CHAPTER V
MULTIPOINT ACTIVE RC NETWORK SYNTHESIS WITH
A MINIMUM NUMBER OF CAPACITORS

5.1 INTRODUCTION

With the publications of Sandberg\[13\] - [132] in 1961, multiport active RC network synthesis has received much attention during the past decade [15, 52, 52, 48, 53, 54, 101, 72]. These papers, with the exception of Hilberman[57], use more number of capacitors (with possibly some of them floating) than the minimum number which is equal to the degree of the given matrix [65].

In the early years efforts were directed towards reducing the number of active elements; but with the advent of integrated circuit technology, the trend is towards reducing the number of passive elements, particularly capacitors and having their one end common and grounded, even if it results in an increase in the number of active devices [103]. It was shown by Mann and Pike[95] that, with the help of a state-variable approach and the reactance extraction principle, it is possible to realize active RC networks using a minimum number of capacitors. Subsequently Melvin and Bickart[98], exploiting the technique of [96], proposed an interesting synthesis procedure to realize active RC network from a given admittance matrix Y(s) using voltage-controlled voltage sources. Later they extended their results[95] to the synthesis of other
types of multiport network functions [15].

This chapter presents a simple and systematic synthesis procedure for the active RC realization of immittance matrices using a similar approach due to Melvin and Bickart [95]. The structures of the realized circuits in terms of the minimum number of elements and grounded ports make them particularly attractive for integrated circuit fabrication.

First, the proposed synthesis approach is briefly discussed in Section (5.2). Then, synthesis of short-circuit (s.c.) admittance matrix, open-circuit (o.c.) impedance matrix, and transfer-impedance matrix using operational amplifiers is considered. Later utilizing this approach and the results of Chapter III, a new passive reciprocal synthesis procedure for SPR immittance matrices using RCT network is evolved.

5.2 PROPOSED APPROACH TO ACTIVE RC MULTIPO RT NETWORK SYNTHESIS

The general idea in the proposed approach is to realize a given multiport network function with the help of capacitive and resistive sub-networks and, by introducing suitable active elements, to force the short-circuit conductance matrix of the resistive sub-network to be hyperdominant which can be easily realized, while ensuring that the state-model of the realized network corresponds to the state-model obtained from the given network function.

Let \( N \) be a multiport network excited at \( p \) of its ports by voltages and/or currents which are elements of the \( p \)-vector \( u(t) \). Let the responses, the voltages and/or currents of \( q \)
of the ports, be elements of the \( q \)-vector \( y(t) \). If \( N \) is excited at a port from which a response is derived, then at that port, if the excitation is a voltage (current), the response must be a current(voltage). Let \( T(s) \) be a \( q \times p \) matrix of real rational functions of the complex variable \( s \) such that

\[
Y(s) = T(s) U(s)
\]

where \( U(s) = \int u(t) \) and \( Y(s) = \int y(t) \); then \( T(s) \) is said to be a multiport network function.

A synthesis procedure, based on the above idea, is to be developed by which \( T(s) \) may be realized as an immittance matrix of an active RC multiport network with a minimum number of grounded capacitors \( n = \delta \left[ T(s) \right] \) and at the most \((p+2n)\) inverting, grounded voltage amplifiers.

In the synthesis method to be presented, \( T(s) \) is assumed to be regular at \( s = \infty \). If it is not so, it can be made regular at infinity by invoking Möbius transformation

\[
s = \frac{\hat{s} \cdot z}{1 - z}
\]

where \( \hat{s} \) is a point of regularity of \( T(s) \) on the negative real axis. The synthesis procedure to be discussed is applied to the newly formed \( T(z) \) matrix which is of the same degree as \( T(s) \). The realization for the original matrix is then obtained by inverse transformation i.e. the final network is obtained by replacing each capacitor of value \( c \) in the realization of \( T(z) \) by a capacitor and a resistor in series having admittance \( \frac{c \cdot \hat{s}}{s + \hat{s}} \).
A minimal realization set \{A, B, C, D\} associated with \(T(s)\) can be easily obtained by applying Ho-Kalman algorithm (Section 2.3.2) to give the state equations of the form
\[
\begin{align*}
\dot{X} &= AX + BU \\
Y &= CX + DU
\end{align*}
\] ...
\(5.2\)
such that
\[
T(s) = D + C(sI-A)^{-1}B
\]
and \(A\) has the minimum dimension \(n\) equal to the degree \([T(s)]\) and \(D = T(\infty)\).

The network that realizes \(T(s)\) will be the interconnection of a \(n\) port grounded capacitive sub-networks, \(N_C\), and a \((p+n)\) port grounded resistive sub-network, \(N_R\), as shown in Fig.5.1. Let \(e_2\) and \(i_2\) denote respectively the \(n\)-vectors of voltages and currents at the ports common to \(N_R\) and \(N_C\) sub-networks. The relationship imposed by \(N_C\) on \(e_2\) and \(i_2\) is
\[
i_2 = -C_0 \dot{e}_2
\] ...
\(5.3\)
where \(C_0\) is an \(nxn\) nonsingular matrix and can be assumed to be diagonal with positive entries only, resulting in a capacitive sub-network in the form of a star of \(n\)-capacitors as shown in Fig.5.2, thereby ensuring that no more than \(n\) capacitors are needed in the realization. That the realization requires at least \(n\) capacitors follows from the fact that \(C_0\) would be singular if the sub-network \(N_C\) contained fewer than \(n\)-capacitors.

Assuming the structure of the sub-network, \(N_R\), consisting of resistors and active elements as shown in Fig.5.3(a,b),
FIG. 5.1 - NETWORK BLOCK DIAGRAMS.

FIG. 5.2 - CAPACITIVE SUB NETWORK $N_c$. 
FIG. 5.3 - BLOCK DIAGRAMS OF NR.
the short-circuit parameter equations of \( \bar{N}_R \) may be written as
\[
I = \bar{G} E
\]...
(5.4)

where \( \bar{G} \) is the short-circuit conductance matrix, of a common ground resistive network, which will be forced to be hyperdominant \(^*\) by incorporating suitable active elements.

With the help of (5.3), (5.4) and the constraints imposed by the active elements [Fig.5.3(a,b)], the state-model \( \{ A, B, C, D \} \) of the given structure for each case is obtained in terms of the sub-matrices of \( \bar{G} \) and the gains of the active elements. Thus, the problem of realization of any multiport network function \( T(s) \) is reduced to that of specifying the various sub-matrices of \( \bar{G} \) subject to the condition that it is hyperdominant.

First, the synthesis of a short-circuit admittance matrix is discussed.

5.2.1 Short-Circuit Admittance Matrix Synthesis

The result established in this section can be enunciated as the following theorem:

**Theorem 5.1**

Any pxp matrix \( T(s) \), of real rational functions of

\(^*\) A matrix is called 'hyperdominant' if it is dominant and all the off-diagonal entries are non-positive. Further, a real matrix is defined to be 'dominant' if each of its main diagonal entries is not less than the sum of the absolute values of all the other entries in the same row [25], [40].
the complex frequency variable $s$ when $T(\infty)$ is the sum of a strictly hyperdominant matrix plus a non-negative matrix, can be realized as the short-circuit admittance matrix of a $p$-port active RC network using a minimum number of $n$ capacitors with a unity capacitance spread, $n = \delta \lceil T(s) \rceil$ and at the most $(p + 2n)$ inverting, grounded voltage amplifiers. All the capacitors, active elements and ports will have the ground as a common terminal.

The proof of the theorem is a logical consequence of the realization procedure for $T(s)$ given as follows:

As $T(s)$ is assumed to be a $pxp$ short-circuit admittance matrix $Y(s)$, it is implied that $p$-vector $U$ is the vector of network port voltages $\textbf{e}_1$ and that $Y$ is the $p$-vector of corresponding port currents $\textbf{i}_1$ (Fig.5.1a). Thus the state equations (5.2) become,

\begin{align*}
\dot{\textbf{e}}_2 &= A \textbf{e}_2 + B \textbf{e}_1 \\
\textbf{i}_1 &= C \textbf{e}_2 + D \textbf{e}_1 \quad \ldots (5.5)
\end{align*}

where $\textbf{e}_2$, an $n$-vector of state variables, is the port voltages at the ports common to $N_R$ and $N_C$ subnetworks (Fig.5.1a).

Assuming the subnetwork, $N_R$, consisting of resistors and inverting, grounded voltage amplifiers to have a structure shown in Fig.5.3(a), where $\overline{N}_R$ is a $(2p + 3n)$ port common terminal resistive network. The short-circuit parameter equations of $\overline{N}_R$ can be written as
where \( \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \) and \( \mathbf{e}_5 \) are each \( p \)-vectors, \( \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5 \) are each \( n \)-vectors; and, the elements of \( \mathbf{G} \) are the submatrices with \( g_{ij} = g'_{ji} \), where prime denotes matrix transposition. \( \mathbf{G} \) being the short-circuit conductance matrix of a common ground resistive network has to be hyperdominant, a necessary and sufficient condition for the \((2p + 3n)\) port common terminal resistive \( \mathbf{N}_R \) to be realizable without internal nodes[140].

The following constraints are imposed due to active elements (Fig. 5.3a)

\[
\begin{align*}
\mathbf{e}_3 &= \mathbf{Q} \mathbf{e}_1 \\
\mathbf{e}_4 &= \mathbf{K} \mathbf{e}_2 \\
\mathbf{e}_5 &= \mathbf{H} \mathbf{K} \mathbf{e}_2
\end{align*}
\]

where the matrices \( \mathbf{Q}, \mathbf{K} \) and \( \mathbf{H} \) are

\[
\begin{align*}
\mathbf{Q} &= \text{diag. } \{q_1, \ldots, q_p\}, \quad \text{with } q_i < 0 \text{ for all } i, \\
\mathbf{K} &= \text{diag. } \{k_1, \ldots, k_n\}, \quad \text{with } k_j < 0 \text{ for all } j, \text{ and} \\
\mathbf{H} &= \text{diag. } \{h_1, \ldots, h_n\}, \quad \text{with } h_j < 0 \text{ for all } j.
\end{align*}
\]

From (5.3), (5.6) and (5.7), we obtain the state equations in the form (5.5) and \( \{A, B, C, D\} \) can be expressed as...
Thus the problem of realization of $T(s)$ has been reduced to that of specifying the $g$ sub-matrices associated with $N_R$ and given by (5.8), subject to the condition that $g$ is hyperdominant. The existence of such a realization is evident from the following steps in the procedure for specifying the various submatrices of $\mathcal{G}$.

Step I

Since $D$ is the sum of a strictly hyperdominant matrix plus a non-negative matrix, while $g_{11}$ is hyperdominant and $g_{13}Q$ is non-negative, we can select suitable values for $g_{11}$ and $Q$ such that $g_{13}$, as specified in (5.8d), has only non-positive entries.

Step II

As (5.8b) contains both non-negative and non-positive elements, therefore, by a suitable choice of $q$, the submatrices $g_{21}$ and $g_{23}$ are obtained such that these have non-positive entries.

Step III

From (5.8c), we have

$$g_{14}K + g_{15}HK = C - g_{12} = P + M$$  \hspace{1cm} (5.9)
where $P = \begin{bmatrix} p_{ij} \end{bmatrix}_{p \times n}$ contains all the non-negative entries of $C - \varepsilon_{12}$, and

$M = \begin{bmatrix} m_{ij} \end{bmatrix}_{p \times n}$ contains all the non-positive entries of $C - \varepsilon_{12}$.

Thus from (5.9) we have

\[ g_{14} \quad K = P \quad \ldots \quad (5.10a) \]
\[ g_{15} \quad H \quad K = M \quad \ldots \quad (5.10b) \]

Obviously, by taking the $|k_j|$ and the $|h_j|$ sufficiently large, the non-zero elements of $g_{14}$ and $g_{15}$ can be made as small in magnitude as is necessary to make the rows of $\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{14} & g_{15} \end{bmatrix}$ hyperdominant. Later it will be shown that it is always possible to select such amplifier gains. If $H^+\begin{bmatrix} K^+ \end{bmatrix}$ denotes the pseudoinverse $[31]$ of $H\begin{bmatrix} K \end{bmatrix}$, then (5.10) yields

\[ g_{14} = P \quad K^+ \quad \ldots \quad (5.11a) \]

provided the consistency condition $P\begin{bmatrix} I - K^+ K \end{bmatrix} = 0$ is satisfied, and

\[ g_{15} = M \quad K^+ \quad H^+ \quad \ldots \quad (5.11b) \]

provided the consistency conditions,

\[ M \begin{bmatrix} I - K^+ K \end{bmatrix} = 0, \quad \text{and} \]
\[ M \quad K^+\begin{bmatrix} I - H^+ H \end{bmatrix} = 0, \quad \text{are satisfied.} \]

Step IV

Rewriting (5.8a) as

\[ g_{24} \quad K + g_{25} \quad H \quad K = - \varepsilon_{44} - \varepsilon_{22} = P_1 + M_1 \quad \ldots \quad (5.12) \]
where \( P_i = \begin{bmatrix} P_{ij} \end{bmatrix}_{nxn} \) contains all the non-negative entries of \(-G^A - \xi_{22}\) and
\[
M_i = \begin{bmatrix} m_{ij} \end{bmatrix}_{nxm} \text{ contains all the non-positive entries of } -G^A - \xi_{22}.
\]

Thus, from (5.12),
\[
\begin{align*}
\xi_{24} K &= P_i \quad \ldots (5.13a) \\
\xi_{25} H K &= M_i \quad \ldots (5.13b)
\end{align*}
\]

Since \( K \) and \( H \) are already known, the submatrices \( \xi_{24} \) and \( \xi_{25} \) can be obtained from (5.12) and (5.13) with a suitable choice of \( \xi_{22} \). It may be noted that \( \xi_{22} \) may be selected such that the modulus of the sum of the rows corresponding to it is just equal to zero, thereby reducing the number of resistors required in the realization.

Thus, having determined the \( \xi \)'s that appear in (5.8), the remaining entries of the \( \Xi \) matrix can be filled in arbitrarily, however, a maximum number of zero entries, such that the hyperdominant nature of \( \Xi \) is retained, is advantageous.

**Amplifier Gain Selection**

The above synthesis procedure was developed on the observation that amplifier gains (\( K \) and \( H \)) exist such that the equations in (5.10) have solutions corresponding to which the rows of \( [\xi_{11}, \ldots, \xi_{15}] \) are hyperdominant. Now some criteria for choosing the amplifier gains will be described.

From (5.8d), \( \xi_{11}, \xi_{13} \) and \( Q \) are found. \( \xi_{21} \) and \( \xi_{23} \) are obtained from (5.8b).
Let $g_{11}$ be a matrix with only positive diagonal entries $g_{ii}$ and $S_i$ be the sum of the magnitude of entries in the $i$th row of $g_{12}$ and $g_{13}$. For dominance,

$$g_{ii} > S_i.$$ 

Let

$$g_{ii} - S_i > \pi_i \text{ where } \pi_i > 0.$$ 

From step III, we have

$$g_{14} K = P \text{, and } g_{15} H K = M$$

Let

$$g_{14} = \begin{bmatrix} d_{ij} \end{bmatrix}_{pxn} \text{ where } d_{ij} < 0$$

$$g_{15} = \begin{bmatrix} e_{ij} \end{bmatrix}_{pxn} \text{ where } e_{ij} < 0$$

From (5.10), we get

$$d_{ij} h_{kj} = p_{ij} \quad \cdots (5.10c)$$

$$c_{ij} h_{kj} = m_{ij} \quad \cdots (5.10d)$$

Choosing $k_j$ in (5.10c) such that

$$|k_j| > \max_i \left\{ \frac{p_{ij} \cdot 2n}{\pi_i} \right\},$$

then

$$d_{ij} < \frac{\pi_i}{2n} < g_{ii}.$$ 

Hence

$$\sum_{j=1}^{n} |d_{ij}| < \frac{\pi_i}{z}.$$ 

Similarly by considering (5.10d), $h_j$ is chosen such that

$$|h_j| > \max_i \left\{ \frac{m_{ij} \cdot 2n}{k_j \cdot \pi_i} \right\},$$

which ensures

$$\sum_{j=1}^{n} |e_{ij}| < \frac{\pi_i}{z}.$$
Therefore \( \frac{n}{j=1} |d_{ij}| + \frac{n}{j=1} |e_{ij}| < \pi_i \).

Since \( \mathcal{G}_{12}, \mathcal{G}_{13}, \mathcal{G}_{14}, \mathcal{G}_{15} \) are the only submatrices (corresponding to the rows denoted by \( \mathcal{G}_{12} \) to \( \mathcal{G}_{1n} \)) in the various expressions, it is clear that the rows of \( \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} & \mathcal{G}_{13} & \mathcal{G}_{14} & \mathcal{G}_{15} \end{bmatrix} \) will be hyperdominant. Similarly it can be easily shown that the rows of \( \begin{bmatrix} \mathcal{G}_{21}, \ldots, \mathcal{G}_{25} \end{bmatrix} \) will also be hyperdominant, thus ensuring that it is always possible to construct a hyperdominant matrix \( \mathcal{G} \) of \( \mathbb{N}_R \) for the proposed structure, Fig. 5.3(a). This completes the specification of the \( \mathcal{G} \) and hence the theorem.

Example 5.1

The example of Melvin and Bickart is taken for illustration.

\[
T(s) = Y(s) = \begin{bmatrix}
\frac{2s+1}{s+1} & \frac{s}{s+1} \\
\frac{s+1/2}{s+1} & \frac{2s-1}{s+1}
\end{bmatrix}
\]

Obviously,
\[
D = T(\infty) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}
\]

Using the Ho-Kalman algorithm an irreducible realization is obtained; thus:

\[
A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ -1/2 & -5/2 \end{bmatrix}
\]
Select
\[ g_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and} \quad Q = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \]

From (5.8a), we obtain
\[ g_{13} = \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix} \]

Choosing \( \Phi = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \), \( K = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \),
\[ H = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}, \quad \text{and} \quad g_{21} = \begin{bmatrix} -1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \]

from (5.8c) and (5.8b), the sub-matrices obtained are
\[ g_{12} = \begin{bmatrix} -1/2 & 0 \\ -1/2 & -1/2 \end{bmatrix} \]
\[ g_{14} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ g_{15} = \begin{bmatrix} -1/8 & 0 \\ 0 & -1/2 \end{bmatrix} \]

and
\[ g_{23} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

It is easily verified that the rows of \[ g_{11} \ g_{12} \ g_{13} \ g_{14} \ g_{15} \] are hyperdominant.

Then select \( g_{22} \) in the manner discussed earlier, thus;
\[ g_{22} = \begin{bmatrix} 7/6 & 0 \\ 0 & 1/2 \end{bmatrix} \]

From (5.8a), set
\[ g_{24} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad g_{25} = \begin{bmatrix} -1/6 & 0 \\ 0 & 0 \end{bmatrix} \]
The remaining entries of \( \bar{G} \) matrix can be filled in arbitrarily such that it remains hyperdominant. A suitable choice for \( \bar{G} \) is given below.

\[
\bar{G} = \begin{bmatrix}
2 & 0 & -1/2 & 0 & 0 & 0 & 0 & -1/8 & 0 \\
0 & 2 & -1/2 & -1/2 & 0 & 0 & 0 & -1/2 & 0 \\
-1/2 & -1/2 & 7/6 & 0 & 0 & 0 & 0 & -1/6 & 0 \\
0 & -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/8 & 0 & -1/6 & 0 & 0 & 0 & 0 & 7/24 & 0 \\
0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\
\end{bmatrix}
\]

Having obtained the \( \bar{G} \)-matrix of \( \bar{W}_R \), the network can be easily constructed as shown in Fig. 5.4. It may be noted that the \( \bar{G} \) matrix obtained above has all zero entries in two rows and two columns, meaning thereby, that the nodes corresponding to them (in this case \( e_{41} \) and \( e_{42} \)) will disappear; so the cascaded active elements can be combined as shown in Fig. 5.4. Further, only 9 resistors are needed in the above realization as against 12 used in [25], while the numbers of capacitors and active elements remain same. In general, this procedure will require at the most \( p+2n \) active elements compared to at the most \( 2p+2n \) elements required by Melvin and Bicka [95] for the \( Y(s) \) under consideration. It may be noted that the network \( \bar{W}_R \) (Fig. 5.4) which realizes the hyperdominant \( \bar{G} \) has no internal nodes as indicated earlier.
FIG. 5.4-Example 5.1: REALIZATION OF Y(s).

FIG. 5.5-Example 5.2: REALIZATION OF Z(s).
5.2.2 Open-Circuit Impedance Matrix Synthesis

When \( T(s) \) is assumed to be a \( p \times p \) open-circuit impedance matrix \( Z(s) \), it is implied that the \( p \)-vector \( U \) is the vector of network port currents \( i_1 \) and that \( Y \) is the \( p \)-vector of corresponding port voltages \( \hat{e}_1 \).

Consider the network block diagram in Fig. 5.1(b), where \( N_R \) and \( N_C \) are specified as in the previous case. \( \hat{e}_1 \), \( e_1 \) and \( i_1 \) are related through the resistor \( R \) as

\[
\hat{e}_1 = e_1 + R i_1 \quad \ldots (5.14)
\]

The state equations (5.2) in this case become

\[
\begin{align*}
\dot{e}_2 &= A e_2 + B i_1 \quad \ldots (5.15a) \\
\dot{e}_1 &= C e_2 + D i_1 \quad \ldots (5.15b)
\end{align*}
\]

We may choose the same structure of \( N_R \) [Fig. 5.3(a)] as in the preceding section, then (5.6) and (5.7) will remain unaltered. From (5.3), (5.6), (5.7), and (5.14), the state-equations in the form (5.15) are obtained and \([A,B,C,D]\) can be expressed as:

\[
\begin{align*}
D &= R + (g_{11} + g_{13} Q)^{-1} \ldots (5.16a) \\
B &= -Q^{-1}(g_{21} + g_{23} Q)(g_{11} + g_{13} Q)^{-1} \ldots (5.16b) \\
C &= -(g_{11} + g_{13} Q)^{-1}(g_{12} + g_{14} K + g_{15} H K) \ldots (5.16c) \\
A &= -Q^{-1} \left\{ (g_{22} + g_{24} H + g_{25} H K) - (g_{21} + g_{23} Q)(g_{11} + g_{13} Q)^{-1}(g_{14} K + g_{15} H K) \right\} \ldots (5.16d)
\end{align*}
\]

Thus the problem of realization of \( Z(s) \) has been reduced to that of identifying the various terms of
g-submatrices of $\overline{R}_R$ given by (5.16) subject to the condition that $G$ is hyperdominant. The existence of such a realization is evident from the following steps in the procedure for specifying the various submatrices of $G$.

Step I.

In (5.16d), if $D$ is non-singular, $R$ can be set as $R = [0]$; otherwise the elements of $R$ can always be specified such that $(D-R)$ is non-singular. Assuming suitable values of $\varepsilon_{11}$ and $Q$, $\varepsilon_{13}$ can be obtained such that it has all negative entries or zeros.

Step II.

Since $\varepsilon_{11}$, $\varepsilon_{13}$ and $Q$ are fixed, then by suitable choice of $G$, $g_{21}$ and $g_{23}$ can be obtained from (5.16b) such that these have non-positive entries.

Step III.

On substituting the values of $\varepsilon_{11}$, $\varepsilon_{13}$, $Q$ and $\varepsilon_{12}$ in (5.16c), we get

$$-(\varepsilon_{11} + \varepsilon_{13} Q) C - \varepsilon_{12} = \varepsilon_{14} K + \varepsilon_{15} H K$$

The right hand side of (5.17) can be split up into matrices $P$ and $M$ where $P \left[ M \right]$ contains all the non-negative elements of $-(\varepsilon_{11} + \varepsilon_{13} Q) C - \varepsilon_{12}$. Thus

$$\varepsilon_{14} K = P$$
$$\varepsilon_{15} H K = M$$

Suitable values for $K$ and $H$ can be assumed such that $\varepsilon_{14}$ and $\varepsilon_{15}$ are as small in magnitude as is necessary to make the
rows of \[
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{14} & \varepsilon_{15}
\end{bmatrix}
\] hyperdominant.

Step IV.

Now (5.16d) can be written as

\[\varepsilon_{24}^{\text{K}+\varepsilon_{25}^{\text{HK}}} = -C_0 A - \varepsilon_{22}^+ (\varepsilon_{21}^{\text{+}} + \varepsilon_{23}^Q) (\varepsilon_{11}^{\text{+}} + \varepsilon_{13}^Q)^{-1} (\varepsilon_{14}^{\text{K}+\varepsilon_{15}^{\text{HK}}}) \quad \ldots \quad (5.18)\]

Substituting in (5.18) the values of the terms obtained earlier, the only unknowns to be determined are \(\varepsilon_{22}, \varepsilon_{24},\) and \(\varepsilon_{25}\). Thus with suitable choice of diagonal sub-matrix \(\varepsilon_{22}, \varepsilon_{24}\) and \(\varepsilon_{25}\) can be found from (5.18) such that these have negative or zero entries. It may be noted that we may select \(\varepsilon_{22}\) such that the modulus of the sum of the rows corresponding to it is just equal to zero; thereby reducing the number of resistors required in the realization.

Thus having determined the terms appearing in (5.16), the remaining entries of the \(\mathcal{G}\) matrix can be filled in arbitrarily; however keeping maximum zero entries such that the hyperdominant nature of \(\mathcal{G}\) is retained is advantageous.

Now, the main result established above can be stated as the following theorem:

**Theorem 5.2**

Any \(pxp\) matrix \(T(s)\), of real rational functions of the complex frequency variable \(s\), can be realized as the open-circuit impedance matrix \(Z(s)\) of a \(p\)-port active RC network using a minimum number of \(n\) capacitors having unity capacitance spread, \(n = \delta \lceil T(s) \rceil\) and at the most \((p + 2n)\) inverting, grounded voltage amplifiers. All the capacitors
and amplifiers will share a common ground.

The procedure described above is illustrated with the help of the same example as in [18] and it is shown that the realization is possible with only four active elements instead of six required in [18].

Example 5.2 [18]

The following 2x2 matrix is to be realized as the open-circuit impedance matrix of a two port:

\[ T(s) = Z(s) = \begin{bmatrix} s & s+1 \\ s-1 & s \end{bmatrix} \]

Since \( T(s) \) is not regular at \( s = \infty \), a Möbius transformation can be invoked to make \( T(s) \) regular at \( s = \infty \).

Let \( s = \frac{z}{1-z} \),

then \( T(z) = \begin{bmatrix} \frac{z}{1-z} & \frac{1}{1-z} \\ \frac{-1+2z}{1-z} & \frac{z}{1-z} \end{bmatrix} \)

will be realized; later on each capacitor having admittance \( cz \) will be replaced by a capacitor and a resistor, having admittance \( \frac{cs}{s+1} \).

Applying Ho and Kalman algorithm, a minimal realization set \([A,B,C,D]\) of \( T(z) \) is obtained as

\[ A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad D = T(\infty) = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix} \]

Since \( D \) is non-singular, \( R \) can be set as \( R = \begin{bmatrix} 0 \end{bmatrix} \).
From (5.16a), with $g_{11} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ and $Q = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$, $g_{13}$ is obtained as $g_{13} = \begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}$.

Choosing $G = \begin{bmatrix} 1 \end{bmatrix}$, from (5.16b), we obtain

$g_{21} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $g_{23} = \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{5} \end{bmatrix}$.

From (5.16c) and (5.17), with $K = \begin{bmatrix} -5 \end{bmatrix}$, $H = \begin{bmatrix} -5 \end{bmatrix}$, we find,

$g_{14} = \begin{bmatrix} -1/5 \\ -1/5 \end{bmatrix}$ and $g_{15} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

It is easily verified that the rows of $[g_{11} \\ g_{12} \\ g_{13} \\ g_{14} \\ g_{15}]$ are hyperdominant. Now substituting the values determined above in (5.16d) or (5.18), we get

$g_{24}K + g_{25}HK = -5 - g_{22}$.

Selecting $g_{22}$ in the manner discussed earlier, thus with

$g_{22} = \begin{bmatrix} 65 \\ 24 \end{bmatrix}$,

we obtain,

$g_{24} = \begin{bmatrix} 0 \end{bmatrix}$ and $g_{25} = \begin{bmatrix} -\frac{37}{120} \end{bmatrix}$.

The remaining entries of $G$ can now be filled in arbitrarily such that it remains hyperdominant. A suitable choice for $G$ is given below.
Having obtained the $\mathbf{G}$-matrix of $\mathbf{N}_R$, the network can now be constructed as shown in Fig. 5.5. The number of active elements used in the realization is four instead of six required in [15]. It may be noted that a capacitor and a resistor are in series at each capacitive port of $\mathbf{N}_R$, as Möbius transformation has been invoked in this example.

5.2.3 Transfer-Impedance Matrix Synthesis using Operational Amplifiers

This section presents a synthesis procedure to realize $\mathbf{T}(s)$ as a $q \times p$ O.C. transfer-impedance matrix of a multi-port active RC filter using commercially available operational amplifiers (OA), and inverting, voltage-controlled voltage sources (VCVS), which can be easily constructed from OA, as active elements. The main result established in this section can be given as the following theorem:

**Theorem 5.3**

Any $q \times p$ matrix $\mathbf{T}(s)$, of real rational functions of the complex frequency variable $s$, can be realized as the
O.C. transfer-impedance matrix of a \((q + p)\) port active RC network containing a minimum number of \(n\) capacitors with unity capacitance spread, \(n = \delta [T(s)]\), \(q\) operational, amplifiers (OA), and at the most \((p + 2n)\) inverting, common ground voltage-controlled voltage sources (VCVS). All the capacitors, ports and active elements will have the ground as a common terminal.

The following proof incorporates a step by step realization procedure for \(T(s)\).

By assuming that \(T(s)\) is a \(q \times p\) O.C. transfer-impedance matrix, it is implied that the \(p\)-vector \(U\) is a vector of source port currents \(i_1\), the \(q\)-vector \(Y\) is a vector of response port voltages \(e_3\), with the response ports open i.e. \(i_3 = 0\) [Fig. 5.1(c)].

The state equations (5.2), in this case, become

\[
\begin{align*}
\dot{e}_2 &= A \cdot e_2 + B \cdot i_1 \\
e_3 &= C \cdot e_2 + D \cdot i_1 \\
\end{align*}
\]

Assuming the sub-network, \(N_R\), consisting of the resistors, OA and inverting VCVS to have a structure shown in Fig.5.3(b), where \(N_R\) is a \((2p + 2q + 3n)\) port grounded sub-network of resistors. The short circuit parameter equations of \(N_R\) can be denoted as

\[
I = \bar{G} \bar{E}
\]

where,

\[
I = \begin{bmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \end{bmatrix}^T,
\]

\[
E = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{bmatrix}^T.
\]
and \[ \mathbf{G} = \begin{bmatrix} g_{ij} \end{bmatrix} (2p+2q+3n) \times (2p+2q+3n) \]

where \( i_1, i_4, e_1 \) and \( e_4 \) are each \( p \)-vectors, \( i_3, i_7, e_3 \) and \( e_7 \) are each \( q \)-vectors, \( i_2, i_5, i_6, e_2, e_5 \) and \( e_6 \) are each \( n \)-vectors and the elements of \( \mathbf{G} \) are the submatrices with \( g_{ij} = g_{ji}^{\dagger} \); \( \mathbf{G} \) being the short circuit conductance matrix of a common ground resistive network has to be hyperdominant, a necessary and sufficient condition for the \((2p+2q+3n)\) port common terminal network \( \mathbf{N}_R \) to be realizable without internal nodes \([4,4] \).

The active elements \([\text{Fig. 5.3(b)}]\) impose the following constraints:

\[
\begin{align*}
\begin{bmatrix}
e_4 = -e_1 \\
e_5 = K e_2 \\
e_6 = H K e_2 \\
e_7 = i_7 = C
\end{bmatrix}
\end{align*}
\]

where the matrices \( K \) and \( H \) are

\[
K = \text{diag.}\{k_1, \ldots, k_n\}, \text{ with } k_i < 0 \text{ for all } i \quad \text{and} \\
H = \text{diag.}\{h_1, \ldots, h_n\}, \text{ with } h_i < 0 \text{ for all } i.
\]

From (5.3), (5.20), and (5.21), we obtain the state equations in the form (5.19), and \( \{A, B, C, D\} \) can be expressed as:

\[
\begin{align*}
D &= -\varepsilon^{-1}_{73} (\varepsilon_{71} - \varepsilon_{74}) (\varepsilon_{11} - \varepsilon_{14})^{-1} \\
B &= -\varepsilon^{-1}_{\phi} (\varepsilon_{21} - \varepsilon_{24}) (\varepsilon_{11} - \varepsilon_{14})^{-1} \\
C &= -\varepsilon^{-1}_{73} \{(\varepsilon_{75} K + \varepsilon_{76}^H K) - (\varepsilon_{71} - \varepsilon_{74})(\varepsilon_{11} - \varepsilon_{14})^{-1}(\varepsilon_{12} + \varepsilon_{15} K + \varepsilon_{16}^H K)\} \\
A &= -\varepsilon^{-1}_{\phi} \{(\varepsilon_{22} + \varepsilon_{25} K + \varepsilon_{26}^H K) - (\varepsilon_{21} - \varepsilon_{24})(\varepsilon_{11} - \varepsilon_{14})^{-1}(\varepsilon_{12} + \varepsilon_{15} K + \varepsilon_{16}^H K)\}
\end{align*}
\]
Thus the problem of realization of \( T(s) \) has been reduced to that of specifying the \( g \) submatrices associated with \( \mathbf{N}_R \) and given by (5.22), subject to the condition that \( \mathbf{G} \) is hyperdominant. The existence of such a realization is evident from the following steps in the procedure for identifying the various submatrices of \( \mathbf{G} \).

**Step I.**

Since \( \varepsilon_{73} \) is assumed non-singular with non-positive entries, while \( \varepsilon_{11} \) is hyperdominant and \( \varepsilon_{14} \) is non-positive, we can select suitable values for \( \varepsilon_{73}, \varepsilon_{11} \) and \( \varepsilon_{14} \) such that \( \varepsilon_{71} \) and \( \varepsilon_{74} \) as specified in (5.22a) have only non-positive entries.

**Step II.**

As \( \varepsilon_{11}, \varepsilon_{14} \) and \( \varepsilon_{73} \) are fixed, then by a suitable choice of \( \mathbf{G}_0 \), the submatrices \( \varepsilon_{21} \) and \( \varepsilon_{24} \) are obtained from (5.22b) such that these have non-positive entries.

**Step III.**

From (5.22c), assuming \( \varepsilon_{75} = \varepsilon_{76} = 0 \), we get

\[
\varepsilon_{15}^K + \varepsilon_{16}^{HK} = (\varepsilon_{71} - \varepsilon_{74})^{-1}(\varepsilon_{11} - \varepsilon_{14})\varepsilon_{73}^C - \varepsilon_{12} = P + M \quad \ldots \quad (5.23)
\]

where,

\[
P = \left[ p_{ij} \right]_{p \times n} \quad \text{contains all the non-negative entries of} \quad (\varepsilon_{71} - \varepsilon_{74})^{-1}(\varepsilon_{11} - \varepsilon_{14})\varepsilon_{73}^C - \varepsilon_{12}
\]

\[
M = \left[ m_{ij} \right]_{p \times n} \quad \text{contains all the non-positive entries of} \quad (\varepsilon_{71} - \varepsilon_{74})^{-1}(\varepsilon_{11} - \varepsilon_{14})\varepsilon_{73}^C - \varepsilon_{12}.
\]

Thus, \( \varepsilon_{15}^K = P \quad \ldots \quad (5.24a) \)

\( \varepsilon_{16}^{HK} = M \quad \ldots \quad (5.24b) \)
It is obvious that, by making \(|k_j|\) and the \(|h_j|\) sufficiently large, the non-zero elements of \(g_{15}\) and \(g_{16}\) can be made as small in magnitude as is necessary to make the rows of \([E_{11}, \ldots, E_{17}]\) hyperdominant. It will be shown later that it is always possible to select such amplifier gains. If \(K^+[H^+]\) denotes the pseudoinverse \([31]\) of \(K[H]\), then (5.24) yields

\[
g_{15} = PK^+ \quad \ldots \quad (5.25a)
\]

provided the consistency condition \(P[I - K^+K] = 0\) is satisfied, and

\[
g_{16} = MK^+H^+ \quad \ldots \quad (5.25b)
\]

provided the consistency conditions \(M[I - K^+K] = 0\), and \(MK^+[I - H^+H] = 0\) are satisfied.

Step IV.

From (5.22d), we get

\[
g_{25}K^+g_{26}HK = -G^+(g_{21} - g_{24})(g_{11} - g_{14})^{-1}(g_{12} + g_{15}K^+g_{16}HK) - g_{22} \quad \ldots \quad (5.26)
\]

Thus \(g_{25}\) and \(g_{26}\) can be found from (5.26) by selecting a suitable value of \(g_{22}\) such that these have non-positive entries. We may also select \(g_{22}\) in such a way that the modulus of the sum of the rows corresponding to it is just equal to zero, thereby reducing the number of resistors required in the realization.

Thus having determined the \(g\)'s that appear in (5.22), the remaining entries of \(\bar{G}\) matrix can be filled in arbitrarily; however, a maximum number of zero entries such that the hyperdominant nature of \(\bar{G}\) is retained, is advantageous.
Amplifier Gain Selection

The above synthesis procedure was developed on the observation that amplifier gains ($K$ and $H$) exist such that the equations in (5.24) have solutions corresponding to which the rows of $[\varepsilon_{11}, \ldots, \varepsilon_{17}]$ are hyperdominant. Now some criteria will be given for choosing the amplifier gains.

From steps I and II, the various submatrices appearing in (5.22a) and (5.22b) are obtained. Let $g_{11}$ be a matrix with only positive diagonal entries $g_{ii}$ and $S_i$ be the sum of the magnitude of entries in the $i$th row of $g_{12}$ and $g_{13}$. For dominance,

$$g_{ii} > S_i.$$  

Let $g_{ii} - S_i > \pi_i$ where $\pi_i > 0$.

From step III, we have

$g_{15}K = P$ and $g_{16}HK = M$

Now let $g_{15} = [d_{ij}]_{p \times n}$, where $d_{ij} < 0$

and $g_{16} = [e_{ij}]_{p \times n}$, where $e_{ij} < 0$

From (5.24) we get,

$$d_{ij}k_j = p_{ij} \quad \ldots \quad (5.24c)$$

and $$e_{ij}h_jk_j = m_{ij} \quad \ldots \quad (5.24d)$$

Choosing $k_j$ in (5.24c) such that

$$|k_j| > \max_i \left\{ \frac{p_{ij} \cdot 2n}{\pi_i} \right\}$$

then

$$d_{ij} < \frac{\pi_i}{2n} < \varepsilon_{ii}.$$
Similarly by considering (5.24d), \( h_j \) is chosen such that

\[
|h_j| > \max_i \left\{ \frac{m_i \cdot 2n}{k_j \cdot \pi_i} \right\}
\]

which ensures

\[
\sum_{j=1}^n |d_{ij}| + \sum_{j=1}^n |e_{ij}| < \frac{\pi_i}{2}
\]

Therefore, \( \sum_{j=1}^n |d_{ij}| \) + \( \sum_{j=1}^n |e_{ij}| \) < \( \pi_i \).

Since \( g_{12}, g_{14}, g_{15}, g_{16}, g_{17} \), are the only submatrices (corresponding to the rows denoted by \( g_{12} \) to \( g_{1n} \)) in the various expressions, it is clear that with the choice of \( g_{1j} = 0 \), the rows of \( \begin{bmatrix} g_{11}, \ldots, g_{17} \end{bmatrix} \) will be hyperdominant. Similarly it can be shown that the rows of \( \begin{bmatrix} g_{21}, \ldots, g_{27} \end{bmatrix} \) will also be hyperdominant; thus ensuring that it is always possible to construct a hyperdominant matrix \( G \) of \( \mathbb{N}_R \) for the proposed structure [Fig.5.3(b)]. This completes the specification of the \( G \) and hence the theorem.

Example 5.3

To illustrate the above result, the following 2x2 matrix will be realized as an open-circuit transfer impedance matrix of a two port:

\[
T(s) = \begin{bmatrix}
\frac{2s-1}{s+1} & \frac{s+1}{s+1} \\
-1 & \frac{2s+1}{s+1}
\end{bmatrix}
\]

Here \( p = q = 2 \) and \( n = 2 \).

Since \( T(s) \) is regular at \( s = \infty \), a Möbius transformation
will not be needed. Using the Ho-Kalman algorithm an irreducible realization of \( T(s) \) is obtained; thus:

\[
A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1/6 \\ 0 & 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} -3 & 0 \\ -1 & -5/6 \end{bmatrix}, \quad D = T(\infty) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]

Select

\[
\xi_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi_{73} = \begin{bmatrix} -1/10 \\ 0 \\ -1/10 \end{bmatrix} \quad \text{and} \quad \xi_{14} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.
\]

From (5.22a), we obtain

\[
\xi_{74} = \begin{bmatrix} -2/10 \\ 0 \\ -2/10 \end{bmatrix} \quad \text{and} \quad \xi_{71} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.
\]

Choosing \( C_0 = \begin{bmatrix} 1/10 \\ 0 \\ 1/10 \end{bmatrix} \), the submatrices obtained from (5.22b) are

\[
\xi_{21} = \begin{bmatrix} -1/10 \\ -1/60 \\ 0 \\ -1/10 \end{bmatrix} \quad \text{and} \quad \xi_{24} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.
\]

Select \( K = \begin{bmatrix} -5 \\ 0 \\ 0 \\ -5 \end{bmatrix} \) and \( H = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \).

From (5.23) and (5.24), we obtain

\[
\xi_{15} = \begin{bmatrix} -27/100 \\ -31/300 \\ -31/300 \end{bmatrix} \quad \text{and} \quad \xi_{16} = \begin{bmatrix} 0 \\ 0 \\ -1/24 \end{bmatrix}.
\]

It is easily verified that the rows of \( [\xi_{11}, \xi_{12}, \xi_{13}, \xi_{14}, \xi_{15}, \xi_{16}, \xi_{17}] \) are hyperdominant. Then select \( \xi_{22} \)
in the manner discussed earlier; thus:

\[ g_{22} = \begin{bmatrix} \frac{1586}{11520} & 0 \\ 0 & \frac{133}{960} \end{bmatrix}. \]

From (5.26), the submatrices obtained are,

\[ g_{25} = \begin{bmatrix} 0 & -\frac{1}{288} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad g_{26} = \begin{bmatrix} -\frac{16244}{57700} & 0 \\ -\frac{1}{100} & -\frac{76}{4800} \end{bmatrix} \]

Therefore, the following \( \mathbf{G} \) composed according to the guidelines set forth earlier will be hyperdominant.

\[
\mathbf{G} = \begin{bmatrix}
1 & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0 & 0 & -\frac{27}{100} & 0 & 0 & -\frac{1}{24} & 0 & 0 \\
0 & 1 & -\frac{1}{60} & -\frac{1}{10} & 0 & 0 & 0 & 0 & -\frac{31}{300} & -\frac{31}{300} & 0 & 0 & 0 & 0 \\
-\frac{1}{10} & -\frac{1}{60} & \frac{1586}{11520} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{288} & \frac{16244}{57700} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{10} & 0 & \frac{133}{960} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & -\frac{76}{4800} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{112}{300} & 0 & 0 & 0 & 0 & \frac{1}{10} & -\frac{1}{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{76}{4800} & 0 & 0 & 0 & \frac{1}{10} & 0 & -\frac{1}{10} & 0 & 0 & 0 \\
-\frac{27}{100} & -\frac{31}{300} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{288} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{31}{300} & -\frac{1}{288} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{16821}{57700} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{16244}{57700} & -\frac{1}{100} & 0 & 0 & 0 & 0 & 0 & \frac{276}{4800} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{24} & 0 & 0 & -\frac{1}{10} & 0 & -\frac{2}{10} & -\frac{1}{10} & 0 & 0 & 0 & \frac{4}{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{10} & 0 & \frac{2}{10} & -\frac{1}{10} & 0 & 0 & 0 & \frac{3}{10} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The realization of \( T(s) \) based on the realization of \( \mathbf{G} \) is shown in Fig. 5.6. Note that two differential output \( CA \) are
FIG. 5.6- Example 5.3: REALIZATION OF $T(s)$. 
shown rather than the cascade of two with gains $k_1=k_2=-5$ and $h_1=h_2=-1$. Thus the total number of active elements used is 6. Further, the network $\mathbf{N}_R$ (Fig. 5.6) which realizes the hyperdominant $\mathbf{G}$ has no internal nodes as indicated earlier.

It may be noted that the proposed method has two distinct advantages over the one due to Bickart and Melvin[18] for the case of C.C. transfer-impedance matrix. First, it uses commercially available CA instead of voltage amplifiers and secondly, it will usually require fewer resistors, because the submatrix $g_{22}$, as discussed in the procedure and illustrated in the example, can always be chosen to be hyperdominant.

5.3 PASSIVE RECIPROCAL RCT MULTIPORT NETWORK SYNTHESIS

The problem of realizing a given SPR immittance matrix with passive reciprocal RLC multiport network (without gyrators) is one of the important and interesting problems in network synthesis and has been studied via state-variable approach by several authors during recent years[14], [11], [13], [15], [13], [15], [16].

In this section, a new synthesis procedure to realize a given SPR immittance matrix using passive reciprocal RCT multiport network with a minimum number of capacitors is presented. The method is essentially an extension of the technique of active RC multiport synthesis discussed in the preceding section and uses ideal transformers in place of active elements. By selecting suitable transformation ratios [149], the hyperdominant matrix $\mathbf{G}$ is again constructed.
Of course, this synthesis is only possible if the minimal realization set \( \{A, B, C, D\} \), associated with the given SPR immittance function, satisfies the following constraints.

\[
\begin{align*}
(1) \quad & M_1 + M'_1 \succ 0 \\
(2) \quad & (I + \sum)M_1 = M'_1(I + \sum)
\end{align*}
\]...

(5.27)
(5.28)

where \( M_1 = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \), and \( \sum \) is an unique diagonal matrix of \( \pm 1 \)'s as defined earlier.

More precisely, the first condition (5.27) says that \( M_1 \) possesses a passive synthesis, while the second condition (5.28) guarantees reciprocal realization [15], [17], [16].

A minimal reciprocal realization, of a given SPR immittance matrix, that fulfills the above conditions can be constructed with the help of the algorithms proposed in Chapter III. Once such a realization is in hand, a passive reciprocal synthesis of the given immittance function can be obtained by using an identical procedure as given in Section 5.2.

In the following, the main result of the passive reciprocal synthesis of SPR short-circuit admittance matrix \( Y(s) \) and SPR open-circuit impedance matrix \( Z(s) \) is stated in the form of the following theorem:

**Theorem 5.4**

Any \( p \times p \) SPR matrix \( T(s) \), of real rational functions of complex frequency variable \( s \), can be realized as the immittance matrix \( [Y(s) \text{ or } Z(s)] \) of a \( p \)-port passive reciprocal RCT network using a minimum number of grounded capacitors \( n = \delta[T(s)] \) and at the most \( (p+2n) \) ideal transformers. In
addition, the ports will be grounded.

Note: If \( T(s) \) is a short-circuit admittance matrix, then 
\( T(\infty) \) must be the sum of a strictly hyperdominant
matrix plus a non-negative matrix.

Since proof of the theorem follows identically to the
one given in Section (5.2.1) when \( T(s) \) is \( Y(s) \) with the above
constraint on \( D \), and in Section (5.2.2) when \( T(s) \) is \( Z(s) \), the
synthesis procedure is illustrated with the help of suitable
examples for both \( Y(s) \) and \( Z(s) \) respectively. It may be noted
that the entries of the diagonal matrices \( Q \), \( K \) and \( H \), in this
case, will correspond to the transformation-ratios rather than
the gains of the amplifiers. Further, the network block dia-
gram is same as shown in Fig.5.1(a,b) whereas, Fig.5.3(c)
depicts the block diagram of \( N_R \) consisting of ideal transfor-
ners and resistors.

First, the synthesis of SPR \( Y(s) \) is illustrated with
the help of the following example:

Example 5.4

The following 2x2 SPR matrix \( T(s) \) is to be realized as
the short-circuit admittance matrix \( Y(s) \) of a 2-port RCT
network:

\[
T(s) = Y(s) = \begin{bmatrix}
\frac{2s+3}{s+1} & \frac{s+2}{s+1} \\
\frac{s+2}{s+1} & \frac{2s^2+4s+3}{(s+1)^2}
\end{bmatrix}.
\]

Obviously, \( D = T(\infty) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \).
A minimal reciprocal realization \{A, B, C, D\} associated with \( T_1(s) \) satisfying (5.27) and (5.28) is obtained by the algorithm presented in Chapter III, [85]. Thus

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & -2
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 \\
0 & C \\
0 & -1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

Now we realize the 2-port RCT network corresponding to this state-model using the procedure of Section (5.2.1).

Select, \( g_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \),

from (5.8d), \( g_{13} \) is obtained as

\[
g_{13} = \begin{bmatrix}
0 & -1/4 \\
-1/4 & 0
\end{bmatrix}.
\]

Choosing \( T = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \), the submatrices obtained from (5.8b) are

\[
g_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/8 \end{bmatrix} \quad \text{and} \quad g_{21} = \begin{bmatrix} -1/2 & -1/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Select \( K = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \) and \( H = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \),

from (5.8c), we obtain

\[
\begin{bmatrix} \frac{-3}{8} & c & c \\ \frac{-3}{8} & 0 & -\frac{1}{4} \end{bmatrix}
\text{ and } \begin{bmatrix} 0 & c & c \\ 0 & c & c \\ 0 & c & c \end{bmatrix}.
\]

Then select \( g_{22} \) in the manner discussed earlier; thus:

\[
\begin{bmatrix} \frac{7}{6} & c & c \\ c & \frac{1}{6} & c \\ 0 & c & \frac{2}{5} \end{bmatrix}.
\]

From (5.8a), the submatrices obtained are

\[
\begin{bmatrix} 0 & c & c & c \\ 0 & c & c & c \\ 0 & -\frac{1}{8} & -\frac{3}{2}c & c \end{bmatrix}
\text{ and } \begin{bmatrix} -\frac{1}{6} & c & c & c \\ c & -\frac{1}{24} & -\frac{1}{8} & c \\ 0 & c & c & c \end{bmatrix}.
\]

The remaining entries of \( \bar{G} \) matrix can be filled in arbitrarily such that it remains hyperdominant. A suitable choice for \( \bar{G} \) is given below.
The realization of $Y(s)$ based on the realization of $\bar{G}$ is shown in Fig. 5.7. Note that the network $\bar{M}_R$ which realizes the hyperdominant $\bar{G}$ has no internal nodes as indicated earlier.

Next the passive reciprocal synthesis of SPR $Z(s)$ is illustrated.

Example 5.5

The following 3x3 SPR matrix $T(s)$ is to be realized as the C.C. impedance matrix of a 3-port RCT network:
FIG. 5.7-Example 5.4: REALIZATION OF SPR $Y(s)$. 
\[ T(s) = Z(s) = \begin{pmatrix}
\frac{2s+3}{s+1} & \frac{s}{s+1} & \frac{s+2}{s+1} \\
\frac{s}{s+1} & \frac{3s+4}{s+1} & \frac{s+1/2}{s+1} \\
\frac{s+2}{s+1} & \frac{s+1/2}{s+1} & \frac{2s+3}{s+1}
\end{pmatrix} \]

A minimal reciprocal realization set \{A, B, C, D\} of the above function has been constructed in Chapter III (Example 3.2); thus,

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -1 & 1 \\
0 & 1/2 & 1/2 \\
0 & -1/2 & 1/2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1/2 & 1/2 \\
1 & 1/2 & -1/2
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 3 & 1 \\
2 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix}.
\]

It is verified that this realization satisfies both passivity and reciprocity conditions i.e. (5.27) and (5.28). Therefore, we can proceed to realize it as an o.c. impedance matrix of a 3-port RCT network following the procedure of Section 5.2.2, and using Fig. 5.1(b) and Fig. 5.3(c).

Since \(D\) is nonsingular, \(R\) is set as \(R = [C]\).

Select
\[
\varepsilon_{11} = \begin{bmatrix}
5/7 & -1/7 & -2/7 \\
-1/7 & 3/7 & -1/7 \\
-2/7 & -1/7 & 5/7
\end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix}
-2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{bmatrix},
\]

\(\varepsilon_{13}\) is obtained from (5.16a) as
\[
\varepsilon_{13} = \begin{bmatrix}
\varepsilon_{31} \\
\varepsilon_{32}
\end{bmatrix}.
\]
Choosing \( \mathbf{C}_\mathcal{P} = \begin{bmatrix} 1/50 & 0 & 0 \\ 0 & 1/50 & 0 \\ 0 & 0 & 1/50 \end{bmatrix} \),

from (5.16b), the submatrices obtained are

\[
\mathbf{g}_{21} = \begin{bmatrix} -2/50 & 0 & -2/50 \\ -1/50 & -2/50 & -3/100 \\ 0 & 0 & -1/100 \end{bmatrix} , \quad \mathbf{g}_{23} = \begin{bmatrix} 0 & -1/100 & 0 \\ 0 & 0 & 0 \\ 0 & -1/100 & 0 \end{bmatrix} .
\]

Select

\[
\mathbf{K} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} , \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} ,
\]

from (5.16c) and (5.17), we obtain

\[
\mathbf{g}_{14} = \begin{bmatrix} 0 & -13/1400 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -277/2800 \end{bmatrix} ,
\]

and

\[
\mathbf{g}_{15} = \begin{bmatrix} -93/2800 & 0 & -1/224 \\ -5/112 & -9/1400 & -1/56 \\ -93/2800 & -197/11200 & 0 \end{bmatrix} .
\]
It can easily be seen that the rows of \[ \vec{e}_{11} \; \vec{e}_{12} \; \vec{e}_{13} \; \vec{e}_{14} \; \vec{e}_{15} \] are hyperdominant.

Now select \( g_{22} \) in the manner discussed earlier; thus:

\[
g_{22} = \begin{bmatrix}
\frac{90961}{10000} & 0 & 0 \\
0 & \frac{83452}{525} & 0 \\
0 & 0 & \frac{25118}{1050}
\end{bmatrix},
\]

from (5.16d) or (5.18), the submatrices obtained are

\[
g_{24} = \begin{bmatrix}
0 & -\frac{2132}{35000} & 0 \\
-\frac{2132}{35000} & 0 & -\frac{133}{14000} \\
0 & 0 & 0
\end{bmatrix},
\]

\[
g_{25} = \begin{bmatrix}
-\frac{224}{4200} & 0 & -\frac{217}{112000} \\
0 & -\frac{249}{21} & 0 \\
-\frac{359}{14000} & -\frac{7}{40000} & -\frac{795}{672000}
\end{bmatrix}.
\]

The remaining entries of \( \overline{G} \) can now be filled in arbitrarily such that its hyperdominant nature is retained.

A suitable choice of \( \overline{G} \) is given below.
Once $\mathcal{G}$ matrix of $\mathcal{N}_R$ is obtained, the network consisting of $R$, $C$ and $T$ can be easily constructed as shown in Fig. 5.8. Further, the network $\mathcal{N}_R$ (Fig. 5.8) which realizes the hyperdominant $\mathcal{G}$ has no internal nodes as indicated earlier.

6.4 CONCLUDING DISCUSSIONS

A simple and systematic synthesis procedure, based on a state-variable approach and the reactance extraction principle, has been developed whereby any qxp matrix $T(s)$, of real rational functions of the complex frequency variable $s$ can be realized as (i) a s.c. admittance matrix, (ii) an o.c.i. impedance matrix, and (iii) a transfer-impedance matrix of an active RC multiport network with the ports grounded. The realized network, in each case, contains a minimum number of $n$ grounded capacitors having unity capacitance spread, $n = \delta[T(s)]$, and at the most $(p + 2n)$ inverting VCVS. Of course, in the case of qxp transfer-impedance matrix synthesis $qOA$ are also needed. The facts that all the minimum number of capacitors have same value and that all the active elements, capacitors and ports are grounded, are very much desirable if the network is to be fabricated as an integrated circuit. The distinct advantages of the proposed method over the one due to Bickart and Melvin [15] are that it requires, in general, less number of active elements and resistors while retaining all the advantages of their method. Also, for the case of transfer-impedance matrix synthesis, this method uses commercially available operational amplifiers in place of finite gain
FIG. 5.8 - Example 5.5: REALIZATION OF SPR \( Z(s) \).
voltage amplifiers used in [15].

Based on the synthesis approach in Section (5.2) and the result of Chapter III, a new passive reciprocal synthesis of a SPR immittance matrix using RCT multiport network with a minimum number of capacitors has been evolved.

Since a minimum number of capacitors is used, it is conjectured that the realization of $T(s)$ will be relatively insensitive to capacitance variations. It is hoped that further investigations of this synthesis procedure will provide a quantitative assessment of the sensitivity of selected network attributes and validity of the conjecture. Finally, the procedure can be reduced to a digital computer program.