CHAPTER 5

Fourth Order Quasilinear Difference Equations-II

5.1 Introduction

In this chapter, we continue the study of oscillatory behavior of fourth order quasilinear difference equation of the form

\[ \Delta^3 (p_{n-1} (\Delta y_{n-1})^\alpha) + q_n y_n^\beta = 0, \quad n \in \mathbb{N}, \]  

(5.1.1)

where \( \{p_n\} \) is a nondecreasing sequence of positive reals such that \( \sum_{n=1}^{\infty} \frac{1}{p_n^{\alpha}} = \infty \), \( \{q_n\} \) is a sequence of non-negative reals and \( \alpha \) and \( \beta \) are ratios of odd positive integers.

In Section 5.2, we state some lemmas with proof which are needed in the sequel to prove our main results.

The content of this chapter has been communicated for publication.
In Section 5.3, we derive some new sufficient conditions for the oscillation of all solutions of equation (5.1.1). Examples are provided to illustrate the results.

5.2 Some Preliminary Lemmas

In this section, we state some lemmas which are needed in the sequel to prove our main results.

Lemma 5.2.1. Let \( \{z_n\} \) be a sequence of real numbers and let \( \{z_n\} \) and \( \{\Delta^3 z_n\} \) be of constant sign with \( \Delta^3 z_n \) not being identically zero on any subset \( \{n, n+1, \cdots\} \) of \( \mathbb{N} \). If

\[
z_n \Delta^3 z_n \leq 0,
\]

then there exists an integer \( n_2 \geq n_1 \in \mathbb{N} \) such that either

\[
\text{sgn} z_n = \text{sgn} \Delta z_n = \text{sgn} \Delta^2 z_n, \text{ for } n \geq n_2, \tag{5.2.1}
\]

or

\[
\text{sgn} z_n = \text{sgn} \Delta^2 z_n \neq \text{sgn} \Delta z_n, \text{ for } n \geq n_2. \tag{5.2.2}
\]

Proof. The proof of the lemma is similar to that of Lemma 2.1 in [64] and hence the details are omitted. \( \square \)

Lemma 5.2.2. Let \( \{z_n\} \) be as defined in Lemma 5.2.1 and such that \( z_n > 0 \) and (5.2.1) holds for all \( n \geq n_1 \). Then

\[
z_n \geq \frac{1}{2} \left( \frac{n}{2} \right)^{(2)} \Delta^2 z_n, \text{ for } n \geq n_2. \tag{5.2.3}
\]
Proof. From (5.2.1), we have

\[ z_n > 0, \Delta z_n > 0, \Delta^2 z_n > 0 \text{ for } \Delta^3 z_n \leq 0 \text{ for } n \geq n_1, \]

and hence

\[ \Delta z_n = \Delta z_{n_1} + \sum_{s=n_1}^{n-1} \Delta^2 z_s \]

\[ \geq (n - n_1) \Delta^2 z_{n_1}. \]

Summing the last inequality from \( n_1 \) to \( n - 1 \), we have

\[ z_n \geq z_{n_1} + \Delta^2 z_{n_1} \sum_{s=n_1}^{n-1} (s - n_1) \]

\[ \geq \frac{1}{2} \left( \frac{n}{2} \right)^{(2)} \Delta^2 z_{n_1} \text{ for } n \geq n_2 = 2n_1, \]

and the proof is complete.

Lemma 5.2.3. Let \( \{w_n\} \) be a positive sequence such that \( \Delta w_n > 0, \Delta^2 w_n < 0, \Delta^3 w_n > 0 \) and \( \Delta^4 w_n \leq 0 \) for \( n \geq n_1 \in \mathbb{N} \). Then

\[ w_n \geq \frac{1}{6} \left( \frac{n}{2} \right)^{(3)} \Delta^3 w_{n_1}, \quad n \geq n_2 = 2n_1. \tag{5.2.4} \]

Proof. We know

\[ \sum_{s=n_1}^{n-1} (s - n_1)^{(3)} \Delta^4 w_{s-3} = (n - n_1)^{(3)} \Delta^3 w_{n-3} - 3 (n - n_1)^{(2)} \Delta^2 w_{n-2} \]

\[ + 6 (n - n_1) \Delta w_{n-1} - 6w_n + 6w_{n_1}. \]

Then using the hypothesis, we obtain

\[ w_n \geq \frac{1}{6} \left( n - n_1 \right)^{(3)} \Delta^3 w_{n-3} \]

\[ \geq \frac{1}{6} \left( n - n_1 \right)^{(3)} \Delta^3 w_n \]

\[ \geq \frac{1}{6} \left( \frac{n}{2} \right)^{(3)} \Delta^3 w_n, \text{ for } n \geq n_2 = 2n_1. \]

This completes the proof of Lemma 5.2.3.
Lemma 5.2.4. Let \( \{w_n\} \) be a positive sequence such that
\[
\Delta w_n > 0, \Delta \left( (\Delta w_n)^{\frac{\alpha}{\beta}} \right) < 0, \Delta^2 \left( (\Delta w_n)^{\frac{\alpha}{\beta}} \right) > 0 \text{ and } \Delta^3 \left( (\Delta w_n)^{\frac{\alpha}{\beta}} \right) \leq 0
\]
for \( n \geq n_1 \in \mathbb{N} \). Then
\[
\frac{\alpha}{\beta} \geq \frac{1}{6} \left( \frac{n}{2} \right)^{(3)} \Delta^2 \left( (\Delta w_n)^{\frac{\alpha}{\beta}} \right) \text{ for } n \geq n_2 \geq 2n_1.
\]
Proof. The proof is similar to that of Lemma 5.2.3 and hence the details are omitted.

Lemma 5.2.5. Suppose \( F_n \geq 0 \) and \( Q_n \geq 0 \) for all \( n \geq n_0 \in \mathbb{N} \). If there exists a positive sequence \( \{W_n\} \) such that
\[
W_{n+1} - Q_n W_n + F_n \leq 0, \ n \geq n_0,
\]
then
\[
\sum_{n=n_0}^{\infty} F_n \exp \left( \sum_{t=n_0}^{n} Q_t \right) < \infty.
\]
Proof. The proof of Lemma 5.2.5 can be found in [74] and hence the details are omitted.

Lemma 5.2.6. Assume that \( \{y_n\} \) is an eventually positive solution of equation (5.1.1). Let
\[
z_n = p_{n-1} (\Delta y_{n-1})^\alpha. \tag{5.2.5}
\]
Then \( z_n > 0 \) eventually.

Proof. Let \( y_{n-1} > 0 \) for all \( n \geq n_1 \in \mathbb{N} \). Then by the equation (5.1.1), we have
\[
\Delta^3 z_n = -q_n y_{n}^\beta \leq 0,
\]
for all \( n \geq n_1 \), which implies that \( \{\Delta z_n\} \), \( \{\Delta^2 z_n\} \) are monotonic and either

\[
\Delta^2 z_n < 0,
\]
or

\[
\Delta^2 z_n > 0.
\]

We claim that \( \Delta^2 z_n > 0 \). Suppose \( \Delta^2 z_n < 0 \) for \( n \geq n_1 \). Then there is a constant \( d > 0 \) and an integer \( n_2 \in \mathbb{N} \) such that

\[
z_n \leq -d \quad \text{for} \quad n \geq n_2.
\]

Therefore, we have

\[
\Delta y_{n-1} \leq -\left(\frac{d}{p_{n-1}}\right)^{\frac{1}{\alpha}}
\]

for all \( n \geq n_2 \). Summing the last inequality from \( n_2 \) to \( n \) and then taking \( n \to \infty \), we see that \( y_n \to -\infty \) as \( n \to \infty \). This is a contradiction and hence \( \Delta^2 z_n > 0 \) for \( n \geq n_1 \).

Next we consider the following three possible cases:

**Case 1.** For \( n \geq n_1 \), we have

\[
z_n < 0 \text{ and } \Delta z_n < 0.
\]

**Case 2.** For \( n \geq n_1 \), we have

\[
z_n > 0, \Delta z_n > 0 \text{ and } \Delta^2 z_n > 0.
\]

**Case 3.** For \( n \geq n_1 \), we have

\[
z_n < 0, \Delta z_n < 0 \text{ and } \Delta^2 z_n > 0.
\]
For Cases 1 and 3, using a similar method to the above, we can obtain a contradiction and so Cases 1 and 3 are impossible. For Case 2, we see that \( z_n > 0 \) eventually. This completes the proof. \( \square \)

### 5.3 Oscillation Results

In this section, we derive some new sufficient conditions for the oscillation of all solutions of equation (5.1.1). We begin with the case \( \alpha = \beta \).

**Theorem 5.3.1.** Assume that \( \alpha = \beta \). If

\[
Q_n = \frac{p_n - \delta \left( \frac{n}{8} \right)^3 q_n}{p_n} \geq 0 \quad \text{where } \delta = 1 \text{ if } \alpha \geq 1 \text{ and } \\
= \alpha \text{ if } \alpha < 1 \text{ and } \\
\sum_{n=n_0}^{\infty} q_n \exp \left( \sum_{i=n_0}^{\infty} Q_i \right) = \infty, \tag{5.3.1}
\]

then every solution of equation (5.1.1) oscillates.

**Proof.** Let \( \{y_n\} \) be a nonoscillatory solution of equation (5.1.1). We may assume without loss of generality that \( y_{n-1} > 0 \) for all \( n \geq n_1 \in \mathbb{N} \). Let \( z_n \) be as in Lemma 5.2.6. Then by equation (5.1.1) and (5.2.5), we have

\[
\Delta^3 z_n = -q_n y_n^{\alpha} \leq 0, \quad n \geq n_1 \tag{5.3.2}
\]

and therefore \( \{z_n\}, \{\Delta z_n\} \) and \( \{\Delta^2 z_n\} \) are strictly monotonic. By Lemma 5.2.6, \( z_n > 0 \) eventually. From Lemma 5.2.1, we have either

\[
z_n > 0, \Delta z_n > 0 \text{ and } \Delta^2 z_n > 0, \tag{5.3.3}
\]

or

\[
z_n > 0, \Delta z_n < 0 \text{ and } \Delta^2 z_n > 0 \text{ for } n \geq n_2 \geq n_1. \tag{5.3.4}
\]
From Lemma 5.2.6, we have $\Delta y_{n-1} > 0$. Hence there exists a constant $M > 0$ and a positive integer $n_3 \geq n_2$ such that

$$y_n \geq M \quad \text{for all } n \geq n_3. \quad (5.3.5)$$

In the case of (5.3.3), we have from Lemma 5.2.2

$$z_n \geq \frac{1}{2} \left( \frac{n}{4} \right)^{(2)} \Delta^2 z_n \quad \text{for all } n \geq N_0. \quad (5.3.6)$$

Let $\ell$ be a positive integer such that $N_0 + \ell < n \leq N_0 + \ell + 1$ for $n \geq N_1 = N_0 + 1$. Then we have

$$n > \frac{1}{2} \left( n + \ell \right) \quad \text{for } n \geq N_1. \quad (5.3.12)$$

From (5.3.6), we obtain

$$z_n > \frac{1}{2} \left( \frac{n + \ell}{8} \right)^{(2)} \Delta^2 z_n \quad (5.3.7)$$

for $n \geq N_1$.

From (5.2.5), we have

$$y_n^\alpha = \left( \left( \frac{z_n}{p_{n-1}} \right)^{1/\alpha} + y_{n-1} \right)^\alpha \geq \delta \left( \frac{z_n}{p_{n-1}} + y_{n-1}^\alpha \right), \quad (5.3.8)$$

where $\delta$ is already defined in the hypothesis.

In view of (5.3.7) and (5.3.8), we obtain

$$y_n^\alpha \geq \frac{\delta}{p_n} \sum_{j=0}^{\ell-1} z_{n-j} + \delta^{\ell-1} y_{n-\ell}^\alpha$$

$$\geq \frac{\delta}{p_n} \sum_{j=0}^{\ell-1} \frac{1}{2} \left( \frac{n + \ell - j}{8} \right)^{(2)} \Delta^2 z_{n-j} + \delta^{\ell-1} M^\alpha$$

$$\geq \frac{\delta}{2p_n} \left( \frac{n}{8} \right)^{(2)} \sum_{j=0}^{\ell-1} \Delta^2 z_{n-j} + M_1, \quad \text{for } n \geq N_1, \quad (5.3.9)$$
where \( M_1 = \delta^{\ell-1}M^\alpha \). Since \( \{\Delta^2z_n\} \) is decreasing and \( \Delta^2z_n > 0 \), we have
\[
\sum_{j=0}^{\ell-1} \Delta^2z_{n-j} \geq \sum_{i=N_2}^{n-1} \Delta^2z_i, \tag{5.3.10}
\]
where \( N_2 = N_1 + 1 \).

Substituting (5.3.9) and (5.3.10) in (5.3.2), we obtain
\[
\Delta^3z_n + \frac{\delta}{2} q_n \left( \frac{n}{8} \right)^{(2)} \sum_{i=N_2}^{n-1} \Delta^2z_i + M_1q_n \leq 0. \tag{5.3.11}
\]

Set \( V_n = \sum_{i=N_2}^{n-1} \Delta^2z_i, n \geq N_2 \). Then, by (5.3.11), we get
\[
\Delta^2V_n + \frac{\delta}{2} q_n \left( \frac{n}{8} \right)^{(2)} V_n + M_1q_n \leq 0. \tag{5.3.12}
\]

From (5.3.3) and (5.3.12), we have
\[ V_n > 0, \Delta V_n > 0, \Delta^2V_n \leq 0 \]
and by Lemma 4.1 of Hooker and Patula [31], there exists an integer \( N_3 \geq N_2 \) such that
\[
V_n \geq \left( \frac{n}{2} \right) \Delta V_n, \quad n \geq N_3. \tag{5.3.13}
\]

From (5.3.12) and (5.3.13), we obtain
\[
\Delta^2V_n + \frac{\delta}{2} q_n \left( \frac{n}{8} \right)^{(2)} \left( \frac{n}{2} \right) \Delta V_n + M_1q_n \leq 0
\]
for \( n \geq N_3 \). Since \( \left( \frac{n}{8} \right)^{(2)} \left( \frac{n}{4} \right) \geq \left( \frac{n}{8} \right)^{(3)} \), we have from the last inequality
\[
\Delta^2V_n + \delta \left( \frac{n}{8} \right)^{(3)} \frac{q_n}{p_n} \Delta V_n + M_1q_n \leq 0, \quad n \geq N_3. \tag{5.3.14}
\]
Next assume (5.3.4) holds. Then we have

\[ y_n^\alpha \geq \frac{\delta}{p_n} \sum_{j=0}^{\ell-1} z_{n-j} + M_1. \]  

(5.3.15)

Since \( z_n > 0 \) and \( \{ z_n \} \) is decreasing, we have

\[ \sum_{j=0}^{\ell-1} z_{n-j} \geq \sum_{i=N_2}^{n-1} z_i. \]  

(5.3.16)

From (5.3.2), (5.3.15) and (5.3.16), we obtain

\[ \Delta^3 z_n + \frac{\delta q_n}{p_n} \sum_{i=N_2}^{n-1} z_i + M_1 q_n \leq 0. \]  

(5.3.17)

Set \( u_n = \sum_{i=N_2}^{n-1} z_i, n \geq N_2. \) Then, by (5.3.17), we get

\[ \Delta^4 u_n + \frac{\delta q_n}{p_n} u_n + M_1 q_n \leq 0. \]  

(5.3.18)

From (5.3.4), we know

\[ u_n > 0, \Delta u_n > 0, \Delta^2 u_n < 0, \Delta^3 u_n > 0 \]

and by Lemma 5.2.3, there exists an integer \( N_4 \geq N_2 \) such that

\[ u_n \geq \frac{1}{6} \left( \frac{n}{2} \right)^{(3)} \Delta^3 u_n, \quad n \geq N_4. \]  

(5.3.19)

Using (5.3.19) in (5.3.18), we obtain

\[ \Delta^4 u_n + \frac{\delta}{6} \left( \frac{n}{2} \right)^{(3)} \frac{q_n}{p_n} \Delta^3 u_n + M_1 q_n \leq 0. \]

Since \( \frac{1}{6} \left( \frac{n}{2} \right)^{(3)} \geq \left( \frac{n}{8} \right)^{(3)} \), we have from the last inequality

\[ \Delta^4 u_n + \delta \left( \frac{n}{8} \right)^{(3)} \frac{q_n}{p_n} \Delta^3 u_n + M_1 q_n \leq 0, \quad n \geq N_4. \]  

(5.3.20)
Let $W_n = \Delta V_n$ for (5.3.14) and $W_n = \Delta^3u_n$ for (5.3.20). Then we have $W_n > 0$ for $n \geq N_5 = \max(N_3, N_4)$.

From (5.3.14) and (5.3.20), we obtain

$$\Delta W_n + \delta \left(\frac{n}{8}\right)^{\alpha} \frac{q_n}{p_n} W_n + M_1 q_n \leq 0, \quad n \geq N_5.$$ 

By Lemma 5.2.5, we have

$$M_1 \sum_{n=N_5}^{\infty} q_s \exp \left( \sum_{i=N_5}^{n} Q_i \right) < \infty,$$

which is a contradiction and hence the proof is complete. 

**Theorem 5.3.2.** Assume that all solutions of equation

$$\Delta W_n + B_n W_n^\beta + M q_n \leq 0 \text{ for } n \geq n_0, \quad (5.3.21)$$

where

$$B_n = \begin{cases} \frac{\delta}{2} n q_n \left(\frac{1}{2p_n} \left(\frac{n}{8}\right)^{2}\right)^{\frac{\beta}{\alpha}}; & \text{if } \alpha \neq \beta \\ \delta q_n \left(\frac{1}{6p_n} \left(\frac{n}{2}\right)^{3}\right)^{\frac{\beta}{\alpha}}; & \text{if } \alpha = \beta \end{cases}$$

are oscillatory. Then every solution of equation (5.1.1) is oscillatory.

**Proof.** Proceeding as in the proof of Theorem 5.3.1, we obtain

$$\Delta^3 z_n = -q_n y_n^\beta \leq 0, \quad n \geq n_1. \quad (5.3.22)$$

Again proceeding as in the proof of Theorem 5.3.1, we have from (5.3.22),

$$\Delta^3 z_n + \delta q_n \left(\frac{n}{8}\right)^{2} \frac{\beta}{\alpha} \sum_{i=N_2}^{n-1} (\Delta^2 z_i)^\frac{\beta}{\alpha} + M_1 q_n \leq 0. \quad (5.3.23)$$
Let \( V_n = \sum_{i=N_2}^{n-1} (\Delta^2 z_i)^{\frac{\beta}{\alpha}} \), \( n \geq N_2 \). Then from (5.3.23) and (5.3.2), we have

\[
\Delta \left( (\Delta V_n)^{\frac{\beta}{\alpha}} \right) + \delta \frac{n}{2} q_n \left( \frac{n}{2p_n} \right)^{\frac{\beta}{\alpha}} \Delta V_n + M_1 q_n \leq 0, \tag{5.3.24}
\]

where \( \Delta V_n > 0 \). Once again proceeding as in the proof of Theorem 5.3.1 and using Lemma 5.2.4, we obtain

\[
\Delta^3 \left( (\Delta u_n)^{\frac{\beta}{\alpha}} \right) + \delta \left( \frac{1}{6} \frac{n}{2p_n} \right)^{\frac{\beta}{\alpha}} q_n \Delta^2 \left( (\Delta u_n)^{\frac{\beta}{\alpha}} \right) + M_1 q_n \leq 0. \tag{5.3.25}
\]

Let \( W_n = (\Delta V_n)^{\frac{\beta}{\alpha}} \), if (5.3.24) holds and \( W_n = \Delta^2 \left( (\Delta u_n)^{\frac{\beta}{\alpha}} \right) \), if (5.3.25) holds. Then in both the cases, \( \{W_n\} \) is a positive solution of either

\[
\Delta W_n + B_n W_n^{\frac{\beta}{\alpha}} + M_1 q_n \leq 0,
\]

or

\[
\Delta W_n + B_n W_n + M_1 q_n \leq 0,
\]

which is a contradiction. This completes the proof. \( \square \)

Finally, we give a easily verifiable condition for the oscillation of all solutions of equation (5.1.1).

**Theorem 5.3.3.** Assume that

\[
\sum_{n=n_0}^{\infty} A_n q_n < \infty, \tag{5.3.26}
\]

where

\[
A_n = \max \left\{ n^{(2)}, \left( \frac{n^{(2)}}{p_n} \right)^{\frac{\beta}{\alpha}} \right\}.
\]

Then every solution of equation (5.1.1) oscillates.
Proof. Let \( \{y_n\} \) be a nonoscillatory solution of equation (5.1.1). Without loss of generality, we may assume that \( y_{n-1} > 0 \) for all \( n \geq n_1 \in \mathbb{N} \). Let \( \{z_n\} \) be as in Lemma 5.2.6. Then, by Lemma 5.2.1, we have either (5.3.3) or (5.3.4) holds. If (5.3.4) holds, then there exists a constant \( M > 0 \) and \( n_2 \geq n_1 \) such that \( y_n^2 \geq M \) for \( n \geq n_2 \). From (5.1.1), we have

\[
\Delta^3 z_n \leq -Mq_n, \quad n \geq n_2. \tag{5.3.27}
\]

Multiplying both sides of (5.3.27) by \( n^{(2)} \) and summing, we get

\[
n^{(2)} \Delta^2 z_{n+1} - n_2^{(2)} \Delta^2 z_{n+2} + 2n \Delta z_{n+2} + 2n_1 \Delta z_{n+2} + 2z_{n+2} + 2z_{n+3} - 2z_{n+2} = -\sum_{s=n_2}^{n-1} s^{(2)} q_s. \tag{5.3.28}
\]

It is easy to see that, in view of (5.3.4), we have from (5.3.28),

\[
\sum_{s=n_2}^{n-1} s^{(2)} q_s < n_2^{(2)} \Delta^2 z_{n_2+1} - 2n_1 \Delta z_{n_2+2} + 2z_{n_2+3},
\]

which contradicts (5.3.26) as \( n \to \infty \). If (5.3.3) holds, then by Discrete Taylor’s formula [1], we have

\[
z_n = z_{n_1} + (n - n_1) \Delta z_1 + \frac{1}{2} \sum_{j=n_1}^{n-2} (n - j - 1)^{(2)} \Delta^2 z_j
\]

\[
\geq (n - n_1) \Delta z_1 \geq \left( \frac{n}{2} \right) \Delta z_1, \quad \text{for } n_2 \geq 2n_1.
\]

Hence, \( z_n \geq M_1 n \) and therefore by (5.2.5),

\[
\Delta y_{n-1} \geq M_1^\frac{1}{\alpha} \left( \frac{n}{p_{n-1}} \right)^\frac{1}{\alpha}, \quad n \geq n_2.
\]

Summing the last inequality from \( n_2 \) to \( n \), we obtain

\[
y_n \geq M_1^\frac{1}{\alpha} \sum_{s=n_2}^{n} \frac{s^{\alpha}}{(p_{s-1})^{\frac{1}{\alpha}}},
\]
or
\[
y_n^\alpha \geq \delta M_1^\alpha \sum_{s=n_2}^{n-1} \frac{s}{p_{s-1}}
\geq \delta M_1^\alpha \frac{(n-n_2)^{(2)}}{2p_n}
\geq \frac{\delta}{2} M_1^\alpha \left( \frac{\left( \frac{n}{2} \right)^{(2)}}{p_n} \right), \ n \geq n_3 \geq 2n_2.
\]

Thus for all \( n \geq n_3 \), we have
\[
y_n^\beta \geq \left( \frac{\delta}{2} \right)^{\frac{\beta}{\alpha}} M_1^\beta \left( \frac{\left( \frac{n}{2} \right)^{(2)}}{p_n} \right)^{\frac{\beta}{\alpha}}, \ n \geq n_2.
\] (5.3.29)

Using (5.3.29) in equation (5.1.1) and summing, we obtain
\[
\sum_{s=n_3}^{n-1} \left( \frac{\left( \frac{s}{2} \right)^{(2)}}{p_s} \right)^{\frac{\beta}{\alpha}} q_s < \infty,
\]
which contradicts (5.3.26). This completes the proof.

We conclude this chapter with the following examples.

**Example 5.3.1.** Consider the difference equation
\[
\Delta^3 \left( (\Delta y_n)^5 \right) + \frac{1}{n^{10}} y_{n-1}^5 = 0, \ n \geq 2.
\] (5.3.30)

Then it is easy to see that all the conditions of Theorem 5.3.1 are satisfied and hence all solutions of equation (5.3.30) are oscillatory.

**Example 5.3.2.** Consider the difference equation
\[
\Delta^3 \left( (n-1) (\Delta y_{n-1})^3 \right) + \frac{1}{n^3} y_n^5 = 0, \ n \geq 2.
\] (5.3.31)

Then all the conditions of Theorem 5.3.3 are satisfied and hence all solutions of equation (5.3.31) are oscillatory.