Wavelets and Superwavelets

This chapter is devoted to basic definitions and results on wavelets and frames that are necessary for our study.

1.1 Frames, Wavelets and MRA Wavelets

Definition 1.1.1. A frame in a Hilbert space \( H \) is a sequence \( \{x_n\} \) of vectors in \( H \) with the property that there exist constants \( A, B > 0 \) such that for \( f \in H \)

\[
A \|f\|^2 \leq \sum_n |\langle f, x_n \rangle|^2 \leq B \|f\|^2.
\]

The greatest possible such \( A \) is the lower frame bound and the least possible such \( B \) is the upper frame bound. If \( A = B \), then the frame is called a tight frame. If \( A = B = 1 \), then the frame is called a normalised tight frame.

Definition 1.1.2. A function \( \psi \in L^2(\mathbb{R}) \) is a wavelet (resp. frame wavelet) if \( \{U^jT^k \psi : j, k \in \mathbb{Z}\} \) is an orthonormal basis (resp. frame) for \( L^2(\mathbb{R}) \), where \( T \) and \( U \) are the translation and dilation unitary operators, respectively, on \( L^2(\mathbb{R}) \) defined by
1.1. Frames, Wavelets and MRA Wavelets

\[(Tf)(t) = f(t - 1)\]  \hspace{1cm} \text{(1.1)}

and

\[(Uf)(t) = \sqrt{2}f(2t).\]  \hspace{1cm} \text{(1.2)}

**Definition 1.1.3.** \(\psi \in L^2(\mathbb{R})\) is a **tight frame wavelet** (resp. **normalized tight frame wavelet**) if \(\{U^jT^k\psi : j, k \in \mathbb{Z}\}\) is a tight frame (resp. normalized tight frame).

**Example 1.1.4.** Consider the function \(\psi : \mathbb{R} \rightarrow \mathbb{R}\) defined by

\[
\psi(t) = \begin{cases} 
1 & \text{if } t \in [0, \frac{1}{2}) \\
-1 & \text{if } t \in [\frac{1}{2}, 1) \\
0 & \text{otherwise}
\end{cases}
\]

Then it can be shown that \(\{U^jT^k\psi : j, k \in \mathbb{Z}\}\) is an orthonormal basis for \(L^2(\mathbb{R})\), so that \(\psi\) is a wavelet. This wavelet is called the **Haar wavelet**.

**Example 1.1.5.** Let \(\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \chi_{[-2\pi, -\pi) \cup [\pi, 2\pi)}\). Then \(\{U^jT^k\psi : j, k \in \mathbb{Z}\}\) is an orthonormal basis for \(L^2(\mathbb{R})\), so that \(\psi\) is a wavelet.

**Example 1.1.6.** The function \(\psi\) whose Fourier transform satisfies

\[
\hat{\psi}(\xi) = e^{i\frac{\xi}{2}} \chi_{[-2\pi, -\pi) \cup [\pi, 2\pi)}
\]

is a wavelet, called the **Shannon wavelet**.

**Example 1.1.7.** Let \(\psi \in L^2(\mathbb{R})\) be such that \(\text{supp}(\hat{\psi})\) is contained in
{s \in \mathbb{R} : \frac{1}{2} \leq |s| \leq 2} \text{ and}

\sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j s) \right|^2 = \frac{1}{2\pi} \text{ for all } s \neq 0.

Then \(\psi\) is a normalised tight frame wavelet and these type of frame wavelets are called Frazier-Jawerth type frame wavelets.

**Example 1.1.8.** Consider the Mexican hat function

\[ \psi(x) = \frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}} (1 - x)^2 e^{-\frac{1}{2}x^2}, \]

which coincides with \(-\frac{d^2}{dx^2} (e^{-\frac{1}{2}x^2})\), when normalized in the space \(L^2(\mathbb{R})\). Ingrid Daubechies [13] has reported frame bounds of 3.223 and 3.596 for the frame obtained by translations and dilations of the Mexican hat function.

An important concept in wavelet theory is the multiresolution analysis established by Mallat [25] which is used to derive wavelets. Such wavelets are called **MRA wavelets**. However, there exist non-MRA wavelets. e.g., Journe’s wavelet [23] and wavelets constructed by Bownik., *et al.* [6]. But Papadakis [26] introduced the concept of generalised frame MRA (GFMRA) and announced that any wavelet in \(L^2(\mathbb{R})\) is derived by GFMRA. A GFMRA may not give wavelets, e.g. [35].

**Definition 1.1.9.** A **Multiresolution Analysis (MRA)** for \(L^2(\mathbb{R})\) consists of a sequence \(\{V_j : j \in \mathbb{Z}\}\) of closed subspaces of \(L^2(\mathbb{R})\) satisfying

1. \(V_j \subset V_{j+1}, \ j \in \mathbb{Z}\),
2. \(\bigcap_{j \in \mathbb{Z}} V_j = \{0\}\),
3. \(\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})\),
4. \( f \in V_j \) if and only if \( Uf \in V_{j+1}, \ j \in \mathbb{Z} \).

5. There is a \( \phi \in V_0 \) such that \( \{T^k\phi : k \in \mathbb{Z}\} \) is an orthonormal basis for \( V_0 \).

The function \( \phi \) given above is called an \textit{orthogonal scaling function} for the multi-resolution analysis.

**Example 1.1.10.** Let \( V_j \) be the closed span of functions in \( L^2(\mathbb{R}) \) which are constant on intervals of the form \([2^{-j}k, 2^{-j}(k+1)]\), \( k \in \mathbb{Z} \). Then \( \{V_j : j \in \mathbb{Z}\} \) is an MRA and we can take the scaling function to be \( \phi = \chi_{[-1,0]} \). As we shall see in Example 1.1.12, this MRA is related to the Haar wavelet.

It was shown by Hernandez and Weiss [23] that if \( \phi \) is a scaling function for an MRA, then there is a \( 2\pi \)-periodic measurable function \( m \) such that

\[
\hat{\phi}(2\xi) = m(\xi) \hat{\phi}(\xi)
\]

for a.e. \( \xi \in \mathbb{R} \). It is known that the function \( \psi \) given by

\[
\hat{\psi}(\xi) = e^{i\frac{\pi}{2}m\left(\frac{1}{2}\xi + \pi\right)} \hat{\phi}\left(\frac{1}{2}\xi\right)
\]

is a wavelet, and moreover, it is known that every function of the form

\[
(1.3) \quad \hat{\psi}(\xi) = e^{i\frac{\pi}{2}k(\xi)m\left(\frac{1}{2}\xi + \pi\right)} \hat{\phi}\left(\frac{1}{2}\xi\right)
\]

where \( k \) is any measurable unimodular \( 2\pi \)-periodic function, gives a wavelet. These are all contained in the difference space \( W_0 = V_1 \ominus V_0 \), and more importantly, every wavelet contained in \( W_0 \) has the form above.

**Definition 1.1.11.** A wavelet which has the form given in (1.3) for some MRA is called an \textit{MRA wavelet}.
Example 1.1.12. The Haar wavelet seen in Example 1.1.4 is constructed (up to a translation) from the MRA generated by the scaling function $\phi = \chi_{[-1,0]}$ associated with the MRA described in Example 1.1.10.

Example 1.1.13. The Shannon wavelet seen in Example 1.1.6 is an MRA wavelet whose scaling vector $\phi$ is given by the Fourier transform $\hat{\phi}(\xi) = \chi_{[-\pi,\pi]}(\xi)$.

Remark 1.1.14. In practice, the wavelets $\psi$ are constructed from the scaling function in such a way that

$$\{T^k\psi : k \in \mathbb{Z}\}$$

is an orthonormal basis for $W_0 := V_1 \ominus V_0$, the orthocomplement of $V_0$ in $V_1$.

Suppose $\psi$ is a wavelet. For $j \in \mathbb{Z}$, let $W_j$ be the subspace generated by $\{U^jT^l\psi : l \in \mathbb{Z}\}$ and let

$$V_j = \oplus_{k<j} W_k.$$  \hfill (1.4)

Then $\{V_j : j \in \mathbb{Z}\}$ satisfies (1) to (4) in the Definition 1.1.9 of MRA. If (5) in Definition 1.1.9 is also satisfied, then $\psi$ is an MRA wavelet.

Definition 1.1.15. Let $\psi$ be a normalized tight frame wavelet. For $j \in \mathbb{Z}$, let $W_j$ be the subspace generated by $\{U^jT^l\psi : l \in \mathbb{Z}\}$ and $V_j$ as in (1.4). Then $\{V_j : j \in \mathbb{Z}\}$ satisfies (1) to (4) in the Definition 1.1.9 of MRA. Also, if there is a function $\phi$ in $V_0$ such that $\{T^l\phi : l \in \mathbb{Z}\}$ is a normalized tight frame for $V_0$, then we call $\psi$ an MRA frame wavelet.

### 1.2 Wavelet Sets and Frame Sets

**Definition 1.2.1.** A unitary system $\mathcal{U}$ is a subset of the set of all unitary operators acting on a separable Hilbert space $H$ which contains the identity
Definition 1.2.2. A vector \( x \in H \) is called a normalised tight frame vector (resp. frame vector with bounds \( A \) and \( B \)) for a unitary system \( U \) if \( Ux = \{Ux : U \in U\} \) forms a tight frame (resp. frame with bounds \( A \) and \( B \)) for \( \text{span}(Ux) \). It is called a complete normalised tight frame vector (resp. complete frame vector with bounds \( A \) and \( B \)) when \( Ux \) is a normalized tight frame (resp. frame with bounds \( A \) and \( B \)) for \( H \).

Notation 1.2.3. Let \( \mathcal{F} \) be the extended Fourier transform on \( L^2(\mathbb{R}) \) and \( U \) and \( T \) be defined as in (1.1) and (1.2). Then we use the following notations:

\[
\hat{U} = \mathcal{F}UF^{-1},
\]

\[
\hat{T} = \mathcal{F}TF^{-1},
\]

and

\[
U^nT^m = \{U^nT^m : n, m \in \mathbb{Z}\}.
\]

Proposition 1.2.4. For \( f \in L^2(\mathbb{R}) \)

\[
\hat{T}f = e^{-i}f
\]

and

\[
\hat{U}f = U^{-1}f
\]

Proof.

\[
\left(\hat{T}f\right)(\xi) = \left(\mathcal{F}TF^{-1}f\right)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(TF^{-1}f\right)(s)e^{-is\xi}ds,
\]
\[ \hat{U}f(\xi) = (\mathcal{F}U\mathcal{F}^{-1}f)(\xi) \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (U\mathcal{F}^{-1}f)(s) e^{-is\xi} ds \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}^{-1}f)(2s) e^{-is\xi} ds \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathcal{F}^{-1}f)(u) e^{-iu\xi} du \]
\[ = \frac{1}{\sqrt{2}} \mathcal{F}(\mathcal{F}^{-1}f)(\frac{\xi}{2}) \]
\[ = \frac{1}{\sqrt{2}} f\left(\frac{\xi}{2}\right) \]
\[ = U^{-1}f(\xi). \]

**Defintion 1.2.5.** A measurable subset \( E \) of \( \mathbb{R} \) is called a wavelet set if \( \frac{1}{\sqrt{2\pi}} \chi_E \), where \( \chi_E \) is the characteristic function of \( E \), is the Fourier transform of a wavelet in \( L^2(\mathbb{R}) \).

**Example 1.2.6.** [14] \( \frac{1}{\sqrt{2\pi}} \chi_{[-2\pi, -\pi) \cup [\pi, 2\pi)} \) is the Fourier transform of the wavelet seen in Example 1.1.5, and hence \([-2\pi, -\pi) \cup [\pi, 2\pi)\) is a wavelet set.
1.3. Superwavelets and Construction of Superwavelets in $L^2(\mathbb{R})^2$

**Definition 1.2.7.** A measurable subset $E$ of $\mathbb{R}$ is called a *frame set* if $\frac{1}{\sqrt{2\pi}} \chi_E$ is a complete normalized tight frame vector for $\mathcal{U}_{\hat{\cdot}, \hat{\cdot}}$. In other words, $E$ is a frame set if $\frac{1}{\sqrt{2\pi}} \chi_E$ is the Fourier transform of a normalized tight frame wavelet.

**Example 1.2.8.** $E = [-\pi, -\frac{\pi}{2}) \cup [\frac{\pi}{2}, \pi)$ is a frame set.

1.3 Superwavelets and Construction of Superwavelets in $L^2(\mathbb{R})^2$

Now we pass on to the super space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and study superwavelets in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

**Definition 1.3.1.** [22] Suppose that $\eta_1, \eta_2$ are normalized tight frame wavelets. The ordered pair $(\eta_1, \eta_2)$ is a *superwavelet* of length 2 if $\{U^kT^l \eta_1 \oplus U^kT^l \eta_2 : k, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

**Definition 1.3.2.** [22] Let $G$, $E$ and $F$ be measurable subsets of $\mathbb{R}$. $G$ is 2-dilation congruent to $F$ if there is a bijection $\tau : G \to F$ such that for any $s \in G$ there is $n \in \mathbb{Z}$ satisfying $\tau(s) = 2^n s$. $G$ is $2\pi-$translation congruent to $E$ if there is a bijection $\phi : G \to E$ such that $\phi(s) - s$ is an integral multiple of $2\pi$ for each $s \in G$.

**Lemma 1.3.3.** [12] Let $E$ and $F$ be bounded measurable sets in $\mathbb{R}$ such that $E$ contains a neighbourhood of 0, and $F$ has nonempty interior and is bounded away from 0. Then there is a measurable set $G \subset \mathbb{R}$, which is 2-dilation congruent to $F$ and $2\pi-$ translation congruent to $E$.

**Theorem 1.3.4.** [22] Let $E$ be a measurable subset of $\mathbb{R}$. Then $E$ is a frame set if and only if $E$ is both $2\pi-$translation congruent to a subset $F$ of $[0, 2\pi]$ and 2-dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$. 
Remark 1.3.5. $(\eta_1, \eta_2)$ is a superwavelet for $\mathcal{U}_{\hat{U}, \hat{T}}$ means $\eta_1$ and $\eta_2$ are normalized tight frame wavelets and

$$\left\{ \hat{U}^n \hat{T}^m (\eta_1 \oplus \eta_2) : n, m \in \mathbb{Z} \right\} = \left\{ \hat{U}^n \hat{T}^m \eta_1 \oplus \hat{U}^n \hat{T}^m \eta_2 : n, m \in \mathbb{Z} \right\}$$

is an orthonormal basis for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Using the above results, we have the following examples:

Let $E_1 = [-\pi, -\frac{1}{2}\pi) \cup [\frac{1}{2}\pi, \pi)$. Consider the mappings $\phi : E_1 \to [\frac{1}{2}\pi, \frac{3}{2}\pi)$ and $\tau : E_1 \to [-2\pi, -\pi) \cup [\pi, 2\pi)$ defined by:

$$\phi(s) = \begin{cases} s + 2\pi, & s \in [-\pi, -\frac{1}{2}\pi) \\ s, & s \in [\frac{1}{2}\pi, \pi) \end{cases}$$

and

$$\tau(s) = 2s, \ s \in E_1$$

Then $\phi$ and $\tau$ are bijective mappings. Hence by Theorem 1.3.4, noting that $[-\pi, \pi]$ is $2\pi$ translation congruent to $[0, 2\pi]$, $E_1$ is a frame set.

Let $E = [-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and $F = [-\pi, -\frac{1}{2}\pi) \cup [\frac{1}{2}\pi, \pi)$. Then by Lemma 1.3.3, there exists a measurable set $G$ such that $G$ is 2-dilation congruent to $F$ and $2\pi$-translation congruent to $E$. Now $G$ is 2-dilation congruent to $F$ and $F$ is 2-dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$. Hence $G$ is 2-dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$. Similarly, $G$ is $2\pi$ translation congruent to $[\frac{1}{2}\pi, \frac{3}{2}\pi)$. Hence, by Theorem 1.3.4, $G$ is a frame set.

Claim: $(\eta_1, \eta_2)$ is a superwavelet, where $\eta_1 = \frac{1}{\sqrt{2\pi}} \chi_{E_1}$ and $\eta_2 = \frac{1}{\sqrt{2\pi}} \chi_{G}$. 
1.3. Superwavelets and Construction of Superwavelets in $L^2(\mathbb{R})^2$

We have, for every $j, l \in \mathbb{Z}, k \neq 0$

$$\left\langle \hat{U}^k \hat{T}^l \eta_1 \oplus \hat{U}^k \hat{T}^l \eta_2, \hat{T}^j \eta_1 \oplus \hat{T}^j \eta_2 \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = \left\langle \hat{U}^k \hat{T}^l \eta_1, \hat{T}^j \eta_1 \right\rangle_{L^2(\mathbb{R})} + \left\langle \hat{U}^k \hat{T}^l \eta_2, \hat{T}^j \eta_2 \right\rangle_{L^2(\mathbb{R})} = 0,$$

as $\left\langle \hat{U}^k \hat{T}^i \eta_i, \hat{T}^j \eta_j \right\rangle = 0$ $(i = 1, 2)$.

Also

$$\left\langle \hat{T}^i \eta_1 \oplus \hat{T}^i \eta_2, \eta_1 \oplus \eta_2 \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases},$$

Hence $\{ \hat{U}^k \hat{T}^i \eta_i : k, l \in \mathbb{Z} \}$ is an ONB for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Hence $(\eta_1, \eta_2)$ is a super-wavelet of length 2.

We can summarise the above discussion as:

**Proposition 1.3.6.** There is a super-wavelet of length 2.

**Definition 1.3.7.** If $E$ and $F$ are frame sets, we call $(E, F)$ a strong complementary pair if $\left( \mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_E \right), \mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_F \right) \right)$ is a super-wavelet.

The following version of Theorem 1.3.4 is needed in the following example.

**Theorem 1.3.8.** Let $E$ be a measurable subset of $\mathbb{R}$. Then $E$ is a frame set if and only if $E$ is both $2\pi$–translation congruent to a subset $F$ of $[0, 2\pi]$ and 2-dilation generator for $\mathbb{R}$.

**Example 1.3.9.** Let $E = [-\pi, -\frac{1}{2}\pi) \cup [\frac{1}{2}\pi, \pi)$. The argument before Proposition 1.3.6 gives us the existence of the strong complement frame set of $E$.

Now we construct a concrete one. Consider a set of type $[a, \frac{\pi}{2}) \cup [2\pi, a + 2\pi)$. This set is a 2-dilation generator of a partition of $[0, \infty)$ if $\frac{a}{2} (a + 2\pi) = 2a$.

So we get $a = \frac{2\pi}{7}$. Thus $[\frac{2\pi}{7}, \frac{\pi}{2}) \cup [2\pi, \frac{16\pi}{7})$ is a 2-dilation generator of a partition of $[0, \infty)$. Symmetrically, $[-\frac{16\pi}{7}, -2\pi) \cup [-\frac{\pi}{2}, -\frac{2\pi}{7})$ is a 2-dilation generator of a partition of $(-\infty, 0]$.
1.3. Superwavelets and Construction of Superwavelets in $L^2(\mathbb{R})^2$

generator of a partition of $(-\infty, 0]$. Writing

$$A = \left[-\frac{16\pi}{7}, -2\pi\right), \quad B = \left[-\frac{\pi}{2}, -\frac{2\pi}{7}\right),$$

$$C = \left[\frac{2\pi}{7}, \frac{\pi}{2}\right), \quad D = \left[2\pi, \frac{16\pi}{7}\right),$$

and we let $L = A \cup B \cup C \cup D$. Then

$$\text{(1.5)} \quad (A + 2\pi) \cup B \cup C \cup (D - 2\pi) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Thus $L$ is $2\pi$ translation congruent to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$L$ is a $2$-dilation generator of partition of $\mathbb{R}$ and since $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $2\pi$ translation congruent to a subset of $[0, 2\pi)$, Theorem 1.3.8 shows that $L$ is a frame set. Also

$$\tau(E) = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

and

$$\tau(L) = [0, \frac{1}{2}\pi) \cup \left[\frac{3}{2}\pi, 2\pi\right).$$

Hence

$$\tau(E) \cup \tau(L) = [0, 2\pi) \quad \text{and} \quad \tau(E) \cap \tau(L) = \Phi.$$

Now we use the following result:

**Proposition 1.3.10.** Let $E$ and $F$ be frame sets. Then $(E, F)$ is a strong complementary pair if and only if both $\tau(E) \cup \tau(F) = [0, 2\pi)$ and $\tau(E) \cap \tau(F)$ has measure zero.

By this proposition, the pair $(E, L)$, where $E$ and $L$ are as obtained above, is a strong complementary pair. Thus $L$ is a strong complementary frame set of $E$ and $\left(\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_E\right), \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}\chi_L\right)\right)$ is a super-wavelet.

The above two crucial results 1.3.6 and 1.3.9 are the essence of what is done in [22].
We conclude this section by giving the definition of superwavelet having length $n$.

**Definition 1.3.11.** [22] Suppose that $\eta_1, \ldots, \eta_n$ are normalized tight frame wavelets. The $n$-tuple $(\eta_1, \ldots, \eta_n)$ is a superwavelet of length $n$ if \( \{ U^k T^l \eta_1 \oplus \cdots \oplus U^k T^l \eta_n : k, l \in \mathbb{Z} \} \) is an orthonormal basis for $L^2(\mathbb{R})^n$.

It is shown in [22] that for any $n$, there is a superwavelet of length $n$. Just like in the $L^2(\mathbb{R})$ case, the superwavelet usually generates a normalized tight frame and to get an orthogonal basis, some extra conditions must be imposed on the initial low-pass filter from which the superwavelet is constructed such as the Cohen condition [10] or Lawton’s condition [34], Theorem 3.9 in [5]. In [18] and [17], a certain affine structure on the space $L^2(\mathbb{R})^n$ was introduced which was shown to admit multiresolution wavelet bases.

### 1.4 MRA Superwavelets

We have seen in the previous section that $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ has a wavelet (superwavelet of length 2). In this section we discuss whether MRA wavelet (i.e., wavelet made out of an MRA) can be found in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

The following theorem tells us that there is no MRA superwavelet in the usual sense, i.e., as given in Definition 1.1.9.

**Theorem 1.4.1.** [22] Let $\phi_1, \phi_2 \in L^2(\mathbb{R})$ and let $V_0$ be the closed subspace generated by $\{ T^l \phi_1 \oplus T^l \phi_2 : l \in \mathbb{Z} \}$. Suppose $V_0 \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and $V_0 \subset (U \oplus U) V_0$ and that

\[ \{ T^l \phi_1 \oplus T^l \phi_2 : l \in \mathbb{Z} \} \]
is an orthonormal basis for $V_0$. Then

$$\bigcup_{j \in \mathbb{Z}} (U^j \oplus U^j) V_0$$

is not dense in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Since MRA superwavelets cannot be defined similar to MRA wavelets, Han and Larson [22] have modified the definition of MRA superwavelet as follows:

**Definition 1.4.2.** A superwavelet $(\eta_1, \eta_2) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is an MRA superwavelet if each $\eta_1$ and $\eta_2$ are MRA frame wavelets.

**Remark 1.4.3.** In the case of MRA waveletes in $L^2(\mathbb{R})$, $\psi$ is an MRA wavelet if $\{U^jT^k\psi : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. However, $(\eta_1, \eta_2) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is an MRA superwavelet only when both $\eta_1$ and $\eta_2$ are MRA frame wavelets in $L^2(\mathbb{R})$.

**Proposition 1.4.4.** [11] For a frame set $G$, $\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}} \chi_G\right)$ is an MRA frame if and only if $G^* := \bigcup_{j=1}^{\infty} 2^{-j} G$ is $2\pi$-translation congruent to a subset of $[-\pi, \pi]$.

The following is an important result. In the proof given below we have explicitly constructed the function $\phi$.

**Proposition 1.4.5.** [11] There is an MRA superwavelet of length 2.

**Proof.** Consider $E$ and $L$ as in Example 1.3.9.

Also, let

$$E^* = \bigcup_{j=1}^{\infty} 2^{-j} E, \quad L^* = \bigcup_{j=1}^{\infty} 2^{-j} L.$$
Then $E^s = \left[ -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right)$ and

$$L^s = \left[ -\frac{8\pi}{7}, -\pi \right) \cup \left[ -\frac{4\pi}{7}, -\frac{\pi}{2} \right) \cup \left[ -\frac{2\pi}{7}, 0 \right) \cup \left[ 0, \frac{2\pi}{7} \right) \cup \left[ \frac{\pi}{2}, \frac{4\pi}{7} \right) \cup \left[ \pi, \frac{8\pi}{7} \right)$$

$E^s$ is $2\pi$-translation congruent to the set itself which is a subset of $[-\pi, \pi]$. The map $\phi$ defined by

$$\phi(s) = \begin{cases} 
    s + 2\pi, & s \in \left[ -\frac{8\pi}{7}, -\pi \right) \\
    s, & s \in \left[ -\frac{4\pi}{7}, -\frac{\pi}{2} \right) \cup \left[ -\frac{2\pi}{7}, 0 \right) \cup \left[ 0, \frac{2\pi}{7} \right) \cup \left[ \frac{\pi}{2}, \frac{4\pi}{7} \right) \\
    s - 2\pi, & s \in \left[ \pi, \frac{8\pi}{7} \right)
\end{cases}$$

shows that the set $L^s$ is $2\pi$-translation congruent to the set

$$\left[ -\pi, \frac{-6\pi}{7} \right) \cup \left[ -\frac{4\pi}{7}, -\frac{\pi}{2} \right) \cup \left[ -\frac{2\pi}{7}, 0 \right) \cup \left[ 0, \frac{2\pi}{7} \right) \cup \left[ \frac{\pi}{2}, \frac{4\pi}{7} \right) \cup \left[ \frac{6\pi}{7}, \pi \right),$$

which is a subset of $[-\pi, \pi]$. Hence by Proposition 1.4.4 both $\mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_E \right)$ and $\mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_L \right)$ are MRA frame wavelets. In Example 1.3.9 we have seen that

$$\left( \mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_E \right), \mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_L \right) \right)$$

is a superwavelet and hence by the discussion above and using the Definition 1.4.2, $\left( \mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_E \right), \mathcal{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \chi_L \right) \right)$ is an MRA superwavelet.

\section{1.5 Superwavelets via MRA}

In this section and the following sections we discuss the theory for the constructions of superwavelets via MRA giving examples [5]. For the discussion we need an abstract version of the situation existent on $L^2(\mathbb{R})$ and we replace $L^2(\mathbb{R})$
by an abstract Hilbert space $H$ and the dilation and translation operators are replaced by two unitaries $U$ and $T$ satisfying the relation $UTU^{-1} = T^N$ for some integer $N \geq 2$ and the integer $N$ is called the scale for $U$ and $T$. The standard translation and dilation operators have scale 2.

We have seen in Theorem 1.4.2 that MRA superwavelets do not exist in the usual sense in $L^2(\mathbb{R})^2$. i.e., the technique of multiresolution analysis breaks down, when multiplexing is required, if one just amplifies the steps used in the construction of MRA wavelets. In [5] multiresolution constructions have been realized for multiple signals, provided some slight modifications are done to the usual dilation and translation operators.

Hereinafter, by superwavelet via MRA we mean a superwavelet made from a multiresolution analysis (MRA).

**Notation 1.5.1.** We often identify functions $f$ on $\mathbb{T}$ with $2\pi$-periodic functions on $\mathbb{R}$ or with functions on the interval $[-\pi, \pi)$. The identification is given by $f(z) \leftrightarrow f(\theta)$ where $z = e^{-i\theta}$.

**Definition 1.5.2.** [5] A wavelet representation is a triple $\tilde{\pi} := (H, U, \pi)$ where $H$ is a Hilbert space, $U$ is a unitary on $H$ and $\pi$ is a representation of $L^\infty(\mathbb{T})$ on $H$ such that

$$U\pi(f)U^{-1} = \pi(f(z^N)), \quad f \in L^\infty(\mathbb{T})$$

(here, by $f(z^N)$ we mean the map $z \rightarrow f(z^N)$).

**Definition 1.5.3.** [5] A wavelet representation is called normal if for any sequence $(f_n)_{n \in \mathbb{N}}$ which converges pointwise a.e. to a function $f \in L^\infty(\mathbb{T})$ and such that $\|f_n\|_\infty \leq M$, $n \in \mathbb{N}$ for some $M > 0$, the sequence $\{\pi(f_n)\}$ converges to $\pi(f)$ in the strong operator topology.
Notation 1.5.4. $U$ is called the dilation and $T := \pi(z)$ the translation of the wavelet representation, where $z$ indicates the identity function on $\mathbb{T}$, $z \mapsto z$. We note that the identity function (denoted by $z$) on $\mathbb{T}$ is an element in $L^\infty(\mathbb{T})$.

Some of the examples in this chapter are outlined in [5]. For the sake of completeness, we give them with details.

Example 1.5.5. A classical example of a normal wavelet representation is the following: $H = L^2(\mathbb{R})$. We choose an integer $N \geq 2$.

$$U\xi(x) = \frac{1}{\sqrt{N}} \xi \left( \frac{x}{N} \right), \quad \xi \in L^2(\mathbb{R}),$$

and $\pi$ is defined by its Fourier transform

$$(1.6) \quad \hat{\pi}(f)(\xi) = f\xi,$$

where $f \in L^\infty(\mathbb{T})$, $\xi \in L^2(\mathbb{R})$.

It is enough to check that for $\xi \in H$

$$(U\pi(f)U^{-1}(\xi))\hat{\cdot} = (\pi(f(z^N))(\xi))\hat{\cdot}$$

Now

$$(U\pi(f)U^{-1}(\xi))\hat{\cdot} = U^{-1}\left( (\pi(f)U^{-1}(\xi))\hat{\cdot} \right)$$

$$= U^{-1}\left( \hat{\pi}(f)\left( U^{-1}(\xi) \right)\hat{\cdot} \right),$$

noting that $$(Tf)\hat{\cdot} = \hat{Tf}, \text{ where } \hat{Tf} = e^{-i\hat{\cdot}}$$

$$= U^{-1}\left( \hat{\pi}(f)U(\hat{\xi}) \right)$$

$$= U^{-1}\left( fU(\hat{\xi}) \right)$$
1.5. Superwavelets via MRA

\[
\begin{align*}
\hat{\mathcal{F}} f(N\cdot) & = \sqrt{N} f(N\cdot) U(\hat{\xi})(N\cdot) \\
& = \sqrt{N} f(N\cdot) \frac{1}{\sqrt{N}} \hat{\xi}(\cdot) \\
& = f(N\cdot) \hat{\xi} \\
& = (\pi(f(z^N))(\xi))^\cdot.
\end{align*}
\]

In particular (1.6) gives

\[\hat{T}\xi(x) = e^{-ix}\xi(x)\]

so that

\[T\xi(x) = \xi(x - 1),\]

for \(\xi \in L^2(\mathbb{R}), \ x \in \mathbb{R}.

We denote this normal wavelet representation by \(\mathcal{R}_0\).

**Example 1.5.6.** If \((H_i, U_i, \pi_i)\) are (normal) wavelet representations for \(i \in \{1, \ldots, n\}\), then \((\oplus_{i=1}^n H_i, \oplus_{i=1}^n U_i, \oplus_{i=1}^n \pi_i)\) is a (normal) wavelet representation called the direct sum of the given wavelet representations.

**Definition 1.5.7.** For a given scale \(N \geq 2\), an \(N\) cycle is a set \(\{z_1, \ldots, z_p\}\) of distinct points in \(\mathbb{T}\), such that \(z_1^N = z_2, z_2^N = z_3, \ldots, z_p^N = z_1\). \(p\) is called the length of the cycle. \(\{1\}\) is called the trivial cycle. \(\{\omega, \omega^2\}\), where \(\omega = e^{-\frac{2\pi i}{3}}\), is a 2-cycle having length 2.

**Example 1.5.8.** Let \(\tilde{\pi} = (H, U, \pi)\) be a (normal) wavelet representation. Let \(C := \{z_1, \ldots, z_p\}\) be a cycle and \(\alpha_1, \ldots, \alpha_p \in \mathbb{T}\). Define

\[H_{C,\alpha} := \underbrace{H \oplus H \oplus \ldots \oplus H}_{p \text{ times}}\]
and, for \( f \in L^\infty(\mathbb{T}) \), \( \xi_1, \ldots, \xi_p \in H \),

\[
U_{C, \alpha} (\xi_1, \ldots, \xi_p) := (\alpha_1 U \xi_2, \alpha_2 U \xi_3, \ldots, \alpha_{p-1} U \xi_p, \alpha_p U \xi_1),
\]

\[
\pi_{C, \alpha} (f) (\xi_1, \ldots, \xi_p) := (\pi (f (z_1 z)) \xi_1, \pi (f (z_2 z)) \xi_2, \ldots, \pi (f (z_p z)) \xi_p).
\]

Then \( \tilde{\pi}_{C, \alpha} := (H_{C, \alpha}, U_{C, \alpha}, \pi_{C, \alpha}) \) is a (normal) wavelet representation which we call the \textit{cyclic amplification} of \( \tilde{\pi} \) with cycle \( C \) and modulation \( \alpha \).

In particular,

\[
T_{C, \alpha} := \pi_{C, \alpha}(z)
\]

and

\[
T := \pi(z),
\]

where \( z \) is the identity map on \( \mathbb{T} \). Since \((\pi \circ (z \to z_1 z))(\xi_1) = z_1 T \xi_1 \), we have

\[
T_{C, \alpha} (\xi_1, \ldots, \xi_p) = (z_1 T \xi_1, \ldots, z_p T \xi_p)
\]

The cyclic amplification with the trivial cycle and \( \alpha_1 = 1 \) is the initial wavelet representation.

\textit{Notation} 1.5.9. When \( \alpha_1 = \ldots = \alpha_p = 1 \) in Example 1.5.8, we use the following notation:

\[
\tilde{\pi}_C := \tilde{\pi}_{C, \alpha}.
\]

\textbf{Example 1.5.10 (Special case of Example 1.5.8).} If \( C \) is a cycle of length \( p \) and \( \alpha_1, \ldots, \alpha_p \) are in \( \mathbb{T} \), we denote by

\[
\mathcal{R}_{C, \alpha} = (L^2(\mathbb{R})_{C, \alpha}, U_{C, \alpha}, \pi_{C, \alpha})
\]

the cyclic amplification \( (\mathcal{R}_0)_{C, \alpha} \) of the main representation \( \mathcal{R}_0 \). When all \( \alpha_i \) are
1.5. Superwavelets via MRA

1. we use the notation

\[ \mathcal{R}_C = (L^2(\mathbb{R})_C, U_C, \pi_C). \]

**Example 1.5.11.** Another important wavelet representation is the direct sum of wavelet representations associated to several \( N \) cycles. That is, if \( C_1, \ldots, C_n \) are distinct cycles and \( \alpha_1, \ldots, \alpha_n \) are some finite sets of numbers in \( \mathbb{T} \), then let 

\[ C := C_1 \cup \ldots \cup C_n, \]

and define

\[ \mathcal{R}_{C,\alpha} := \mathcal{R}_{C_1,\alpha_1} \oplus \ldots \oplus \mathcal{R}_{C_n,\alpha_n}, \]

and is called the wavelet representation associated to the cycles \( C_1, \ldots, C_n \) and the numbers \( \alpha_1, \ldots, \alpha_n \).

**Definition 1.5.12.** [5] Let \( (H, U, \pi) \) be a wavelet representation. A sequence \( (V_n)_{n\in\mathbb{Z}} \) of closed subspaces of \( H \) with the properties

i. \( V_n \subset V_{n+1} \);

ii. \( \bigcup_{n\in\mathbb{Z}} V_n = H \);

iii. \( \bigcap_{n\in\mathbb{Z}} V_n = \{0\} \);

iv. \( U(V_n) = V_{n-1} \);

v. There is a \( \phi \in V_0 \) such that \( \{ T^k \phi : k \in \mathbb{Z} \} \) is an orthonormal basis for \( V_0 \),

is called a *multiresolution analysis (MRA)* for the wavelet representation.

A vector \( \phi \) as in (v) for which there exists an MRA such that (i) to (iv) hold is called *orthogonal scaling vector*.

**Theorem 1.5.13.** [5] Let \( \mathcal{R}_{C,\alpha} \) be the wavelet representation in Example 1.5.11. Denote by \( e^{-i\theta_j} \) the \( j \)-th point of the cycle \( C_i \). Then \( \varphi_C = \varphi_{C_1} \oplus \varphi_{C_2} \oplus \ldots \oplus \varphi_{C_n} \)

is an orthogonal scaling vector for this wavelet representation if and only if the following conditions are satisfied:
1.5. Superwavelets via MRA

i. [Orthogonality] \[ \sum_{i=1}^{n} \sum_{k=1}^{p_i} \left( \text{Per} \left( |\hat{\phi}_{C_{i,k}}|^2 \right) \right) (\xi - \theta_{i,k}) = 1 \text{ for a.e. } \xi \in \mathbb{R}; \]

ii. [Scaling equation] There exists a function \( m_0 \in L^\infty(\mathbb{T}) \) such that for a.e. \( \xi \in \mathbb{R} \) and for all \( i \in \{1, \ldots, n\} : \)

\[
\alpha_{i,1} \sqrt{N} \hat{\phi}_{C_{i,2}}(N\xi) = m_0(\theta_{i,1} + \xi) \hat{\phi}_{C_{i,1}}(\xi), \\
\alpha_{i,2} \sqrt{N} \hat{\phi}_{C_{i,3}}(N\xi) = m_0(\theta_{i,2} + \xi) \hat{\phi}_{C_{i,2}}(\xi), \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\alpha_{i,p_i} \sqrt{N} \hat{\phi}_{C_{i,1}}(N\xi) = m_0(\theta_{i,p_i} + \xi) \hat{\phi}_{C_{i,p_i}}(\xi); 
\]

iii. [Cyclicity] For each \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, p_i\}, \hat{\phi}_{C_i} \) does not vanish on any subset \( E \) of \( \mathbb{R} \) invariant under dilations by \( N^{p_i} \) (i.e., \( N^{p_i}E = E \)) of positive measure.

Proposition 1.5.14. [5] Choose a scale \( N \geq 2 \). Let \( \hat{\pi} \) be a normal wavelet representation having an orthogonal scaling function \( \phi \) with non-degenerate filter \( m_0 \). Denote by \( (V_n)_{n \in \mathbb{Z}} \) the associated MRA. Assume that there are given the "high-pass filters" \( m_1, \ldots, m_{N-1} \in L^\infty(\mathbb{T}) \) satisfying

\[
(1.7) \quad \frac{1}{\sqrt{N}} \begin{bmatrix}
  m_0(z) & m_0(\rho z) & \cdots & m_0(\rho^{N-1}z) \\
  m_1(z) & m_1(\rho z) & \cdots & m_1(\rho^{N-1}z) \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{N-1}(z) & m_{N-1}(\rho z) & \cdots & m_{N-1}(\rho^{N-1}z)
\end{bmatrix}
\]

is unitary for a.e. \( z \in \mathbb{T}, \quad \left( \rho = e^{\frac{2\pi i}{N}} \right) \), and define \( \psi_i \in H \) by

\[
(1.8) \quad \psi_i =: U^{-1} \pi(m_i) \phi, \quad (i \in \{1, \ldots, N - 1\}).
\]

Then \( \{T^k \psi_i : k \in \mathbb{Z}, i \in \{1, \ldots, N - 1\}\} \) is an orthonormal basis for \( V_1 \ominus V_0 \).
and

\[ \{U^m T^n \psi_i : m \in \mathbb{Z}, \ n \in \mathbb{Z}, \ i \in \{1, \ldots, N-1\} \} \] is an orthonormal basis for \( H. \)

**Remark 1.5.15.** Thus by starting with a scaling vector \( \phi \), we can work out \( N - 1 \) wavelets, where \( N \) is the scale.