Chapter 3

A unified family of super-Halley method

3.1 Introduction

The aim of this CHAPTER\textsuperscript{2} is to develop a new cubically convergent unified family of super-Halley method based on power means to solve nonlinear equations (1.1). The proposed family contains many well-known methods as its particular cases. Further, we have derived many new classes of higher order (cubic and quartic) multipoint iterative methods free from second-order derivative by semi-discrete modifications of proposed methods. It is shown that super-Halley method is the only method which produces fourth-order multipoint iterative methods. Numerical examples are presented to demonstrate performance of proposed one-point as well as multipoint iterative methods.

A family of Chebyshev-Halley type methods [74] which improves Newton’s method is given by

\[ x_{n+1} = x_n - \left[ 1 + \frac{1}{2} \left\{ \frac{L_f(x_n)}{1 - \lambda L_f(x_n)} \right\} \right] \frac{f(x_n)}{f'(x_n)}, \]

where \( \lambda \in \mathbb{R} \) and \( L_f(x_n) = \frac{f''(x_n)f'(x_n)}{(f'(x_n))^2} \).

This family converges cubically and is a close relative of Newton’s method. This family includes classical Chebyshev’s method (when \( \lambda = 0 \)) [7, 30–32, 59–61], the famous Halley’s method (when \( \lambda = 0.5 \)) [7, 30–32, 60–73] and super-Halley method (when \( \lambda = 1 \)) [30–32, 61, 71–73] as its special cases.

However, Chebyshev-Halley type methods have two disadvantages which sometimes restrict their practical applications. First, these methods require an evaluation of second-\textsuperscript{2}The main contents of this CHAPTER have been accepted for publication (see [207]).
order derivative in computing process. Secondly, like Newton’s method, requirement of 
\( f'(x_n) \neq 0 \) is an essential condition for their convergence.

The purpose of this work is to provide some alternative derivations through power means and to revisit some well-known root finding iterative methods.

### 3.2 Development of the unified super-Halley family

The well-known Newton’s methods for simple roots \([7, 11–16]\) and for multiple roots \([86]\) are given by

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},
\]

(3.2)

and

\[
x_{n+1} = x_n - \frac{f(x_n) f'(x_n)}{\{f'(x_n)\}^2 - f(x_n) f''(x_n)},
\]

(3.3)

respectively.

From equations (3.2) and (3.3), one can obtain

\[
x_{n+1} = x_n - \frac{1}{2} \left[ \left( \frac{f(x_n)}{f'(x_n)} \right) + \left( \frac{f(x_n) f'(x_n)}{\{f'(x_n)\}^2 - f(x_n) f''(x_n)} \right) \right].
\]

(3.4)

This is an alternative form of the well-known super-Halley method \([30–32, 61, 71–73]\). This can be further rewritten as

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) \{\{f'(x_n)\}^2 - f(x_n) f''(x_n)\}} \left[ \frac{\{f'(x_n)\}^2 + \{\{f'(x_n)\}^2 - f(x_n) f''(x_n)\}}{2} \right],
\]

(3.5)

which is no different from formula (3.1) when \( \lambda = 1 \).

Let \( a = \{f'(x_n)\}^2 \) and \( b = \{\{f'(x_n)\}^2 - f(x_n) f''(x_n)\} \). For quantities ‘a’ and ‘b’ to be positive and different from zero, one should have

\[
f(x_n) f''(x_n) < \{f'(x_n)\}^2.
\]

(3.6)

The quantity \( a = \{f'(x_n)\}^2 \) is obviously positive being square of a non-zero real number. In case, if \( x = x_n \) is a very good approximation to the required root, then \( f(x_n) \) will be sufficiently close to zero and consequently the quantity

\[
\left| \frac{f(x_n) f''(x_n)}{\{f'(x_n)\}^2} \right|,
\]

(3.7)
would be sufficiently small.

The researcher wish to generalize formula (3.5) by $\gamma$th – power mean. For this, let us take $a = \{f'(x_n)\}^2$ and $b = \{\{f'(x_n)\}^2 - f(x_n)f''(x_n)\}$. Clearly the quantity ‘$a$’ is positive and the quantity ‘$b$’ is positive in view of (3.7). Now approximating correction factor in (3.5) by $\gamma$th – power mean as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) \{\{f'(x_n)\}^2 - f(x_n)f''(x_n)\}} \left[\frac{a^\gamma + b^\gamma}{2}\right]^{\frac{1}{\gamma}}.$$ (3.8)

This is called the $\gamma$th – power mean iterative family of super-Halley method. The formula (3.8) may be considered as a **unification of several existing** cubically convergent iterative methods. For different values of ‘$\gamma$’, these well-known super-Halley formulae have been recovered in foregoing analysis. It is clear that formula (3.8) requires three evaluations of the function per iteration and has an efficiency index [13] equal to $\sqrt[3]{3} \approx 1.442$. Therefore, family (3.8) of one-point methods does not have optimal order of convergence according to the Kung-Traub conjecture [9].

### 3.2.1 Special cases:

It is interesting to note that for different values of ‘$\gamma$’, various well-known methods can be deduced from formula (3.8) as follows:

(i). For $\gamma = 1$ (*Arithmetic mean*), formula (3.8) corresponds to the well-known cubically convergent super-Halley method [30–32, 61, 71–73] given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) \{f'(x_n)^2 - f(x_n)f''(x_n)\}} \left[\frac{a^1 + b^1}{2}\right]^{\frac{1}{1}}.$$ (3.9)

(ii). For $\gamma = -1$ (*Harmonic mean*), formula (3.8) corresponds to the well-known cubically convergent Halley’s method [7, 30–32, 60–73] given by

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2\{f'(x_n)^2 - f(x_n)f''(x_n)\}}.$$ (3.10)

(iii). For $\gamma \to 0$ (*Geometric mean*), formula (3.8) corresponds to the well-known cubically convergent Ostrowski’s square-root method [14] given by

$$x_{n+1} = x_n - \frac{f(x_n)}{\text{sign}(f'(x_0)) \sqrt{\{f'(x_n)^2 - f(x_n)f''(x_n)\}}}.$$ (3.11)
(iv). For \( \gamma = \frac{3}{2} \), formula (3.8) reduces to

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{1}{\left\{ \frac{\sqrt{\{f'(x_n)\}^2 + \sqrt{\{f'(x_n)\}^2 - f(x_n) f''(x_n)\}}}{2} \right\}^2}.
\]  

(3.12)

Note that this formula asymptotically reduces to the well-known cubically convergent Chebyshev’s method \([7, 30–32, 59–61]\), by applying the binomial theorem.

(v). For \( \gamma = 2 \) (Root – mean square), formula (3.8) reduces to a new cubically convergent method given by

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{1}{\left\{ \frac{\sqrt{\{f'(x_n)\}^4 + \{\{f'(x_n)\}^2 - f(x_n) f''(x_n)\}^2}{2} \right\}^2}.
\]  

(3.13)

Next, Theorem 3.2.1 presents a mathematical proof for an order of convergence of the proposed iterative family (3.8).

**Theorem 3.2.1.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a sufficiently differentiable function defined on an open interval \( I \), enclosing a simple zero of \( f(x) \) (say \( x = r \in I \)). Assume that initial guess \( x = x_0 \) is sufficiently close to ‘\( r \)’, then for \( \gamma \in \mathbb{R} \), an iteration scheme defined by formula (3.8) is cubically convergent and satisfies the following error equation:

\[
e_{n+1} = \frac{1}{2}((\gamma - 1) C_2^2 + 2C_3) e_n^3 + O(e_n^4),
\]  

(3.14)

where \( e_n = x_n - r \) is an error at \( n^{th} \) iteration and \( C_k = \left( \frac{1}{R} \right)^{\frac{k}{k}} f^{(r)}(r) \), \( k = 2, 3, 4, \ldots \)

**Proof.** Since \( f(x) \) is sufficiently differentiable function, therefore, expanding \( f(x_n), f'(x_n) \) and \( f''(x_n) \) about \( x = r \) by means of Taylor’s series expansion and taking into account that \( f(r) = 0 \) and \( f'(r) \neq 0 \), one can have

\[
f(x_n) = f'(r) [e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)],
\]  

(3.15)

\[
f'(x_n) = f'(r) [1 + 2C_2 e_n + 3C_3 e_n^2 + O(e_n^3)],
\]  

(3.16)

and

\[
f''(x_n) = f'(r) [2C_2 + 6C_3 e_n + 12C_4 e_n^2 + O(e_n^3)].
\]  

(3.17)

Upon using equations (3.15)-(3.17), one obtains

\[
\frac{f(x_n)}{f'(x_n)} = e_n - C_2 e_n^2 + 2(C_2 - C_3) e_n^3 + O(e_n^4),
\]  

(3.18)
and
\[
\frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2} = 2C_2e_n + (6C_3 - 6C_2^2)e_n^2 + (16C_3^2 + 12C_4 - 28C_2C_3)e_n^3 + O(e_n^4). \tag{3.19}
\]

Furthermore, using equation (3.19) and expanding by the binomial theorem, one can have
\[
\left(1 - \frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2}\right)^{-1} = 1 + 2C_2e_n + (6C_3 - 2C_2^2)e_n^2 + (12C_4 - 4C_2C_3)e_n^3 + O(e_n^4), \tag{3.20}
\]
and
\[
\left[1 + \left(1 - \frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2}\right)^{-1}\right]^{\frac{1}{\gamma}} = 1 - C_2e_n + \frac{1}{2} \left(-6C_3 + C_2^2(5 + \gamma)\right)e_n^2 + O(e_n^3). \tag{3.21}
\]

**Case (i)**

For \(\gamma \in \mathbb{R} - \{0\}\), formula (3.8) may be rewritten as
\[
x_{n+1} = x_n - \left(\frac{f'(x_n)}{f''(x_n)}\right) \left(1 - \frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2}\right)^{-1} \left[1 + \left(1 - \frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2}\right)^{-1}\right]^{\frac{1}{\gamma}}. \tag{3.22}
\]

Substituting equations (3.18), (3.20) and (3.21) in formula (3.22), one obtains
\[
e_{n+1} = -\frac{1}{2} \left((\gamma - 1)C_2^2 + 2C_3\right)e_n^3 + O(e_n^4). \tag{3.23}
\]
This proves cubic convergence of formula (3.8).

**Case (ii)**

For \(\gamma \to 0\), it can be seen that formula (3.8) reduces to Ostrowski’s square-root method, which converges cubically and proof of which is given in [14]. Therefore, it can be concluded that for all \(\gamma \in \mathbb{R}\), the \(\gamma\text{-th}\) power mean family (3.8) of super-Halley method converges cubically. This completes proof of the theorem.

It is worth to note here that third-order convergence of \(\gamma\text{-th}\)-power mean family can be shown in more general case, not only for integral values of \(\gamma\), but also for all non-zero finite real values of \(\gamma\). Because the non-integral values of \(\gamma\) needlessly increases computational cost of the algorithm, therefore we shall not consider these cases here. Thus for all non-zero \(\gamma \in \mathbb{R}\), \(\gamma\text{-th}\)-power mean family (3.8) of super-Halley method has cubic convergence.
Some other new cubically convergent iterative methods based on Heronian mean \((H_oM)\), contra-harmonic mean \((C_oM)\), centroidal mean \((C_eM)\) and logarithmic mean \((L_0M)\) can also be derived from formula (3.5) respectively as follows:

(i) Iterative method based on Heronian mean \((IMH_oM)\)

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left[ 1 + \frac{f'(x_n)}{\sqrt{f'(x_n)^2 - f(x_n)f''(x_n)}} + \frac{f'(x_n)^2}{f'(x_n)^2 - f(x_n)f''(x_n)} \right].
\]

(3.24)

Putting \(a = \{f'(x_n)\}^2\) and \(b = \{f'(x_n)^2 - f(x_n)f''(x_n)\}\), in the above mentioned equation and after some simplification, one can obtain

\[
x_{n+1} = x_n - \frac{1}{3} \frac{f(x_n)}{f'(x_n)} \left[ 1 + \frac{f'(x_n)}{\sqrt{f'(x_n)^2 - f(x_n)f''(x_n)}} + \frac{f'(x_n)^2}{f'(x_n)^2 - f(x_n)f''(x_n)} \right].
\]

(3.25)

A mathematical proof for an order of convergence of the proposed iterative method (3.25) \((IMH_oM)\) is as under:

**Theorem 3.2.2.** Let \(f : I \subseteq \mathbb{R} \to \mathbb{R}\) be a sufficiently differentiable function defined on an open interval \(I\), enclosing a simple zero of \(f(x)\) (say \(x = r \in I\)). Assume that initial guess \(x = x_0\) is sufficiently close to ‘\(r\)’, then for \(\gamma \in \mathbb{R}\), an iteration scheme defined by formula (3.25) is cubically convergent and satisfies the following error equation:

\[
e_{n+1} = \frac{1}{6} (C_2^2 - 6C_3) e_n^3 + O(e_n^4).
\]

(3.26)

**Proof.** Upon using equations (3.15)-(3.17), one obtains

\[
f'(x_n) = e_n - C_2 e_n^2 + 2(C_2^2 - C_3)e_n^3 + O(e_n^4),
\]

(3.27)

\[
\frac{f'(x_n)}{\sqrt{f'(x_n)^2 - f(x_n)f''(x_n)}} = 1 + C_2 e_n + \frac{3}{2} (-C_2^2 + 2C_3)e_n^2 + O(e_n^3),
\]

(3.28)

and

\[
\frac{f'(x_n)}{\{f'(x_n)^2 - f(x_n)f''(x_n)\}} = 1 + 2C_2 e_n + (-2C_2^2 + 6C_3)e_n^2 + O(e_n^3).
\]

(3.29)

Substituting equations (3.27)-(3.29) in formula (3.25), one can get

\[
e_{n+1} = \frac{1}{6} (C_2^2 - 6C_3) e_n^3 + O(e_n^4).
\]

(3.30)

This completes proof of the theorem.
(ii) **Iterative method based on contra-harmonic mean (IMC<sub>0</sub>M)**

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\left(\{f'(x_n)\}^2 - f(x_n)f''(x_n)\right)} \left[ \frac{a^2 + b^2}{a + b} \right].
\]

(3.31)

Putting \(a = \{f'(x_n)\}^2\) and \(b = \{\{f'(x_n)\}^2 - f(x_n)f''(x_n)\}\), in the above mentioned equation, one gets

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\left(\{f'(x_n)\}^2 - f(x_n)f''(x_n)\right)} \left[ \frac{f(x_n)f''(x_n)}{2\{f'(x_n)\}^2 - f(x_n)f''(x_n)} \right].
\]

(3.32)

(iii) **Iterative method based on centroidal mean (IMC<sub>c</sub>M)**

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\left(\{f'(x_n)\}^2 - f(x_n)f''(x_n)\right)} \left[ \frac{2(a^2 + ab + b^2)}{3(a + b)} \right].
\]

(3.33)

Substituting \(a = \{f'(x_n)\}^2\) and \(b = \{\{f'(x_n)\}^2 - f(x_n)f''(x_n)\}\), in the above mentioned equation, one can immediately get

\[
x_{n+1} = x_n - \frac{2}{3} \frac{f(x_n)}{f'(x_n)} \left[ 1 + \frac{\{f'(x_n)\}^4}{\left(\{f'(x_n)\}^2 - f(x_n)f''(x_n)\right)^2} \right].
\]

(3.34)

(iv) **Iterative method based on logarithmic mean (IML<sub>0</sub>M)**

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\left(\{f'(x_n)\}^2 - f(x_n)f''(x_n)\right)} \left[ \frac{a - b}{\log(a) - \log(b)} \right].
\]

(3.35)

Letting \(a = \{f'(x_n)\}^2\) and \(b = \{\{f'(x_n)\}^2 - f(x_n)f''(x_n)\}\), in the above mentioned equation, one can obtain

\[
x_{n+1} = x_n + \frac{\{f(x_n)\}^2 f''(x_n)}{f'(x_n)\left(\{f'(x_n)\}^2 - f(x_n)f''(x_n)\right)} \left[ \frac{1}{\log(1 - \frac{f(x_n)f''(x_n)}{\{f'(x_n)\}^2})} \right].
\]

(3.36)

The order of convergence of methods namely, method (3.32), method (3.34) and method (3.36) can be proved on the similar lines as in Theorem 3.2.2. They satisfy the following error equations:

\[
e_{n+1} = -(C_2 + 3) e_n^3 + O(e_n^4),
\]

(3.37)

\[
e_{n+1} = -\frac{1}{3} (C_2 + 3C_3) e_n^3 + O(e_n^4),
\]

(3.38)

and

\[
e_{n+1} = \frac{1}{3} (7C_2 + 3C_3) e_n^3 + O(e_n^4),
\]

(3.39)

respectively.
3.2.2 Worked examples

In this section, let us consider some numerical results obtained by employing the existing classical methods namely, Newton’s method (\(NM\)), Chebyshev’s method (\(CM\)) (3.1) for \(\lambda = 0\), Halley’s method (\(HM\)) (3.1) for \(\lambda = 0.5\), super-Halley method (\(SHM\)) (3.1) for \(\lambda = 1\) and newly developed methods namely, method (3.25) (\(IMHoM\)), method (3.32) (\(IMCoM\)), method (3.34) (\(IMCeM\)) and method (3.36) (\(IMLoM\)) to solve nonlinear equations given in Table 3.1. The results are summarized in Table 3.2 (Number of iterations), Table 3.3 (Computational order of convergence) and Table 3.4 (Total number of function evaluations) respectively. Computations have been performed using \MATLAB\® version 7.5 (R2007b) in double precision arithmetic. We use \(\varepsilon = 10^{-15}\) as a tolerance error. The following stopping criteria are used for computer programs:

\[
(i) |x_{n+1} - x_n| < \varepsilon,
(ii) |f(x_{n+1})| < \varepsilon.
\]

<table>
<thead>
<tr>
<th>Example No.</th>
<th>Examples</th>
<th>I</th>
<th>Initial guesses</th>
<th>Root (r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.1</td>
<td>((x - 1)^6 - 1 = 0).</td>
<td>[1,3]</td>
<td>1.7 3.0</td>
<td>2.000000000000000</td>
</tr>
<tr>
<td>3.2.2</td>
<td>(x^3 + 4x^2 - 10 = 0).</td>
<td>[0,2]</td>
<td>0.1 2.0</td>
<td>1.3653013414097</td>
</tr>
<tr>
<td>3.2.3</td>
<td>(\cos(x) - x = 0).</td>
<td>[0,2]</td>
<td>0.0 2.0</td>
<td>0.739085133215167</td>
</tr>
<tr>
<td>3.2.4</td>
<td>(\arctan(x) = 0).</td>
<td>[-1,2]</td>
<td>-1.0 1.0</td>
<td>0.000000000000000</td>
</tr>
<tr>
<td>3.2.5</td>
<td>(x^3 + 4x^2 + \cos(x - 1) - 6.</td>
<td>[0.5,3]</td>
<td>1.8 3.0</td>
<td>1.000000000000000</td>
</tr>
<tr>
<td>3.2.6</td>
<td>(\exp(x) - 4x^2 = 0).</td>
<td>[0.5,2]</td>
<td>0.5 2.0</td>
<td>0.714805912362778</td>
</tr>
<tr>
<td>3.2.7</td>
<td>(x^2 - \exp(x) - 3x + 2 = 0).</td>
<td>[0,1]</td>
<td>0.0 1.0</td>
<td>0.257530285439860</td>
</tr>
<tr>
<td>3.2.8</td>
<td>(\sin^2(x) - x^2 + 1 = 0).</td>
<td>[1,3]</td>
<td>1 3.0</td>
<td>1.404491648215341</td>
</tr>
<tr>
<td>3.2.9</td>
<td>(\exp(x^2 + 7x - 30) - 1 = 0).</td>
<td>[2.9,3.5]</td>
<td>2.9 3.1</td>
<td>3.000000000000000</td>
</tr>
</tbody>
</table>
Table 3.2

Number of iterations (D below stands for divergent)

<table>
<thead>
<tr>
<th>Example No.</th>
<th>Example</th>
<th>NM</th>
<th>CM</th>
<th>HM</th>
<th>SHM</th>
<th>IMH_oM (3.25)</th>
<th>IMC_oM (3.32)</th>
<th>IMC_eM (3.34)</th>
<th>IML_oM (3.36)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.1</td>
<td>8</td>
<td>5*</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>13</td>
<td>6</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2.2</td>
<td>10</td>
<td>46</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3.2.3</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2.4</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3.2.5</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2.6</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2.7</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3.2.8</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3.2.9</td>
<td>7</td>
<td>D</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

* Converges to undesired root.
Table 3.3

Computational order of convergence (COC)

<table>
<thead>
<tr>
<th>Example No.</th>
<th>NM</th>
<th>CM</th>
<th>HM</th>
<th>SHM</th>
<th>IMH₀M</th>
<th>IMC₀M</th>
<th>IMCₑM</th>
<th>IML₀M</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.1</td>
<td>2.00</td>
<td>—</td>
<td>3.03</td>
<td>3.00</td>
<td>2.88</td>
<td>2.97</td>
<td>3.04</td>
<td>2.95</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>2.96</td>
<td>2.95</td>
<td>3.05</td>
<td>2.99</td>
<td>2.99</td>
<td>3.00</td>
<td>2.90</td>
</tr>
<tr>
<td>3.2.2</td>
<td>2.00</td>
<td>3.02</td>
<td>3.01</td>
<td>2.81</td>
<td>2.85</td>
<td>2.99</td>
<td>2.80</td>
<td>3.02</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>2.99</td>
<td>2.83</td>
<td>2.74</td>
<td>2.74</td>
<td>2.96</td>
<td>2.75</td>
<td>3.04</td>
</tr>
<tr>
<td>3.2.3</td>
<td>2.00</td>
<td>3.08</td>
<td>3.00</td>
<td>2.78</td>
<td>2.59</td>
<td>3.09</td>
<td>2.56</td>
<td>2.83</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>2.96</td>
<td>2.99</td>
<td>3.02</td>
<td>3.01</td>
<td>3.10</td>
<td>3.03</td>
<td>3.00</td>
</tr>
<tr>
<td>3.2.4</td>
<td>3.00</td>
<td>3.01</td>
<td>3.00</td>
<td>2.99</td>
<td>3.00</td>
<td>2.96</td>
<td>2.97</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>3.00</td>
<td>3.01</td>
<td>3.00</td>
<td>2.99</td>
<td>3.00</td>
<td>2.96</td>
<td>2.97</td>
<td>3.00</td>
</tr>
<tr>
<td>3.2.5</td>
<td>2.00</td>
<td>2.97</td>
<td>2.99</td>
<td>2.62</td>
<td>2.70</td>
<td>3.02</td>
<td>2.60</td>
<td>3.02</td>
</tr>
<tr>
<td></td>
<td>1.99</td>
<td>3.00</td>
<td>2.88</td>
<td>2.28</td>
<td>2.80</td>
<td>3.22</td>
<td>3.02</td>
<td>3.02</td>
</tr>
<tr>
<td>3.2.6</td>
<td>2.00</td>
<td>3.03</td>
<td>2.68</td>
<td>3.19</td>
<td>2.78</td>
<td>3.52</td>
<td>3.56</td>
<td>3.01</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>2.76</td>
<td>2.84</td>
<td>3.11</td>
<td>3.00</td>
<td>2.67</td>
<td>2.89</td>
<td>2.94</td>
</tr>
<tr>
<td>3.2.7</td>
<td>2.00</td>
<td>2.87</td>
<td>2.93</td>
<td>2.97</td>
<td>2.96</td>
<td>2.99</td>
<td>2.98</td>
<td>2.96</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>3.35</td>
<td>3.31</td>
<td>3.28</td>
<td>3.29</td>
<td>3.26</td>
<td>3.28</td>
<td>3.29</td>
</tr>
<tr>
<td>3.2.8</td>
<td>2.00</td>
<td>3.05</td>
<td>3.01</td>
<td>4.06</td>
<td>3.91</td>
<td>2.89</td>
<td>2.94</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>2.81</td>
<td>2.88</td>
<td>3.03</td>
<td>3.49</td>
<td>2.64</td>
<td>2.54</td>
<td>3.11</td>
</tr>
<tr>
<td>3.2.9</td>
<td>2.00</td>
<td>...</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>3.01</td>
<td>2.90</td>
<td>2.98</td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>2.94</td>
<td>3.00</td>
<td>3.00</td>
<td>3.00</td>
<td>2.99</td>
<td>3.00</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Here, the symbols namely

(i) ‘—’ represents COC for a case of undesired root.

(ii) ‘…’ represents COC for a case of divergence.
### Table 3.4

**Total number of function evaluations (TNOFE)**

<table>
<thead>
<tr>
<th>Example No.</th>
<th>NM</th>
<th>CM</th>
<th>HM</th>
<th>SHM</th>
<th>IMH$_o$M (3.25)</th>
<th>IMC$_o$M (3.32)</th>
<th>IMC$_e$M (3.34)</th>
<th>IML$_o$M (3.36)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.1</td>
<td>16</td>
<td>12</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td>30</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>18</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td>39</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>3.2.2</td>
<td>20</td>
<td>138</td>
<td>18</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>3.2.3</td>
<td>10</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.4</td>
<td>10</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>15</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>3.2.5</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>15</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.6</td>
<td>10</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.7</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>3.2.8</td>
<td>12</td>
<td>15</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>15</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.9</td>
<td>14</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

Here, the symbols namely

(i) ‘~’ represents TNOFE for a case of undesired root.

(ii) ‘~’ represents TNOFE for a case of divergence.
3.3 Generalized families of multipoint iterative methods without memory

The practical difficulty associated with recently proposed third-order methods given by formula (3.8) may be evaluation of second-order derivative. In past and in recent years, many new modifications of Newton’s method free from second-order derivative have been proposed by researchers [7, 27, 34, 35, 82, 9, 47–55, 57, 58, 75–81, 87–109] by discretization of second-order derivative or by considering different quadrature formulae for computation of an integral arising from Newton’s theorem [8, 52–55] or by considering Adomian decomposition method [39–44]. All these modifications are targeted at increasing a local order of convergence with the view of increasing their efficiency index [13]. Some of these multipoint iterative methods are of third-order [7, 52–55, 57, 58, 75–81, 87–95], while others are of order four [7, 34, 35, 82, 9, 47–51, 96–109]. Recently, fourth-order methods proposed by Nedzhibov et al. [101], Basu [102], Maheswari [103], Cordero et al. [104], Petković and Petković (a survey) [105], Ghanbari [106], Soleymani [47, 107], Cordero and Torregrosa [108], Chun et al. [109] etc. have optimal order of convergence [9]. The efficiency index [13] of these optimal multipoint iterative methods is equal to $\sqrt[3]{4} \approx 1.587$. These multipoint methods calculate new approximations to a zero of $f(x)$ by sampling $f(x)$ and possibly its derivatives for number of values of independent variable, per step. Nedzhibov et al. [101] have modified Chebyshev-Halley type methods [74] to derive several cubic and quartic order convergent multipoint iterative methods free from second-order derivative. Recently, Wang and Li [93] further derived a family of new third-order multipoint iterative methods free from first-order derivatives for solving nonlinear equations numerically.

Here, it is intended to develop and unify the general class of multipoint iterative methods free from second-order derivative. The main idea of proposed methods lies in discretization of second-order derivative involved in $\gamma^h$ – power mean family of super-Halley type methods (3.8). This work can be viewed as a generalization over Nedzhibov et al. [101], Wang and Li [93] families of multipoint iterative methods. In this section, three different families of multipoint iterative methods free from second-order derivative have been derived.
3.3.1 First family

Expanding the function \( f(x_n - \theta u) \) about a point \( x = x_n \) with \( f(x_n) \neq 0 \), by Taylor’s series expansion, where \( \theta \in \mathbb{R} \setminus \{0\} \) one can have

\[
f(x_n - \theta u) = f(x_n) - \theta uf'(x_n) + \frac{\theta^2 u^2}{2} f''(x_n) + O(u^3).
\] (3.40)

Let us take \( u = \frac{f(x_n)}{f'(x_n)} \), and inserting this into equation (3.40), one obtains

\[
f(x_n) f''(x_n) \approx 2 \frac{f'(x_n)^2}{\theta^2 f(x_n)} \left\{ f(x_n - \theta u) - (1 - \theta) f(x_n) \right\}.
\] (3.41)

Using this approximate value of \( f(x_n) f''(x_n) \) into formula (3.8), one gets

\[
x_{n+1} = x_n - \frac{\theta^2 \{ f'(x_n) \}^2}{\{ f''(x_n) \}^3 \left[ (\theta^2 - 2\theta + 2) f'(x_n) - 2 f(x_n - \theta u) \right]} \left[ \frac{(a_1^* \gamma + b_1^* \gamma)^{\frac{1}{2}}}{2} \right],
\] (3.42)

where \( a_1^* = \{ f'(x_n) \}^2 \) and \( b_1^* = \left\{ \{ f'(x_n) \}^2 - \frac{2 f'(x_n)}{\theta f(x_n)} \left\{ f(x_n - \theta u) - (1 - \theta) f(x_n) \right\} \right\} \).

This is a modification over Nedzhibov et al. formula (2.1) in [101]. It is seen that this family depends on real parameters ‘\( \gamma \)’ and ‘\( \theta \).

**Special cases:**

For different specific values of ‘\( \gamma \)’ and ‘\( \theta \)’, following various families of multipoint iterative methods can be derived from formula (3.42):

(i) For \( (\gamma, \theta) = (1, 1) \), one gets formula

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left\{ \frac{f(x_n) - f(x_n - u)}{f(x_n) - 2 f(x_n - u)} \right\}.
\] (3.43)

This is a well-known fourth-order Traub-Ostrowski’s method [7, 82, 101].

(ii) For \( (\gamma, \theta) = (-1, 1) \), one can get formula

\[
x_{n+1} = x_n - \frac{\{ f(x_n) \}^2}{f'(x_n) \{ f(x_n) - f(x_n - u) \}}.
\] (3.44)

This is a well-known cubically convergent Newton-Secant method [7, 82, 101].

(iii) For \( \gamma \rightarrow 0 \) and \( \theta = 1 \), one obtains

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \sqrt{\frac{f(x_n)}{f(x_n) - 2 f(x_n - u)}}.
\] (3.45)

This formula is a new cubically convergent multipoint iterative method.

Note that the family (3.42) can produce many more new cubically convergent multipoint
iterative methods by choosing different values of the parameters. The next theorem presents a mathematical proof for an order of convergence of the proposed family (3.42).

**Theorem 3.3.1.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function defined on an open interval $I$, enclosing a simple zero of $f(x)$ (say $x = r \in I$). Assume that initial guess $x = x_0$ is sufficiently close to ‘$r$’, then for $\gamma \in \mathbb{R}$, an iteration family defined by formula (3.42) has cubic order of convergence for

\[
\gamma \neq 1 \quad \text{and} \quad \theta = 1,
\]

\[
\gamma = 1 \quad \text{and} \quad \theta \neq 1,
\]

\[
\gamma \neq 1 \quad \text{and} \quad \theta \neq 1,
\]

and quartic order of convergence for

\[
(\gamma, \theta) = (1, 1).
\]

**Proof.** Since $f(x)$ is sufficiently differentiable function, therefore, expanding $f(x_n)$ and $f'(x_n)$ about $x = r$ by means of Taylor’s series expansion and taking into account that $f(r) = 0$ and $f'(r) \neq 0$, one can have

\[
f(x_n) = f'(r) [e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)],
\]

(3.46)

and

\[
f'(x_n) = f'(r) [1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5)].
\]

(3.47)

Upon using equations (3.46) and (3.47), one gets

\[
u(x_n) = \frac{f(x_n)}{f'(x_n)} = e_n - C_2 e_n^2 - 2(C_3 - C_2^2) e_n^3 - (3C_4 - 7C_2 C_3 + 4C_3^3) e_n^4 + O(e_n^5).
\]

(3.48)

Also using equation (3.48), one can obtain

\[
e_n - \theta u = (1 - \theta)e_n + \theta C_2 e_n^2 + 2\theta(C_3 - C_2^2) e_n^3 + \theta(3C_4 - 7C_2 C_3 + 4C_3^3) e_n^4 + O(e_n^5),
\]

(3.49)
Substituting equation (3.56) in the formula (3.55), one can have

\[ f(e_n - \theta u) = f'(r)[(1 - \theta)e_n + C_2(1 - \theta + \theta^2)e_n^2 - (2C_2^2\theta^3 + C_3(-1 + \theta - 3\theta^2 + \theta^3))e_n^3 + O(e_n^4)]. \]  

Further, upon using equations (3.46)-(3.48) and (3.50), one can get

\[ \frac{\theta^2 [f(x_n)]^2}{\{f'(x_n)\}^3((\theta^2 - 2\theta + 2)f(x_n) - 2f(x_n - \theta u))} = \frac{1}{\{f'(r)\}^2}[e_n - 3C_2e_n^2 + O(e_n^3)], \]  

and

\[ b_1^* = \{f'(r)\}^2[1 + 2C_2e_n + 2(C_2^2 + C_3\theta)e_n^2 + O(e_n^3)]. \]  

**Case (i)** For \( \gamma \in \mathbb{R} \setminus \{0\} \), putting equations (3.51) and (3.52) in formula (3.42) and using the binomial theorem in formula (3.42), one can obtain

\[ e_{n+1} = \left\{ \left( \frac{\gamma - 1}{2} \right) C_2^3 - C_3(\theta - 1) \right\} e_n^3 + \left\{ (2\gamma - 1)C_2^3 + ((\gamma - 2)\theta + 3(1 - \gamma))C_2C_3 - (\theta - 1)(\theta - 3)C_4 \right\} e_n^4 + O(e_n^5). \]  

For different cases namely: \( \gamma \neq 1 \& \theta = 1, \gamma = 1 \& \theta \neq 1 \) and \( \gamma \neq 1 \& \theta \neq 1 \), it is clear that formula (3.42) has cubic convergence.

While for \( (\gamma, \theta) = (1, 1) \), one gets

\[ e_{n+1} = (C_2^3 - C_2C_3)e_n^4 + O(e_n^5), \]  

which proves quartic order of convergence of the family (3.42).

**Case (ii)** For \( \gamma \to 0 \), formula (3.42) can be rewritten as

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \sqrt{\frac{\theta^2 f(x_n)}{(\theta^2 - 2\theta + 2)f(x_n) - 2f(x_n - \theta u)}}. \]  

Upon using equations (3.46)-(3.48) and (3.50) and making use of the binomial theorem in formula (3.55), one can obtain

\[ \frac{f(x_n)}{f'(x_n)} \sqrt{\frac{\theta^2 f(x_n)}{(\theta^2 - 2\theta + 2)f(x_n) - 2f(x_n - \theta u)}} = e_n + \frac{1}{2}(-C_2^2 + 2(1 - \theta)C_3)e_n^3 + O(e_n^4). \]  

Substituting equation (3.56) in the formula (3.55), one can have

\[ e_{n+1} = -\frac{1}{2}(C_2^2 + 2C_3(\theta - 1))e_n^3 + O(e_n^4), \]  

79
which proves cubic order of convergence of the formula (3.42) for \( \gamma \to 0 \) and \( \theta \in \mathbb{R} \). This completes proof of the theorem.

### 3.3.2 Second family

Replacing second-order derivative in (3.8) by following definition:

\[
    f''(x_n) \approx \frac{f''(x_n) - f'(x_n - \theta u)}{\theta u}, \theta \in \mathbb{R} - \{0\},
\]

one gets the following new generalized family as

\[
    x_{n+1} = x_n - \frac{\theta f(x_n)}{\{f'(x_n)\}^2 \left[ (\theta - 1) f'(x_n) + f'(x_n - \theta u) \right]} \left[ \frac{(a_2^*)^\gamma + (b_2^*)^\gamma}{2} \right]^\frac{1}{\gamma},
\]

where \( a_2^* = \{f'(x_n)\}^2 \) and \( b_2^* = \left\{ \frac{(\theta - 1) \{f'(x_n)\}^2 + f'(x_n) f'(x_n - \theta u)}{\theta u} \right\} \), respectively.

#### Special cases:

(I) For \( \gamma = 1 \) in formula (3.59), one obtains families based on arithmetic mean given by

\[
    x_{n+1} = x_n - \frac{f(x_n)}{2 f'(x_n)} \left\{ \frac{2(\theta - 1) f''(x_n) + f''(x_n - \theta u)}{(\theta - 1) f'(x_n) + f'(x_n - \theta u)} \right\}.
\]

Some interesting particular cases of (3.60) are:

(i). For \( \theta = 1 \) in (3.60), one gets formula

\[
    x_{n+1} = x_n - \frac{f(x_n)}{2} \left\{ \frac{1}{f'(x_n)} + \frac{1}{f'(x_n - u)} \right\}.
\]

This method is same as derived by Traub [7, pp. 165] independently.

(ii). For \( \theta = \frac{2}{3} \) in (3.60), one can get formula

\[
    x_{n+1} = x_n - \frac{f(x_n)}{2 f'(x_n)} \left\{ \frac{3 f'(x_n - \frac{2}{3} u) + f'(x_n)}{3 f'(x_n - \frac{2}{3} u) - f'(x_n)} \right\}.
\]

This is a well-known quartically convergent Jarratt’s method [34, 35].

(II) For \( \gamma = -1 \) in (3.59), one obtains the family based on harmonic mean given by

\[
    x_{n+1} = x_n - \frac{2\theta f(x_n)}{(2\theta - 1) f''(x_n) + f'(x_n - \theta u)}.
\]

This is a modification over the method (3.8) of Weerakoon and Fernando [8].

Some interesting particular cases of (3.63) are:
(i). For $\theta = 1$ in (3.63), one gets formula

$$ x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - u)}. \tag{3.64} $$

This formula was independently derived by Traub [7, pp. 164], Weerakoon and Fernando [8]. Some new cubically convergent multipoint iterative methods based on Heronian mean, contra-harmonic mean, centroidal mean etc. can also be obtained from formula (3.64) as follows:

Cubically convergent multipoint method based on Heronian mean is

$$ x_{n+1} = x_n - \frac{3f(x_n)}{\left\{f'(x_n) + \sqrt{f''(x_n)f''(x_n - u) + f'(x_n - u)}\right\}}. \tag{3.65} $$

Cubically convergent multipoint method based on contra-harmonic mean is

$$ x_{n+1} = x_n - \frac{f(x_n)\{f'(x_n) + f'(x_n - u)\}}{\left\{f'(x_n)\right\}^2 + \left\{f'(x_n - u)\right\}^2}. \tag{3.66} $$

Cubically convergent multipoint method based on centroidal mean is

$$ x_{n+1} = x_n - \frac{3f(x_n)\{f'(x_n) + f'(x_n - u)\}}{2\left\{\left\{f'(x_n)\right\}^2 + \left\{f'(x_n - u)\right\}^2 + f'(x_n)f'(x_n - u)\right\}}. \tag{3.67} $$

(ii). For $\theta = \frac{1}{2}$ in (3.63), one gets formula

$$ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - \frac{1}{2}u)}. \tag{3.68} $$

This is a well-known cubically convergent iterative method [7, pp. 164].

(iii). For $\theta = 2$ in (3.63), one obtains another new cubically convergent method given by

$$ x_{n+1} = x_n - \frac{4f(x_n)}{3f'(x_n) + f'(x_n - 2u)}. \tag{3.69} $$

(III) For $\gamma \to 0$ in equation (3.59), one can obtain the family based on geometric mean

$$ x_{n+1} = x_n - \text{sign}(f'(x_0))f(x_n)\sqrt{\theta \over (\theta - 1)\{f'(x_n)\}^2 + f'(x_n)f'(x_n - \theta u)}. \tag{3.70} $$

Some interesting particular cases of family (3.70) are:

(i). For $\theta = 1$ and $\theta = -1$, one gets the formulae

$$ x_{n+1} = x_n - \frac{f(x_n)}{\text{sign}(f'(x_0))\sqrt{f'(x_n)f'(x_n - u)}}, \tag{3.71} $$

81
and

$$x_{n+1} = x_n - \frac{f(x_n)}{\text{sign}(f'(x_0)) \sqrt{2(f'(x_n))^2 - f'(x_n)f'(x_n-u)}},$$  \hspace{1cm} (3.72)$$

where positive sign is taken if $x_n < r$ and negative sign is taken if $x_n > r$.

These are cubically convergent multipoint iterative methods. Method (3.71) is also derived by Lukić and Ralević [92] independently.

Other modification can directly be obtained from formula (3.6). To do this, replacing second-order derivative in (3.5) by a finite difference (3.58), one can obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \left[ \frac{1}{f'(x_n)} + \frac{\theta}{\left(\theta - 1\right)f'(x_n) + f'(x_n - \theta u)} \right].$$ \hspace{1cm} (3.73)

If one can rewrite the formula (3.73) as

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \left\{ \frac{1}{f'(x_n)} + \frac{\theta}{2 \left[ f'(x_n - \theta u) + f'(x_n) \right] + (\theta - 2)f'(x_n)} \right\},$$ \hspace{1cm} (3.74)

and replace an arithmetic mean $[\{f'(x_n - \theta u) + f'(x_n)\} / 2$ with midpoint value $f' \{(x_n - \theta u) + x_n \} / 2$ in (3.74), one can obtain a new family of methods given by

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \left\{ \frac{1}{f'(x_n)} + \frac{\theta}{2 \left[ f'(x_n - 0.5\theta u) + (\theta - 2)f'(x_n) \right]} \right\}.$$ \hspace{1cm} (3.75)

(i) For $\theta = 1$ in (3.75), one gets a new cubically convergent method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \left\{ \frac{1}{f'(x_n)} + \frac{1}{2 \left[ f'(x_n - 0.5\theta u) - f'(x_n) \right]} \right\}. \hspace{1cm} (3.76)$$

This method can also be obtained from the family (3.73), if $\theta = 1/2$.

(ii) For $\theta = 1/4$ and $\theta = -1$ in (3.73), one can get the formulae

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \left\{ \frac{1}{f'(x_n)} + \frac{1}{4 f'(x_n - 0.25\theta u) - 3f'(x_n)} \right\}; \hspace{1cm} (3.77)$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \left\{ \frac{1}{f'(x_n)} + \frac{1}{2 f'(x_n) - f'(x_n + u)} \right\}, \hspace{1cm} (3.78)$$

respectively.

These are other new cubically convergent iterative methods.

A mathematical proof for an order of convergence of the proposed iterative family (3.59) is presented in the following theorem :
**Theorem 3.3.2.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a sufficiently differentiable function defined on an open interval \( I \), enclosing a simple zero of \( f(x) \) (say \( x = r \in I \)). Assume that initial guess \( x = x_0 \) is sufficiently close to ‘\( r \)’, then for \( \gamma \in \mathbb{R} \), an iteration family defined by formula (3.59) has cubic order of convergence for

\[
\gamma \neq 1 \quad \& \quad \theta = 2/3, \\
\gamma = 1 \quad \& \quad \theta \neq 2/3, \\
\gamma \neq 1 \quad \& \quad \theta \neq 2/3, 
\]

and

\[
\gamma \to 0 \quad \& \quad \theta \in \mathbb{R},
\]

and quartic order of convergence for

\[
(\gamma, \theta) = (1, 2/3).
\]

**Proof.** Upon using equations (3.49) and (3.50), one can have

\[
f'(x_n - \theta u) = f'(r) \left[ 1 + 2(1 - \theta)C_2 e_n + (2 \theta C_2^2 + 3(1 - \theta)^2 C_3) e_n^2 \\
+ \left( 10 \theta C_2 C_3 - 6 \theta^2 C_2 C_3 - 4 \theta C_2 + 4 C_4 (1 - \theta)^3 \right) e_n^3 + O(e_n^4) \right].
\] (3.79)

From equations (3.47) and (3.79), one gets

\[
\frac{f'(x_n - \theta u)}{f'(x_n)} = 1 - 2 \theta C_2 e_n + (6 \theta C_2^2 - 3(2 \theta - \theta^2) C_3) e_n^2 \\
+ \left( 4 C_2 C_3 \theta (7 - 3 \theta) - 16 C_2^3 \theta + 4 C_4 ((1 - \theta)^3 - 1) \right) e_n^3 + O(e_n^4).
\] (3.80)

Further, using equations (3.46)-(3.48) and (3.79), one can get

\[
\frac{\theta f(x_n)}{\{f'(x_n)\}^2 ((\theta - 1)f'(x_n) + f'(x_n - \theta u))} = \frac{1}{\{f'(r)\}^2} \left[ e_n - 3 C_2^2 e_n^2 + (6 C_2^2 - (2 + 3 \theta) C_3) e_n^3 + O(e_n^4) \right],
\] (3.81)

and

\[
b_2^* = \{f'(r)\}^2 \left[ 1 + 2 C_2 e_n + (2 C_2^2 + 3 C_3 \theta) e_n^2 + 4 (C_2 C_3 - C_4 (\theta^2 - 3 \theta + 1)) e_n^3 + O(e_n^4) \right].
\] (3.82)
**Case (i)** For $\gamma \in \mathbb{R} - \{0\}$, making use of equations (3.46), (3.79) and (3.80) in the formula (3.59) and using the binomial theorem in formula (3.59), finally one can obtain

$$e_{n+1} = -\frac{1}{2} \left( C_2^2 \gamma - 1 + C_3 (2 - 3\theta) \right) e_n^3 + \frac{1}{2} \left( C_2^3 (-2 + 4\gamma) \right) + 3 C_2 C_3 \left\{ 2 (1 - \gamma) + (\gamma - 2) \theta - 2 C_4 (2\theta^2 - 6\theta + 3) \right\} e_n^4 + O(e_n^5).$$

(3.83)

For different cases namely: $\gamma \neq 1 \& \theta = 2/3$, $\gamma = 1 \& \theta \neq 2/3$, and $\gamma \neq 1 \& \theta \neq 2/3$, it is clear that formula (3.59) has cubic convergence.

While for $(\gamma, \theta) = (1, 2/3)$, one gets

$$e_{n+1} = \frac{1}{9} (9C_2^2 - 9C_2 C_3 + C_4) e_n^4 + O(e_n^5),$$

(3.84)

which proves quartic order of convergence of the family (3.59).

**Case (ii)** For $\gamma \to 0$, formula (3.59) can be rewritten as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \sqrt{\frac{\theta f'(x_n)}{(\theta - 1) f'(x_n) + f'(x_n - \theta u)}}.$$  

(3.85)

Upon using equations (3.48) and (3.80) in (3.85) and making use of the binomial theorem in formula (3.85), one can obtain

$$\frac{f(x_n)}{f'(x_n)} \sqrt{\frac{\theta f'(x_n)}{(\theta - 1) f'(x_n) + f'(x_n - \theta u)}} = e_n + \frac{1}{2} \left( -C_2^2 + (2 - 3\theta)C_3 \right) e_n^3 + O(e_n^4).$$

(3.86)

Substituting equation (3.86) in the formula (3.85), one can have

$$e_{n+1} = \frac{1}{2} \left( C_2^2 + (-2 + 3\theta)C_3 \right) e_n^3 + O(e_n^4),$$

(3.87)

which proves cubic order of convergence of the family (3.59) for $\gamma \to 0$ and $\theta \in \mathbb{R}$.

This completes proof of the theorem.  

\[\square\]
3.3.3 Third family

Replacing second-order derivative in (3.8) by a finite difference as

\[
f''(x_n) \approx \frac{5f'(x_n) - 4f'(x_n - \frac{\theta u}{2}) - f'(x_n - \theta u)}{3\theta u}, \quad \theta \in \mathbb{R} - \{0\},
\]

one gets the following new generalized family

\[
x_{n+1} = x_n - \frac{3\theta f(x_n)}{\{f'(x_n)\}^2 \{3\theta - 5\} f'(x_n) + 4f'(x_n - \frac{\theta u}{2}) + f'(x_n - \theta u)} \left[ \frac{(a_3^*)^\gamma + (b_3^*)^\gamma}{2} \right] \quad (3.89)
\]

where \(a_3^* = \{f'(x_n)\}^2\) and \(b_3^* = \left\{ \frac{(3\theta - 5)f'(x_n)^2 + 4f'(x_n - \frac{\theta u}{2})f'(x_n) + f'(x_n - \theta u)f'(x_n)}{3\theta} \right\} \).

For particular values of ‘\(\gamma\)’ and ‘\(\theta\)’, some interesting particular cases of this family are:

(i). For \((\gamma, \theta) = (1, 1)\) in (3.89), one gets formula

\[
x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)} \left\{ \frac{f'(x_n) + 4f'(x_n - \frac{u}{2}) + f'(x_n - u)}{f'(x_n) - u} \right\}.
\]

This is an order four new multipoint iterative method. It is clear that the method (3.90) requires four evaluations per iteration and has an efficiency index equal to \(\frac{\sqrt[4]{4}}{4} \approx 1.144\), which is same as that of classical Newton’s method.

(ii). For \((\gamma, \theta) = (-1, 1)\) in (3.89), one can get formula

\[
x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'(x_n - \frac{u}{2}) + f'(x_n - u)}.
\]

This is a cubically convergent method explored by Hasanov et al. [230]. It is clear that formula (3.91) requires four evaluations per iteration and has an efficiency index equal to \(\frac{\sqrt[3]{3}}{3} \approx 1.316\). Therefore, efficiency index of method (3.90) is better than Hasanov et al. method (3.91).

A mathematical proof for an order of convergence of the proposed family (3.89) is as under:
Theorem 3.3.3. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a sufficiently differentiable function defined on an open interval \( I \), enclosing a simple zero of \( f(x) \) (say \( x = r \in I \)). Assume that initial guess \( x = x_0 \) is sufficiently close to \( r \), then for \( \gamma \in \mathbb{R} \), an iteration family defined by formula (3.89) has cubic order of convergence for

\[
\gamma \neq 1 \quad \& \quad \theta = 1,
\]
\[
\gamma = 1 \quad \& \quad \theta \neq 1,
\]
\[
\gamma \neq 1 \quad \& \quad \theta \neq 1,
\]
and

\[
\gamma \to 0 \quad \& \quad \theta \in \mathbb{R},
\]
and quartic order of convergence for

\[
(\gamma, \theta) = (1, 1).
\]

Proof. Upon using equations (3.47) and (3.79), one can have

\[
\frac{f'(x_n - \theta u)}{f''(x_n)} = 1 - 2\theta C_2 e_n + (6\theta C_2^2 - 3(2\theta - \theta^2)C_3) e_n^2 \\
+ (4C_2 C_3 (7 - 3\theta) - 16C_2^3 \theta + 4C_4 (3(1 - \theta)^3 - 1)) e_n^3 + O(e_n^4),
\]

(3.92)
and

\[
\frac{f'(x_n - \theta u/2)}{f''(x_n)} = 1 - \theta C_2 e_n + \left\{ 3\theta C_2^2 - 3\left( \theta - \frac{\theta^2}{4} \right) C_3 \right\} e_n^2 + \left\{ 2C_2 C_3 \left( 7 - \frac{3\theta}{2} \right) \\
- 8C_2^3 \theta + 4C_4 \left( \frac{1 - \theta}{2} \right)^3 - 1 \right\} e_n^3 + O(e_n^4).
\]

(3.93)
Using equations (3.47), (3.48), (3.92) and (3.93), one gets

\[
\frac{3\theta f(x_n)}{\{f''(x_n)\}^2 \{ (3\theta - 5)f'(x_n) + 4f'(x_n - \frac{\theta u}{2}) + f'(x_n - \theta u) \}} = \frac{1}{\{f''(r)\}^2} \left[ e_n^2 - 3C_2 e_n^2 \\
+ ( -3C_2^2 + (1 + \theta)C_3 ) e_n^3 + O(e_n^4) \right],
\]

(3.94)
and

\[
b_3^* = \frac{\{f'(r)\}^2 [ 1 + 2C_2 e_n + 2(C_2^2 + C_3 \theta) e_n^2 + 2(2C_2 C_3 - (\theta^2 - 4\theta + 2)C_4) e_n^3 + O(e_n^4) ]}{\{f''(r)\}^2}.
\]

(3.95)
**Case (i)** For $\gamma \in \mathbb{R} - \{0\}$, using equations (3.48), (3.92) and (3.93) in the formula (3.89) and making use of the binomial theorem in formula (3.89), one can obtain

$$e_{n+1} = - \left\{ \left( \frac{\gamma - 1}{2} \right) C_2^2 - C_3(\theta - 1) \right\} e_n^3 + \left\{ (2\gamma - 1)C_2^3 - [(2 - \gamma)\theta + 3(\gamma - 1)]C_2C_3 - (\theta - 1)(\theta - 3)C_4 \right\} e_n^4 + O(e_n^5). \quad (3.96)$$

For different cases namely: $\gamma \neq 1$ & $\theta = 1$, $\gamma = 1$ & $\theta \neq 1$ and $\gamma \neq 1$ & $\theta \neq 1$, it is clear that formula (3.89) has cubic convergence.

While for $(\gamma, \theta) = (1, 1)$, one gets

$$e_{n+1} = (C_2^3 - C_2C_3)e_n^4 + O(e_n^5), \quad (3.97)$$

which proves quartic order of convergence of the family (3.89).

**Case (ii)** For $\gamma \to 0$, formula (3.89) can be rewritten as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \sqrt{\frac{3\theta}{(3\theta - 5) + 4\frac{f'(x_n - \theta u)}{f'(x_n)} + 4\frac{f'(x_n - \theta u)}{f'(x_n)}}}, \quad (3.98)$$

Upon using equations (3.49), (3.92) and (3.93) in (3.98) and making use of the binomial theorem in formula (3.98), one can obtain

$$\frac{f'(x_n)}{f'(x_n)} \sqrt{\frac{3\theta}{(3\theta - 5) + 4\frac{f'(x_n - \theta u)}{f'(x_n)} + 4\frac{f'(x_n - \theta u)}{f'(x_n)}}} = e_n - \frac{1}{2}(C_2^2 + 2(\theta - 1)C_3)e_n^3 + O(e_n^4). \quad (3.99)$$

Substituting equation (3.99) in formula (3.98), one gets

$$e_{n+1} = \frac{1}{2}(C_2^2 + 2(\theta - 1)C_3)e_n^3 + O(e_n^4), \quad (3.100)$$

which again proves cubic order of convergence of the family (3.89) for $\gamma \to 0$ and $\theta \in \mathbb{R}$.

This completes proof of the theorem.
3.3.4 Worked examples

Now, let us present some numerical results obtained by employing classical existing methods namely, Traub’s method \( (TM) \) \((3.61)\), Weerkoon and Fernando method \( (WFM) \) \((3.64)\), Hasanov et al. method \( (HM^*) \) \((3.91)\), Lukić and Ralević method \( (LRM) \) \((3.71)\) and newly developed methods namely, method \( (3.45) \) \( (NM_1) \), method \( (3.65) \) \( (NM_2) \), method \( (3.66) \) \( (NM_3) \), method \( (3.76) \) \( (NM_4) \), method \( (3.77) \) \( (NM_5) \) and method \( (3.90) \) \( (NM_6) \) respectively to solve nonlinear equations given in Table 3.1. The results are summarized in Table 3.5 (Number of iterations), Table 3.6 (Computational order of convergence) and Table 3.7 (Total number of function evaluations) respectively. Computations have been performed using MATLAB® version 7.5(R2007b) in double precision arithmetic. We use \( \epsilon = 10^{-15} \) as a tolerance error. The following stopping criteria are used for computer programs:

\[
(i)|x_{n+1} - x_n| < \epsilon, \\
(ii)|f(x_{n+1})| < \epsilon.
\]
<table>
<thead>
<tr>
<th>Example No.</th>
<th>TM (3.61)</th>
<th>WFM (3.64)</th>
<th>HM' (3.91)</th>
<th>LRM (3.71)</th>
<th>NM&lt;sub&gt;1&lt;/sub&gt; (3.45)</th>
<th>NM&lt;sub&gt;2&lt;/sub&gt; (3.65)</th>
<th>NM&lt;sub&gt;3&lt;/sub&gt; (3.66)</th>
<th>NM&lt;sub&gt;4&lt;/sub&gt; (3.76)</th>
<th>NM&lt;sub&gt;5&lt;/sub&gt; (3.77)</th>
<th>NM&lt;sub&gt;6&lt;/sub&gt; (3.90)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.1</td>
<td>4</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3.2.2</td>
<td>5</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3.2.3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>3.2.4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3.2.5</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3.2.6</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>3.2.7</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3.2.8</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3.2.9</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
Table 3.6
Computational order of convergence (COC) (ND below stands for not defined)

<table>
<thead>
<tr>
<th>Example No.</th>
<th>TM (3.61)</th>
<th>WFM (3.64)</th>
<th>HM’ (3.91)</th>
<th>LRM (3.71)</th>
<th>NM1 (3.45)</th>
<th>NM2 (3.65)</th>
<th>NM3 (3.66)</th>
<th>NM4 (3.76)</th>
<th>NM5 (3.77)</th>
<th>NM6 (3.90)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.1</td>
<td>3.02</td>
<td>3.03</td>
<td>3.03</td>
<td>3.02</td>
<td>3.00</td>
<td>3.09</td>
<td>3.05</td>
<td>2.59</td>
<td>2.77</td>
<td>3.88</td>
</tr>
<tr>
<td></td>
<td>3.03</td>
<td>2.98</td>
<td>2.99</td>
<td>2.85</td>
<td>2.94</td>
<td>2.98</td>
<td>3.00</td>
<td>2.79</td>
<td>2.78</td>
<td>3.54</td>
</tr>
<tr>
<td>3.2.2</td>
<td>3.05</td>
<td>3.08</td>
<td>3.16</td>
<td>3.02</td>
<td>3.01</td>
<td>3.10</td>
<td>3.10</td>
<td>3.01</td>
<td>2.51</td>
<td>3.70</td>
</tr>
<tr>
<td></td>
<td>3.09</td>
<td>2.78</td>
<td>2.80</td>
<td>2.86</td>
<td>2.88</td>
<td>2.80</td>
<td>2.99</td>
<td>3.00</td>
<td>2.48</td>
<td>3.83</td>
</tr>
<tr>
<td>3.2.3</td>
<td>3.28</td>
<td>3.26</td>
<td>2.86</td>
<td>2.91</td>
<td>3.14</td>
<td>3.58</td>
<td>3.03</td>
<td>3.22</td>
<td>2.92</td>
<td>4.30</td>
</tr>
<tr>
<td></td>
<td>3.01</td>
<td>2.97</td>
<td>2.19</td>
<td>3.75</td>
<td>2.42</td>
<td>3.48</td>
<td>3.02</td>
<td>3.02</td>
<td>3.02</td>
<td>ND</td>
</tr>
<tr>
<td>3.2.4</td>
<td>3.01</td>
<td>2.98</td>
<td>4.56</td>
<td>2.99</td>
<td>7.21</td>
<td>2.98</td>
<td>2.98</td>
<td>2.91</td>
<td>3.00</td>
<td>4.72</td>
</tr>
<tr>
<td></td>
<td>3.01</td>
<td>2.98</td>
<td>4.56</td>
<td>2.99</td>
<td>7.21</td>
<td>2.98</td>
<td>2.98</td>
<td>2.91</td>
<td>3.00</td>
<td>4.72</td>
</tr>
<tr>
<td>3.2.5</td>
<td>3.07</td>
<td>2.98</td>
<td>2.98</td>
<td>2.99</td>
<td>2.99</td>
<td>2.99</td>
<td>2.96</td>
<td>3.05</td>
<td>2.58</td>
<td>3.70</td>
</tr>
<tr>
<td></td>
<td>3.07</td>
<td>2.81</td>
<td>2.84</td>
<td>2.91</td>
<td>2.93</td>
<td>2.84</td>
<td>2.98</td>
<td>4.80</td>
<td>2.95</td>
<td>3.07</td>
</tr>
<tr>
<td>3.2.6</td>
<td>3.51</td>
<td>3.21</td>
<td>3.21</td>
<td>3.10</td>
<td>3.10</td>
<td>3.10</td>
<td>3.01</td>
<td>3.56</td>
<td>3.33</td>
<td>4.20</td>
</tr>
<tr>
<td></td>
<td>2.58</td>
<td>2.98</td>
<td>2.94</td>
<td>2.15</td>
<td>2.97</td>
<td>2.99</td>
<td>2.92</td>
<td>3.16</td>
<td>3.12</td>
<td>3.00</td>
</tr>
<tr>
<td>3.2.7</td>
<td>2.97</td>
<td>2.99</td>
<td>3.04</td>
<td>2.98</td>
<td>3.03</td>
<td>2.98</td>
<td>3.00</td>
<td>2.99</td>
<td>2.97</td>
<td>ND</td>
</tr>
<tr>
<td></td>
<td>3.09</td>
<td>3.02</td>
<td>2.69</td>
<td>3.05</td>
<td>2.82</td>
<td>3.03</td>
<td>2.97</td>
<td>3.12</td>
<td>3.91</td>
<td>3.88</td>
</tr>
<tr>
<td>3.2.8</td>
<td>3.67</td>
<td>3.04</td>
<td>3.03</td>
<td>3.02</td>
<td>3.17</td>
<td>3.03</td>
<td>3.12</td>
<td>4.48</td>
<td>4.10</td>
<td>4.34</td>
</tr>
<tr>
<td></td>
<td>3.38</td>
<td>1.91</td>
<td>2.94</td>
<td>3.01</td>
<td>2.97</td>
<td>1.93</td>
<td>2.95</td>
<td>3.11</td>
<td>3.05</td>
<td>2.90</td>
</tr>
<tr>
<td>3.2.9</td>
<td>3.01</td>
<td>3.09</td>
<td>3.02</td>
<td>3.02</td>
<td>3.07</td>
<td>3.05</td>
<td>3.01</td>
<td>2.96</td>
<td>2.98</td>
<td>4.23</td>
</tr>
<tr>
<td></td>
<td>3.01</td>
<td>2.94</td>
<td>2.96</td>
<td>2.98</td>
<td>2.99</td>
<td>2.95</td>
<td>2.86</td>
<td>2.43</td>
<td>3.01</td>
<td>3.84</td>
</tr>
</tbody>
</table>
### Table 3.7

Total Number of function evaluations (TNOFE)

<table>
<thead>
<tr>
<th>Example No.</th>
<th>TM (3.61)</th>
<th>WFM (3.64)</th>
<th>HM (3.91)</th>
<th>LRM (3.71)</th>
<th>NM1 (3.45)</th>
<th>NM2 (3.65)</th>
<th>NM3 (3.66)</th>
<th>NM4 (3.76)</th>
<th>NM5 (3.77)</th>
<th>NM6 (3.90)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.1</td>
<td>12</td>
<td>21</td>
<td>18</td>
<td>15</td>
<td>15</td>
<td>18</td>
<td>27</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>18</td>
<td>24</td>
<td>15</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.2</td>
<td>15</td>
<td>24</td>
<td>28</td>
<td>15</td>
<td>15</td>
<td>21</td>
<td>33</td>
<td>15</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>3.2.3</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>3.2.4</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.5</td>
<td>9</td>
<td>12</td>
<td>16</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>12</td>
<td>16</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>15</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>3.2.6</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>12</td>
<td>16</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.7</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>3.2.8</td>
<td>9</td>
<td>12</td>
<td>16</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>9</td>
<td>16</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>3.2.9</td>
<td>12</td>
<td>15</td>
<td>20</td>
<td>15</td>
<td>12</td>
<td>15</td>
<td>21</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>12</td>
<td>16</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>
3.4 Discussion and conclusions

This work proposes a family of super-Halley type methods based on nonlinear means. Proposed family (3.8) unifies most of the known cubically convergent iterative methods for solving nonlinear equations and they also provide many more unknown processes. Further, we have also presented many new cubically and quartically convergent multipoint iterative methods free from second-order derivative by discretization. Super-Halley method is the only method which produces multipoint iterative methods of order four. Numerical examples presented here prove that proposed one-point as well as multipoint iterative methods can compete with any of the existing methods including classical Newton’s method. A reasonably close initial guess is necessary for multipoint methods to converge. This condition, however, applies to practically all the iterative methods for solving nonlinear equations. Cubically convergent multipoint methods of first and second families have efficiency indices \[ \sqrt[3]{3} \approx 1.442 \], which are better than the one of Newton’s method \[ \sqrt{2} \approx 1.414 \].

In third family, a new quartically convergent iterative method (3.90), has been obtained. Although, an efficiency index of this method is the same as that of Newton’s method but it is better than Hasanov et al. method (3.91). Since all these proposed iterative methods (one-point as well as multipoint) are close variants of Newton’s method therefore, they may fail miserably like Newton’s method if at any stage of computation, the derivative of the function is zero i.e. \( f'(x_n) = 0 \) or very small in the vicinity of required root.

Further, one can also construct new families of multipoint iterative methods free from first and second-order derivatives in computing process by the forward-difference approximations. Furthermore, in next chapter this limitation has been removed by proposing the iterative methods permitting \( f'(x_n) = 0 \) or very small in the vicinity of the required root.