{n-k}^\alpha \]
\[ \Delta y_n = b_n z_{n-\ell}^\beta \]
\[ \Delta z_n = -c_n x_{n-m}^\gamma \]  

(3.1.1)

where \( n \in \mathbb{N}(n_0) \) subject to the following conditions:

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(c₁) \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are nonnegative real sequences such that 
\[
\sum_{n=n_0}^{\infty} a_n = \infty, \quad \sum_{n=n_0}^{\infty} b_n = \infty, \quad \text{and} \quad c_n \neq 0 \quad \text{for infinitely many values of} \ n;
\]

(c₂) \( k, \ell \) and \( m \) are nonnegative integers and \( \alpha, \beta \) and \( \gamma \) are ratios of odd positive integers.

Let \( \theta = \max\{k, \ell, m\} \). By a solution of the system (3.1.1), we mean a real sequence \( \{(x_n, y_n, z_n)\} \) defined for all \( n \geq n_0 - \theta \) and satisfies the system (3.1.1) for all \( n \in \mathbb{N}(n_0) \). A solution \( \{(x_n), \{y_n\}, \{z_n\}\} \) of the system (3.1.1) is nonoscillatory if each of its component is either eventually positive or eventually negative and oscillatory otherwise.

If \( \{a_n\} \) and \( \{b_n\} \) are positive, then the system (3.1.1) can be reduced to a third order difference equation of the form
\[
\Delta \left( \frac{1}{b_{n-k}} \Delta \left( \frac{1}{a_n} \Delta x_n \right)^{1/\alpha} \right)^{1/\beta} + c_{n-k-\ell} x_{n-k-\ell-m} = 0,
\]
whose oscillatory behavior has been studied extensively in the literature, see for example [1,2,23,51] and the references cited therein. However for the system (3.1.1), the oscillatory behavior is studied in [65] without delay arguments and all the results obtained here state that "every solution \( \{(x_n, y_n, z_n)\} \) of the system (3.1.1) is either oscillatory or \( \lim \inf_{n \to \infty} |x_n| = 0 \) and \( \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = 0. \)" In this case, the solution of the system (3.1.1) is said to be almost oscillatory.

Motivated by the above observation, in this chapter, we obtain conditions under which all solutions of the system (3.1.1) are oscillatory. For related results corresponding to two-dimensional system, one can
refer to [24,33,40,41,52] and the references cited therein. In Section 3.2, we state and prove some lemmas which will be used in establishing our main results. In Section 3.3, we obtain sufficient conditions for the oscillation of all solutions of the system (3.1.1). Examples are provided to illustrate the relevance of the results discussed.

3.2 Some Preliminary Lemmas

In this section, we state and prove some lemmas, which will be used in establishing our main results.

**Lemma 3.2.1.** Let $\{(x_n, y_n, z_n)\}$ be a solution of the system (3.1.1) with $\{x_n\}$ be nonoscillatory for $n \in \mathbb{N}(n_0)$. Then $\{(x_n, y_n, z_n)\}$ is nonoscillatory and $\{x_n\}, \{y_n\}, \{z_n\}$ are monotone for $n \in \mathbb{N}(n_0)$.

**Proof.** Let $\{(x_n, y_n, z_n)\}$ be a solution of the system (3.1.1) with $\{x_n\}$ be nonoscillatory for $n \in \mathbb{N}(n_0)$. Then, without loss of generality, assume that $x_n > 0$ for $n \in \mathbb{N}(n_0)$ and hence from the third equation of the system (3.1.1), we have $\Delta z_n < 0$ for $n \geq N$. Thus, $\{z_n\}$ is nonincreasing sequence for $n \geq N$ and therefore eventually of one sign for $n \geq N$. Since $\{a_n\}$ and $\{b_n\}$ have positive subsequences in view of condition $(c_1)$, applying similar arguments to the second and the first equation of the system (3.1.1), we see that $\{y_n\}$ and $\{x_n\}$ are monotone for $n \geq N$. Hence $\{(x_n, y_n, z_n)\}$ is nonoscillatory and the proof is complete. \qed

**Lemma 3.2.2.** Let $\{(x_n, y_n, z_n)\}$ be a nonoscillatory solution of the system (3.1.1). Then there are only the following two cases, for $n \in \mathbb{N}(n_0)$ sufficiently large:
(I) \( \text{sgn} \ x_n = \text{sgn} \ y_n = \text{sgn} \ z_n; \)

(II) \( \text{sgn} \ x_n = \text{sgn} \ z_n \neq \text{sgn} \ y_n; \)

hold.

Proof. The proof is similar to that of Lemma 2.2 of [65] and hence the details are omitted.

Lemma 3.2.3. [29] If \( X \) and \( Y \) are nonnegative, then

\[
X^\lambda + (\lambda - 1)Y^\lambda - \lambda XY^{\lambda - 1} \geq 0, \quad \lambda > 1,
\]

where equality holds if and only if \( X = Y \).

3.3 Oscillation Results

In this section, we establish conditions for the oscillation of all solutions of the system (3.1.1). We begin with the following theorem.

Theorem 3.3.1. With respect to the difference system (3.1.1), assume that

\[
\alpha = \beta = \gamma = 1, \quad (3.3.1)
\]
\[
\sum_{n=n_0}^{\infty} c_n = \infty, \quad (3.3.2)
\]

and

\[
\sum_{t=n}^{n+m} c_t \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] > 1. \quad (3.3.3)
\]

Then every solution \( \{(x_n, y_n, z_n)\} \) of the system (3.1.1) is oscillatory.
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Proof. Let \( \{(x_n, y_n, z_n)\} \) be a nonoscillatory solution of the system (3.1.1). Then choose an integer \( N \in \mathbb{N}(n_0) \) such that for all \( n \geq N \), the solutions \( \{(x_n, y_n, z_n)\} \) of system (3.1.1) satisfy either Case (I) or (II) of Lemma 3.2.2.

Case (I) First assume that the solution \( \{(x_n, y_n, z_n)\} \) satisfies Case (I) of Lemma 3.2.2 for \( n \geq N \). Without loss of generality, we may assume that \( x_{n-m} > 0 \) for \( n \geq N \). Define

\[
w_n = \frac{z_n}{x_{n-\ell}}, \quad n \geq N.
\]

Then, for \( n \geq N \), we have

\[
\Delta w_n = \frac{\Delta z_n}{x_{n-\ell}} - \frac{z_{n+1}\Delta x_{n-\ell}}{x_{n-\ell}x_{n-\ell+1}} \leq -c_n.
\]

Summing the above inequality from \( N \) to \( j \geq N \), we obtain

\[
\sum_{n=N}^{j} c_n \leq w_N,
\]

which contradicts (3.3.2) as \( j \to \infty \).

Case (II) Let \( s \in \mathbb{N}(n_0) \) be fixed. Then, summing the third equation of the system (3.1.1) from \( s \) to \( n-1 \), we obtain

\[
z_n - z_s + \sum_{t=s}^{n-1} c_t x_{t-m} = 0,
\]

or

\[
-b_n z_{n-\ell} + b_n \sum_{t=n-\ell}^{\infty} c_t x_{t-m} \leq 0.
\]

or

\[
-\Delta y_n + b_n \sum_{t=n}^{\infty} c_t x_{t-m} \leq 0.
\]
Summing the last inequality from \( s \) to \( n \) and rearranging, we obtain

\[
y_n + \sum_{t=n}^{\infty} \left( \sum_{s=n}^{t} b_s \right) c_t x_{t-m} \leq 0,
\]

or

\[
y_{n-k} + \sum_{t=n-k}^{\infty} \left( \sum_{s=n}^{t} b_s \right) c_t x_{t-m} \leq 0,
\]

\[
\Delta x_n + a_n \sum_{t=n}^{\infty} \left( \sum_{s=n}^{t} b_s \right) c_t x_{t-m} \leq 0.
\]

A final summation of the last inequality yields

\[
\sum_{t=n}^{\infty} \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] c_t x_{t-m} \leq x_n,
\]

or

\[
\sum_{t=n}^{n+m} \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] c_t x_{t-m} \leq x_n.
\]  \hspace{1cm} (3.3.4)

Since \( \{x_n\} \) is decreasing, (3.3.4) yields,

\[
\sum_{t=n}^{n+m} c_t \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] \leq 1,
\]

which contradicts (3.3.3). The proof is compete. \( \square \)

**Example 3.3.1.** Consider the difference system

\[
\begin{align*}
\Delta x_n &= 4y_{n-k} \\
\Delta y_n &= \frac{1}{2}z_{n-\ell} \\
\Delta z_n &= -4x_{n-m}, \quad n \geq 1,
\end{align*}
\]  \hspace{1cm} (3.3.5)

where \( k, \ell \) and \( m \) are even positive integers. All conditions of Theorem 3.3.1 are satisfied and hence all solutions of the system (3.3.5) are oscillatory. In fact, \( \{(x_n, y_n, z_n)\} = \{(-1)^n, \frac{(-1)^{n+1}}{2}, 2(-1)^n\} \) is one such solution of the system (3.3.5).
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**Theorem 3.3.2.** With respect to the difference system (3.1.1), assume condition (3.3.2),

\[
\alpha = \beta = 1 \quad \text{and} \quad 0 < \gamma < 1, \tag{3.3.6}
\]

and

\[
\lim_{n \to \infty} \sup_{t=n} \sum_{s=n}^{n+m} c_t \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] = \infty \tag{3.3.7}
\]

hold. Then every solution \( \{(x_n, y_n, z_n)\} \) of the system (3.1.1) is almost oscillatory.

**Proof.** Let \( \{(x_n, y_n, z_n)\} \) be a nonoscillatory solution of the system (3.1.1). Then proceeding as in the proof of Theorem 3.3.1, we choose \( N \in \mathbb{N}(n_0) \) so that Lemma 3.2.2 holds for \( n \geq N \).

First we consider Case (I).

**Case (I)** Define

\[
w_n = \frac{z_n}{x_{n-\ell}^{\gamma}}, \quad n \geq N.
\]

Then

\[
\Delta w_n = \frac{\Delta z_n}{x_{n-\ell}^{\gamma}} - \frac{z_{n+1}^{\gamma} \Delta x_{n-\ell}^{\gamma}}{x_{n-\ell}^{\gamma} x_{n-\ell+1}^{\gamma}} \leq -c_n, \quad n \geq N.
\]

Summing the last inequality from \( N \) to \( j \geq N \), we obtain

\[
\sum_{n=N}^{j} c_n \leq w_N,
\]

which contradicts (3.3.2) as \( j \to \infty \).

**Case (II)** Proceeding as in the proof of Theorem 3.3.1, we obtain

\[
\sum_{t=n}^{n+m} c_t \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] x_{t-m}^{\gamma} \leq x_n. \tag{3.3.8}
\]
Since \( \{x_n\} \) is positive decreasing and \( \gamma \) is such that \( 0 < \gamma < 1 \), we have from (3.3.8),

\[
\sum_{t=n}^{n+m} c_t \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] \leq x_n^{1-\gamma}.
\]

Now taking limit supremum in the last inequality we see that

\[
\limsup_{n \to \infty} \sum_{t=n}^{n+m} c_t \left[ \sum_{s=n}^{t} a_s \left( \sum_{j=s}^{t} b_j \right) \right] < \infty,
\]

which contradicts (3.3.7). This completes the proof of the theorem. \( \square \)

**Example 3.3.2.** Consider the difference system

\[
\begin{align*}
\Delta x_n &= 2ny_{n-1} \\
\Delta y_n &= \frac{2n+3}{n+2} z_{n-2} \\
\Delta z_n &= -\frac{2n+7}{(n+3)(n+4)} x_n^{\frac{1}{3}}, \quad n \geq 3.
\end{align*}
\]

(3.3.9)

(3.3.13)

All conditions of Theorem 3.3.2 are satisfied and hence all solutions of the system (3.3.9) are oscillatory. In fact

\[
\{(x_n, y_n, z_n)\} = \left\{ \left( (-1)^n, \frac{(-1)^n}{n+1}, \frac{(-1)^{n+1}}{n+3} \right) \right\}
\]

is one such solution of the system (3.3.9).

**Theorem 3.3.3.** With respect to the difference system (3.1.1), assume that

\[
\sum_{n=n_0}^{\infty} c_n \left[ \sum_{s=n_0}^{n-m-1} a_s \left( \sum_{t=n_0}^{s-k-1} b_t \right) \right]^{\alpha} \gamma = \infty,
\]

(3.3.10)

\[
\limsup_{n \to \infty} \sum_{t=n}^{n+m} a_t \left( \sum_{s=n}^{t} b_s \left( \sum_{j=s}^{t} c_j \right) \right)^{\beta} \alpha = \infty
\]

(3.3.11)
and
\[ \alpha \beta \gamma < 1. \] (3.3.12)

Then every solution \( \{(x_n, y_n, z_n)\} \) of the system (3.1.1) is oscillatory.

**Proof.** Let \( \{(x_n, y_n, z_n)\} \) be a nonoscillatory solution of the system (3.1.1). Then proceeding as in the proof of Theorem 3.3.1, we choose \( N \in \mathbb{N}(n_0) \) so that Lemma 3.2.2 holds for \( n \geq N \).

First we consider Case (I).

**Case (I)** Summing the second equation of the system (3.1.1) from \( N \) to \( n - k - 1 \), we obtain
\[ y_{n-k} - y_N = \sum_{s=N}^{n-k-1} b_s z_{s-\ell}^\beta, \quad n \geq N + k + 1, \]
or
\[ y_{n-k} \geq \sum_{s=N}^{n-k-1} b_s z_{s-\ell}^\beta, \quad n \geq N_1 \geq N + k + 1. \] (3.3.13)

Using the monotonicity of \( \{z_n\} \) in (3.3.13), we have
\[ y_{n-k}^\alpha \geq z_{s-k-\ell}^{\alpha \beta} \left( \sum_{s=N}^{n-k-1} b_s \right)^\alpha, \quad n \geq N_1. \] (3.3.14)

Summing the first equation of the system (3.1.1) from \( N_1 \) to \( n - m - 1 \) and using (3.3.14), we obtain
\[ x_{n-m} \geq \sum_{s=N_1}^{n-m-1} a_s z_{s-k-\ell}^{\alpha \beta} \left( \sum_{t=N}^{s-k-1} b_t \right)^\alpha, \quad n \geq N_1 + m + 1. \] (3.3.15)

From (3.3.15) and the monotonicity of \( \{z_n\} \), we have
\[ x_{n-m} \geq z_{n-(k+\ell+m)}^{\alpha \beta} \sum_{s=N_1}^{n-m-1} a_s \left( \sum_{t=N}^{s-k-1} b_t \right)^\alpha, \quad n \geq N_2 \geq N_1 + m + 1, \]
or

\[ x_{n-m}^\gamma \geq z_n^{\alpha \beta \gamma} \left( \sum_{s=N_1}^{n-m-1} a_s \left( \sum_{t=N}^{s-k-1} b_t \right)^\alpha \right)^\gamma, \quad n \geq N_2. \tag{3.3.16} \]

Multiplying (3.3.16) by \( \frac{c_n}{z_n^{\alpha \beta \gamma}} \) and using the third equation of the system (3.1.1), and then summing from \( N_2 \) to \( n - 1 \), we obtain

\[ \sum_{s=N_2}^{n-1} \frac{-\Delta z_s}{z_s^{\alpha \beta \gamma}} \geq \sum_{s=N_2}^{n-1} c_s \left( \sum_{t=N_1}^{s-m-1} a_t \left( \sum_{j=N}^{t-k-1} b_j \right)^\alpha \right)^\gamma, \quad n \geq N_2. \tag{3.3.17} \]

For \( z_{n+1} < u < z_n \), we have

\[ \int_{z_{n+1}}^{z_n} \frac{du}{u^{\alpha \beta \gamma}} \geq -\frac{\Delta z_n}{z_n^{\alpha \beta \gamma}}, \quad n \geq N_2. \tag{3.3.18} \]

Combining (3.3.17) and (3.3.18), we obtain

\[ \int_{0}^{z_{N_2}} \frac{du}{u^{\alpha \beta \gamma}} \geq \sum_{n=N_2}^{\infty} c_n \left( \sum_{s=N_1}^{n-m-1} a_s \left( \sum_{t=N}^{s-k-1} b_t \right)^\alpha \right)^\gamma, \]

which is a contradiction in view of (3.3.10) and (3.3.11).

**Case (II)** Let \( s - \ell \in \mathbb{N}(n_0) \) be fixed. Summing the third equation of the system (3.1.1) from \( s - \ell \) to \( n - 1 \), we obtain

\[ z_n - z_{s-\ell} + \sum_{j=s-\ell}^{n-1} c_j x_j^\gamma = 0, \]

or

\[ \left( \sum_{j=n}^{\infty} c_j x_j^\gamma \right)^\beta \leq z_n^\beta. \]

Multiplying both sides of the last inequality by \( b_n \) and then using the second equation of the system (3.1.1) and then summing from \( s - k \in \)
\[ N(n_0) \text{ to } n - 1 \text{ and rearranging, we obtain} \]
\[
\left( \sum_{t=n}^{\infty} b_t \left( \sum_{s=n}^{t} c_s \right)^{\beta} x_{t-m}^{\gamma} \right)^{\alpha} \leq -y_{n-k}^{\alpha}. \]

Multiplying the above inequality by \( a_n \) and using the first equation of the system (3.1.1) and then summing, we obtain
\[
\sum_{t=n}^{\infty} a_t \left( \sum_{s=n}^{t} b_s \left( \sum_{j=s}^{t} c_j \right)^{\beta} \right)^{\alpha} x_{t-m}^{\alpha \beta \gamma} \leq x_n, \]
or
\[
\sum_{t=n}^{n+m} a_t \left( \sum_{s=n}^{t} b_s \left( \sum_{j=s}^{t} c_j \right)^{\beta} \right)^{\alpha} x_{t-m}^{\alpha \beta \gamma} \leq x_n. \quad (3.3.19)
\]

Since \( \{x_n\} \) is decreasing, from (3.3.12) and (3.3.19), we have
\[
\lim_{n \to \infty} \sup_{t=n} \sum_{t=n}^{n+m} a_t \left( \sum_{s=n}^{t} b_s \left( \sum_{j=s}^{t} c_j \right)^{\beta} \right)^{\alpha} < \infty,
\]
which contradicts (3.3.11). This completes the proof of the theorem. \( \square \)

**Example 3.3.3.** Consider the difference system
\[
\Delta x_n = 2(n + 1)^{\frac{1}{3}} y_{n-3}^{\frac{1}{3}},
\]
\[
\Delta y_n = \frac{2n + 3}{n + 1} z_{n-2},
\]
\[
\Delta z_n = -\frac{2n + 7}{(n + 3)(n + 4)} x_{n-1}^{\frac{1}{3}}, \quad n \geq 3. \quad (3.3.20)
\]

All conditions of Theorem 3.3.3 are satisfied and hence all solutions of the system (3.3.20) are oscillatory.

**Theorem 3.3.4.** With respect to the difference system (3.1.1), assume that
\[
\alpha \beta \gamma = 1, \quad (3.3.21)
\]
and
\[ \sum_{t=n}^{n+m} a_t \left( \sum_{s=n}^{t} b_s \left( \sum_{j=s}^{t} c_j \right)^{\beta} \right)^{\alpha} > 1. \] (3.3.22)

If there exists a positive decreasing sequence \( \{ \phi_n \} \) such that
\[ \limsup_{n \to \infty} \sum_{n=n_0}^{n} \left( c_n \phi_n - \frac{1}{(\gamma + 1)^{\gamma}} (\Delta \phi_n)^{\gamma + 1} \right) = \infty, \] (3.3.23)
where
\[ \eta_n = a_n \left( \sum_{s=n_0}^{n-1} b_s \right)^{\alpha} > 0, \text{ for all } n \in \mathbb{N}(n_0), \] (3.3.24)
then all solutions \( \{(x_n, y_n, z_n)\} \) of the system (3.1.1) are oscillatory.

**Proof.** Let \( \{(x_n, y_n, z_n)\} \) be a nonoscillatory solution of the system (3.1.1). Then proceeding as in the proof of Theorem 3.3.1, we choose \( N \in \mathbb{N}(n_0) \) so that Lemma 3.2.2 holds for \( n \geq N \).

First we consider Case (I).

**Case (I)** Define
\[ w_n = \frac{\phi_n z_n}{x_{n-m-1}^{\gamma}}, \quad n \geq N_1 \geq N + m + 1. \]

Then, for \( n \geq N_1 \), we have
\[ \Delta w_n = -c_n \phi_n + \frac{\Delta \phi_n}{\phi_{n+1}} w_{n+1} - \frac{\phi_n z_n \Delta x_{n-m-1}^{\gamma}}{x_{n-m-1}^{\gamma} x_{n-m-1}^{\gamma}}. \] (3.3.25)

Using the mean value theorem to the function \( r(t) = t^{\gamma} \), we have
\[ \Delta x_{n-m-1}^{\gamma} \geq \begin{cases} \gamma x_{n-m-1}^{\gamma-1} \Delta x_{n-m-1}, & \text{if } r \geq 1 \\ \gamma x_{n-m-1}^{\gamma-1} \Delta x_{n-m-1}, & \text{if } r < 1. \end{cases} \] (3.3.26)
From (3.3.25), (3.3.26) and in view of the behavior of \( \{x_n\} \) and \( \{z_n\} \), we obtain
\[
\Delta w_n = -c_n \phi_n + \frac{\Delta \phi_n}{\phi_{n+1}} w_{n+1} - \frac{\gamma \phi_n w_{n+1} \Delta x_{n-m-1}}{\phi_{n+1} x_{n-m}}, \quad n \geq N_1. \tag{3.3.27}
\]
Summing the second equation of the system (3.1.1) from \( N_1 \) to \( n - k - 1 \) and then using the nonincreasing behavior of \( \{z_n\} \), we obtain
\[
y_{n-k} \geq z_n^\beta \left( \sum_{s=N_1}^{n-k-1} b_s \right)^{\frac{1}{\gamma}} z_n^{\frac{1}{\gamma}}, \quad n \geq N_2 \geq N_1. \tag{3.3.28}
\]
Now, from the first equation of (3.1.1), (3.3.28) and (3.3.19), we have
\[
\Delta x_n \geq a_n \left( \sum_{s=N_1}^{n-k-1} b_s \right)^{\frac{1}{\alpha}} z_n^{\frac{1}{\gamma}}, \quad n \geq N_1,
\]
or
\[
\Delta x_{n-m-1} \geq \eta_{n-m-1} \frac{1}{2} z_{n-m-1} \geq \eta_{n-m-1} z_{n+1}^{\frac{1}{\gamma}}, \quad n \geq N_1, \tag{3.3.29}
\]
since \( \{z_n\} \) is nonincreasing. Using (3.3.29) in (3.3.27) and simplifying, we obtain
\[
\Delta w_n \leq -c_n \phi_n + \frac{\Delta \phi_n}{\phi_{n+1}} w_{n+1} - \frac{\gamma \phi_n \eta_{n-m-1}^{1+\frac{1}{\gamma}} w_{n+1}}{\phi_{n+1}^{1+\frac{1}{\gamma}}}, \quad n \geq N_2 \geq N_1.
\]
Set
\[
X = \left( \gamma \phi_n \eta_{n-m-1} \right)^{\frac{1}{\gamma+1}} \frac{w_{n+1}}{\phi_{n+1}}, \quad \lambda = \frac{\gamma + 1}{\gamma} > 1
\]
and
\[
Y = \left( \frac{\gamma}{\gamma + 1} \right)^\gamma \left( \frac{\Delta \phi_n}{\phi_{n+1}} \right)^\gamma \left[ \gamma^{-\left(\frac{1}{\gamma+1}\right)} \left( \phi_n \eta_{n-m-1} \right)^{-\frac{1}{\gamma+1}} \phi_{n+1} \right]^{\gamma}
\]
in Lemma 3.2.3 to conclude that
\[
\frac{\Delta \phi_n}{\phi_{n+1}} w_{n+1} - \frac{\gamma \phi_n \eta_{n-m-1}^{1+\frac{1}{\gamma}} w_{n+1}^{1+\frac{1}{\gamma}}}{\phi_{n+1}^{1+\frac{1}{\gamma}}} \leq \frac{1}{\left( \gamma + 1 \right)^\gamma} \left( \frac{\Delta \phi_n}{\phi_{n+1}} \right)^{\gamma+1} \eta_{n-m-1}^{\frac{1}{\gamma}} \phi_n^{\frac{1}{\gamma}}
\]
and therefore
\[ \Delta w_n \leq -c_n \phi_n + \frac{1}{(\gamma + 1)^\gamma} \eta_{n-m-1}^{m+1} \phi_n^\gamma, \quad n \geq N_2. \]

Summing both sides of the last inequality from \( N_2 \) to \( j \geq N_1 \), we obtain
\[ w_{j+1} - w_{N_2} \leq -\sum_{n=N_2}^{j} \left[ c_n \phi_n + \frac{1}{(\gamma + 1)^\gamma} \eta_{n-m-1}^{m+1} \phi_n^\gamma \right] \to -\infty \]
as \( j \to \infty \), which is a contradiction to the fact that \( w_j > 0 \) for \( j \geq N_2 \).

Case (II) Proceeding as in the proof of Case (II) of Theorem 3.3.3, we obtain (3.3.19). Now using the nonincreasing behavior of \( \{x_n\} \) and condition (3.3.21), we obtain a contradiction to (3.3.22). This completes the proof of the theorem.

In the case of \( \alpha \beta \gamma > 1 \), we are unable to find conditions under which all solutions of the system (3.1.1) are oscillatory. However, we establish the following result.

**Theorem 3.3.5.** With respect to the difference system (3.1.1), assume
\[ \alpha \beta \gamma > 1, \quad (3.3.30) \]
\[ \sum_{n=n_0}^{\infty} b_n \left( \sum_{s=n}^{\infty} c_s \right)^\alpha = \infty \quad (3.3.31) \]
and
\[ \sum_{n=n_0}^{\infty} a_n \left( \sum_{s=n_0}^{n-k-1} b_s \right)^\alpha \left( \sum_{s=n+m+1}^{\infty} c_s \right)^\beta = \infty \quad (3.3.32) \]
hold. Then every solution \( \{(x_n, y_n, z_n)\} \) of the system (3.1.1) is either oscillatory or \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = 0 \).
3.3. Oscillation Results

Proof. Let \( \{(x_n, y_n, z_n)\} \) be a nonoscillatory solution of the system (3.1.1). Then proceeding as in the proof of Theorem 3.3.1, we see that \( \{(x_n, y_n, z_n)\} \) satisfies one of the two cases of Lemma 3.2.2 for \( n \geq N \).

First consider Case (I).

Case (I) In this case, from the third equation of the system (3.1.1) and using the nondecreasing behavior of \( \{x_n\} \), we have

\[
z_n \geq x_{n-m}^\gamma \sum_{s=n}^{\infty} c_s, \quad n \geq N. \tag{3.3.33}
\]

Further, summing the second equation of the system (3.1.1) from \( N \) to \( n - 1 \) and then using the nonincreasing behavior of \( \{z_n\} \), we obtain

\[
y_n \geq z_{n-\ell}^\beta \left( \sum_{s=N}^{n-1} b_s \right), \quad n \geq N,
\]
or

\[
y_{n-k} \geq z_{n-k-\ell}^\beta \left( \sum_{s=N}^{n-k-1} b_s \right), \quad n \geq N_1 \geq N + k + 1. \tag{3.3.34}
\]

From (3.3.33), (3.3.34) and the first equation of system (3.1.1), we have

\[
\Delta x_n \geq a_n \left( \sum_{s=N}^{n-k-1} b_s \right)^\alpha \left( \sum_{s=n+m+1}^{\infty} c_s \right)^\beta x_{n+1}^{\alpha \beta \gamma},
\]
or

\[
\sum_{s=N}^{n-1} \frac{\Delta x_s}{x_s^{\alpha \beta \gamma}} \geq \sum_{s=N_1}^{n-1} a_s \left( \sum_{t=N}^{s-k-1} b_t \right)^\alpha \left( \sum_{t=s+m+1}^{\infty} c_t \right)^\beta, \quad n \geq N_1. \tag{3.3.35}
\]

For \( x_n < u < x_{n+1} \), we have

\[
\int_{x_n}^{x_{n+1}} \frac{du}{u^{\alpha \beta \gamma}} \geq \frac{\Delta x_n}{x_{n+1}^{\alpha \beta \gamma}}, \quad n \geq N_1. \tag{3.3.36}
\]
Combining (3.3.35) and (3.3.36), we obtain
\[
\int_{x_{N_1}}^{\infty} \frac{du}{u^{\alpha+\beta}} \geq \sum_{n=N_1}^{\infty} a_n \left( \sum_{s=N}^{n-k-1} b_s \right)^{\alpha} \left( \sum_{s=n+m+1}^{\infty} c_s \right)^{\beta},
\]
which is a contradiction in view of (3.3.30) and (3.3.32).

**Case (II)** From the first equation of the system (3.1.1), we see that \( \{x_n\} \) is nonincreasing for \( n \geq N \) and therefore \( \lim_{n \to \infty} x_n = L_1 \leq \infty \). Hence from Lemma 2.3 of [65], we have
\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = 0.
\]
We shall prove that \( \lim_{n \to \infty} x_n = 0 \). Let \( L_1 > 0 \). Then there is an integer \( N_1 > N + m \) such that \( x_{n-m} > d_1 > 0 \) for \( m \geq N_1 \). Now summing the third equation the system (3.1.1) from \( n \) to \( \infty \) and then using \( x_{n-m} > d_1 \) for \( m \geq N_1 \), we obtain
\[
z_n \geq d_1^{\gamma} \sum_{s=n}^{\infty} c_s, \quad n \geq N_1.
\]
Suppose \( \beta \) is a ratio of odd positive integers and \( \{z_n\} \) is nonincreasing. We have from the last inequality
\[
z_{n-\ell}^{\beta} \geq d_1^{\gamma \beta} \left( \sum_{s=n}^{\infty} c_s \right)^{\beta}, \quad n \geq N_1. \tag{3.3.37}
\]
Summing the second equation the system (3.1.1) from \( N_1 \) to \( n-1 \) and then using (3.3.37), we obtain
\[
y_n \geq y_{N_1} + d_1^{\gamma \beta} \sum_{s=N_1}^{n-1} b_s \left( \sum_{t=s}^{\infty} c_t \right)^{\beta}, \quad n \geq N_1.
\]
In view of (3.3.31), the last inequality implies that \( \lim_{n \to \infty} y_n = \infty \), which is a contradiction. Therefore \( \lim_{n \to \infty} x_n = 0 \). This completes the proof. \( \square \)
We conclude this chapter with the following example.

**Example 3.3.4.** Consider the difference system

\[
\begin{align*}
\Delta x_n &= (1 + (-1)^n)y_{n-2}^3 \\
\Delta y_n &= n z_{n-3}^{\frac{1}{3}} \\
\Delta z_n &= -\frac{1}{n(n+1)}x_{n-1}^3, \quad n \geq 3.
\end{align*}
\]  

(3.3.38)

All conditions of Theorem 3.3.5 are satisfied for the system (3.3.38) and hence every solution \(\{(x_n, y_n, z_n)\}\) of the system (3.3.38) is almost oscillatory.