Frames Generated by Translation of a Function in $L^2(\mathbb{R})^n$

As mentioned in the introduction, our aim is to define frame MRA in $L^2(\mathbb{R})^n$. We first study the situation in $L^2(\mathbb{R})^2$ and these can be extended naturally to $L^2(\mathbb{R})^n$.

4.1 Translation and Dilation Operators on $L^2(\mathbb{R})^n$

Considering the Hilbert space $L^2(\mathbb{R})^n$ and taking the $N$ cycle $C := \{z_1, \ldots, z_n\}$ and $\alpha_1 = \ldots = \alpha_n = 1$ in Example 1.5.8, we have the translation operator given by

$$T_C (f_1 \oplus \cdots \oplus f_n) = z_1 T f_1 \oplus \cdots \oplus z_n T f_n$$

and the dilation operator given by

$$U_C (f_1 \oplus \cdots \oplus f_n) = U f_2 \oplus \cdots \oplus U f_n \oplus U f_1.$$
where we take $T$ and $U$ as the translation and dilation operators on $L^2(\mathbb{R})$ given by (1.1) and (1.2).

We begin with two simple basic theorems which will be helpful in proving results in this chapter.

**Theorem 4.1.1.** Let $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R})^n$. Then

$$\|f_1 \oplus \cdots \oplus f_n\|_{L^2(\mathbb{R})^n}^2 = \sum_{j=1}^{n} \|f_j\|_{L^2(\mathbb{R})}^2$$

Proof. The inner product on $L^2(\mathbb{R})^n$ is defined by

$$\langle f_1 \oplus \cdots \oplus f_n, g_1 \oplus \cdots \oplus g_n \rangle_{L^2(\mathbb{R})^n} = \sum_{j=1}^{n} \langle f_j, g_j \rangle_{L^2(\mathbb{R})}$$

Thus, we have

$$\|f_1 \oplus \cdots \oplus f_n\|_{L^2(\mathbb{R})^n}^2 = \langle f_1 \oplus \cdots \oplus f_n, f_1 \oplus \cdots \oplus f_n \rangle_{L^2(\mathbb{R})^n}$$

$$= \sum_{j=1}^{n} \langle f_j, f_j \rangle_{L^2(\mathbb{R})}$$

$$= \sum_{j=1}^{n} \|f_j\|_{L^2(\mathbb{R})}^2$$

\[\square\]

**Theorem 4.1.2.** For $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and $T_C$ and $U_C$ as in (2.4) and (2.5), we have

$$(T_C^k(f_1 \oplus f_2))^\sim = e_k(z_1^k f_1 \oplus z_2^k f_2)^\sim$$

and

$$(U_C(f_1 \oplus f_2))^\sim = U^{-1} \widehat{f_2} \oplus U^{-1} \widehat{f_1}$$
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Proof.

\[
(T^k_C(f_1 \oplus f_2)^\sim = (z_1^k T^k f_1 \oplus z_2^k T^k f_2)^\sim \\
= z_1^k T^k f_1 \oplus z_2^k T^k f_2, \text{ as } \hat{g_1} \oplus \hat{g_2} = \hat{g_1} \oplus \hat{g_2} \\
= z_1^k T^k f_1 \oplus z_2^k T^k f_2,
\]

by the linearity property of Fourier transform

\[
= z_1^k e_k \hat{f_1} \oplus z_2^k e_k \hat{f_2}, \text{ as } \hat{T^k f} = e_k \hat{f} \\
= e_k (z_1^k \hat{f_1} \oplus z_2^k \hat{f_2}) \\
= e_k (z_1^k \hat{f_1} \oplus z_2^k \hat{f_2})
\]

Also,

\[
(U_C(f_1 \oplus f_2)^\sim = (Uf_2 \oplus Uf_1)^\sim \\
= (Uf_2) \oplus (Uf_1) \\
= U^{-1} \hat{f_2} \oplus U^{-1} \hat{f_1}, \text{ as } \hat{Uf} = U^{-1} \hat{f} \text{ for } f \in L^2(\mathbb{R})
\]

\[
\square
\]

4.2 Frames Generated by Translation of a Function in $L^2(\mathbb{R})^n$

In the following theorem we associate two bounded $2\pi$ periodic function with a given family $X$ in $L^2(\mathbb{R})^2$ and prove that whenever these associated $2\pi$ periodic functions are bounded, then the Bessel map $L$ corresponding to $X$ is also bounded. The converse is also proved.

Theorem 4.2.1. Let $X = \{T^k_C\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and define
4.2. Frames Generated by Translation of a Function in \(L^2(\mathbb{R})^n\)

\[\Phi_1(\gamma) = \sum_{k \in \mathbb{Z}} |\hat{\phi}_1(\gamma + 2\pi k)|^2 \quad \text{and} \quad \Phi_2(\gamma) = \sum_{k \in \mathbb{Z}} |\hat{\phi}_2(\gamma + 2\pi k)|^2.\] Assume that the Bessel map \(\mathcal{L}\) associated with \(X\) exists. If \(\Phi_1 \leq A\) and \(\Phi_2 \leq B\) a.e. then \(\|\mathcal{L}\| \leq (A + B)^{\frac{1}{2}}.\) Conversely, \(\|\mathcal{L}\| \leq C^{\frac{1}{2}}\) implies \(\Phi_1 \leq C\) and \(\Phi_2 \leq C\) a.e.

**Proof.** By 3.1.12, we have \(\|\mathcal{L}\| = \|\mathcal{L}^*\|\). Let \(c\) be a finitely generated sequence . Then

\[
\|\mathcal{L}^*(c)\|^2 = \|\mathcal{L}^*(c)\|^2, \text{as Fourier transform is unitary}
\]

\[
= \left\| \left( \sum_{k \in \mathbb{Z}} c_k T_{\mathcal{L}}^k \phi_1 + \phi_2 \right) \right\|^2_{L^2(\mathbb{R})^2} = \left\| \left( \sum_{k \in \mathbb{Z}} c_k z_1^k T_k \phi_1 + z_2^k T_k \phi_2 \right) \right\|^2_{L^2(\mathbb{R})^2} = \left\| \left( \sum_{k \in \mathbb{Z}} c_k z_1^k T_k \phi_1 \right) \right\|^2_{L^2(\mathbb{R})} + \left\| \left( \sum_{k \in \mathbb{Z}} c_k z_2^k T_k \phi_2 \right) \right\|^2_{L^2(\mathbb{R})},
\]

using Theorem 4.1.1

\[
\begin{align*}
&= \left\| \sum_{k \in \mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1 \right\|^2_{L^2(\mathbb{R})} + \left\| \sum_{k \in \mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2 \right\|^2_{L^2(\mathbb{R})} \\
&= \int \left\| \sum_{k \in \mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1 \right\|^2_{L^2(\mathbb{R})} + \int \left\| \sum_{k \in \mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2 \right\|^2_{L^2(\mathbb{R})} \\
&= \int \left\| \sum_{k \in \mathbb{Z}} c_k z_1^k e_k(\gamma) \hat{\phi}_1(\gamma) \right\|^2_{L^2(\mathbb{R})} d\gamma + \int \left\| \sum_{k \in \mathbb{Z}} c_k z_2^k e_k(\gamma) \hat{\phi}_2(\gamma) \right\|^2_{L^2(\mathbb{R})} d\gamma \\
&= \sum_{l \in \mathbb{Z}} \int \left\| \sum_{k \in \mathbb{Z}} c_k z_1^k e_k(\gamma + 2\pi l) \right\|^2_{L^2(\mathbb{R})} |\hat{\phi}_1(\gamma + 2\pi l)|^2 d\gamma + \sum_{l \in \mathbb{Z}} \int \left\| \sum_{k \in \mathbb{Z}} c_k z_2^k e_k(\gamma + 2\pi l) \right\|^2_{L^2(\mathbb{R})} |\hat{\phi}_2(\gamma + 2\pi l)|^2 d\gamma 
\end{align*}
\]
4.2. Frames Generated by Translation of a Function in $L^2(\mathbb{R})^n$

\[
\int_T \left| \sum_{k \in \mathbb{Z}} c_k z^k_1 e_k(\gamma) \right|^2 \left| \sum_{l \in \mathbb{Z}} \hat{\phi}_1(\gamma + 2\pi l) \right|^2 d\gamma
+ \int_T \left| \sum_{k \in \mathbb{Z}} c_k z^k_2 e_k(\gamma) \right|^2 \left| \sum_{l \in \mathbb{Z}} \hat{\phi}_2(\gamma + 2\pi l) \right|^2 d\gamma,
\]

as $e_k$ is $2\pi$-periodic.

\[
\int_T \left| \sum_{k \in \mathbb{Z}} c_k z^k_1 e_k \right|^2 \Phi_1 + \int_T \left| \sum_{k \in \mathbb{Z}} c_k z^k_2 e_k \right|^2 \Phi_2
\]

We note that $\sum_{k \in \mathbb{Z}} c_k z^k_1 e_k$ is the Fourier transform of the sequence $(c_k z^k_1) \in l^2(\mathbb{Z})$ and $\sum_{k \in \mathbb{Z}} c_k z^k_2 e_k$ is the Fourier transform of the sequence $(c_k z^k_2) \in l^2(\mathbb{Z})$. Hence, by the Parseval-Plancherel theorem for $\mathbb{T}$ [27]

\[
\langle (c_k z^k_1), (c_k z^k_1) \rangle_{l^2(\mathbb{Z})} = \langle (c_k z^k_1), (c_k z^k_1) \rangle_{L^2(\mathbb{T})},
\]

we have

\[
\| (c_k) \|_{l^2(\mathbb{Z})}^2 = \int_T \left| \sum_{k \in \mathbb{Z}} c_k z^k_1 e_k \right|^2
\]

and, similarly,

\[
\| (c_k) \|_{l^2(\mathbb{Z})}^2 = \int_T \left| \sum_{k \in \mathbb{Z}} c_k z^k_2 e_k \right|^2.
\]

Thus, if $\Phi_1 \leq A$ and $\Phi_2 \leq B$ a.e. on $\mathbb{T}$, then

\[
\| L^* \|^2 = \sup \left\{ \| L^*(c) \|^2 : c = (c_k) \in l^2(\mathbb{Z}), \| c \| \leq 1 \right\}
\]

\[
< A \sup \left\{ \| (c_k z^k_1) \|^2 : \| (c_k) \| \leq 1 \right\} + B \sup \left\{ \| (c_k z^k_2) \|^2 : \| (c_k) \| \leq 1 \right\}
\]

\[
< A \sup \left\{ \| (c_k) \|^2 : \| (c_k) \| \leq 1 \right\} + B \sup \left\{ \| (c_k) \|^2 : \| (c_k) \| \leq 1 \right\}
\]

\[
< A + B
\]
4.2. Frames Generated by Translation of a Function in $L^2(\mathbb{R})^n$

so that

$$\|L\| \leq (A + B)^{\frac{1}{2}}.$$  

For the converse, consider for $\delta > 0$ the set $\Gamma = [\Phi_1 \geq C + \frac{\delta}{2}] \cup [\Phi_2 \geq C + \frac{\delta}{2}]$. Now, for any measurable set $\Gamma \subseteq \mathbb{T}$, there exists a sequence $\{p_n\}$ of trigonometric polynomials with $\|p_n\|_{L^2(\mathbb{T})} \leq |\Gamma|$ such that $\{p_n\}$ converges to $1_\Gamma$ (the characteristic function of the set $\Gamma$) except on a set of arbitrarily small measure. Thus, if the measure $|\Gamma|$ of $\Gamma$ was strictly greater than 0, there would be a finitely supported sequence $c$ with $\|c\|_{l^2(\mathbb{Z})} \leq |\Gamma|$ such that

$$\|L^* (c)\|^2 > |\Gamma| \left( C + \frac{\delta}{2} \right)$$

implies

$$\|L^* \left( \frac{c}{|\Gamma|} \right)\|^2 > |\Gamma| \left( C + \frac{\delta}{2} \right)$$

Hence

$$\|L^*\|^2 = \sup \left\{ \|L^* (c)\|^2 : \|c\|_{l^2(\mathbb{Z})} \leq 1 \right\} > C.$$

We can naturally extend the above result to $L^2(\mathbb{R})^n$ as follows:

**Proposition 4.2.2 (General version of Proposition 4.2.1).** Let

$$X = \{ T^k C \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z} \} \subset L^2(\mathbb{R})^n$$

and define $\Phi_i(\gamma) = \sum_{k \in \mathbb{Z}} \left| \hat{\phi_i}(\gamma + 2\pi k) \right|^2$, $i = 1, \ldots, n$. Denote the Bessel map associated with $X$ by $L$. If $\Phi_i \leq A_i < \infty$, $i = 1, \ldots, n$ a.e., then $\|L\| \leq (A_1 + \ldots + A_n)^{\frac{1}{2}}$. Conversely, $\|L\| \leq C^{\frac{1}{2}}$ implies $\Phi_i \leq C$, $i = 1, \ldots, n$ a.e.

The next theorem deals with a situation in which the given collection becomes
4.2. Frames Generated by Translation of a Function in $L^2(\mathbb{R}^n)$

a frame.

**Theorem 4.2.3.** Let $X = \{T_k^c \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and assume

\[ \Phi_1(\gamma) = \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_1(\gamma + 2\pi k) \right|^2 \quad \text{and} \quad \Phi_2(\gamma) = \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_2(\gamma + 2\pi k) \right|^2 \in L^\infty(\mathbb{T}). \]

Then $X$ has a well defined Bessel map $\mathcal{L}$ and

\[ N(\mathcal{L}^*) = \{ c = (c_k) \in l^2(\mathbb{Z}) : (c_k \omega^k) = 0 \text{ on } [\Phi_1 > 0] \text{ and } (c_k \omega^{2k}) = 0 \text{ on } [\Phi_2 > 0] \}. \]

Further, taking

\[ A_1 = \inf \{ a : [\Phi_1 \leq a] \cap [\Phi_1 > 0] > 0 \}, \]

\[ A_2 = \inf \{ a : [\Phi_2 \leq a] \cap [\Phi_2 > 0] > 0 \}, \]

\[ \underline{\text{esssup}} \Phi_1 = B_1 < \infty, \quad \text{and} \quad \underline{\text{esssup}} \Phi_2 = B_2 < \infty, \]

$X$ is a frame for $V_0 = \text{span} \{ T_k^c \phi_1 \oplus \phi_2 : k \in \mathbb{Z} \}$ with lower frame bound greater than or equal to $A_1 + A_2 > 0$ and upper frame bound less than or equal to $B_1 + B_2 < \infty$. This frame is called the frame generated by $\phi_1 \oplus \phi_2$.

**Proof.** $\Phi_1$ and $\Phi_2$ are essentially bounded implies that there are $C_1 > 0$ and $C_2 > 0$ such that

\[ |\Phi_1(x)| < C_1 \quad \text{and} \quad |\Phi_2(x)| < C_2 \quad \text{a.e.} \]

Taking

\[ \frac{C}{2} = \max \{ C_1, C_2 \}. \]

we have

\[ \Phi_1 \leq \frac{C}{2} \quad \text{and} \quad \Phi_2 \leq \frac{C}{2} \quad \text{a.e.} \]

Hence by Proposition 4.2.1, we have

\[ \| \mathcal{L} \| < C^{1/2}. \]
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Hence $\mathcal{L}$ takes values in $l^2(\mathbb{Z})$ and hence $\mathcal{L}$ is well-defined.

By the definition of $V_0$, $\{T^k_\mathcal{C}\phi_1 \oplus \phi_2 : k \in \mathbb{Z}\}$ is complete in $V_0$.

Let $c \in l^2(\mathbb{Z})$ be arbitrary. Then

$$\|\mathcal{L}^*(c)\|^2 = \left\| \left( \sum_{k \in \mathbb{Z}} c_k T^k_\mathcal{C}\phi_1 \oplus \phi_2 \right) \right\|^2_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})}$$

$$= \left\| \left( \sum_{k \in \mathbb{Z}} c_k z^k_1 T^k_\mathcal{C}\phi_1 \oplus z^k_2 T^k_\phi_2 \right) \right\|^2_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})}$$

$$= \left\| \left( \sum_{k \in \mathbb{Z}} c_k z^k_1 T^k_\mathcal{C}\phi_1 \right) \right\|^2_{L^2(\mathbb{R})} + \left\| \left( \sum_{k \in \mathbb{Z}} c_k z^k_2 T^k_\phi_2 \right) \right\|^2_{L^2(\mathbb{R})}$$

$$= \int_\mathbb{R} \left| \sum_{k \in \mathbb{Z}} c_k z^k_1 e_k \right|^2 \Phi_1 + \int_\mathbb{R} \left| \sum_{k \in \mathbb{Z}} c_k z^k_2 e_k \right|^2 \Phi_2$$

Hence

$$N(\mathcal{L}^*) = \left\{ c = (c_k) \in l^2(\mathbb{Z}) : (c_k z^k_1) = 0 \text{ on } [\Phi_1 > 0] \text{ and } (c_k z^k_2) = 0 \text{ on } [\Phi_2 > 0] \right\}.$$
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implies

$$A_1 \int_T \left| \sum_{k \in \mathbb{Z}} c_k z_1^k e_k \right|^2 + A_2 \int_T \left| \sum_{k \in \mathbb{Z}} c_k z_2^k e_k \right|^2 \leq \int_T \left| \sum_{k \in \mathbb{Z}} c_k z_1^k e_k \right|^2 \Phi_1 + \int_T \left| \sum_{k \in \mathbb{Z}} c_k \omega^k e_k \right|^2 \Phi_2 \leq B_1 \int_T \left| \sum_{k \in \mathbb{Z}} c_k z_1^k e_k \right|^2 + B_2 \int_T \left| \sum_{k \in \mathbb{Z}} c_k z_2^k e_k \right|^2$$

Hence

$$(A_1 + A_2) \|c\|_{\ell^2(\mathbb{Z})}^2 \leq \|\mathcal{L}^*(c)\|^2 \leq (B_1 + B_2) \|c\|_{\ell^2(\mathbb{Z})}^2$$

Consequently, by Proposition 3.1.13, $X$ is a frame with lower frame bound greater than or equal to $A_1 + A_2$ and upper frame bound less than or equal to $B_1 + B_2$. □

**Corollary 4.2.4.** For $X$, $V_0$, $A_1$, $A_2$, $B_1$ and $B_2$ as in Theorem 4.2.3, if $A_1 + A_2 = B_1 + B_2$, then $X$ is a tight frame for $V_0$ with bound $A_1 + A_2$.

**Corollary 4.2.5.** Take $X$, $V_0$, $A_1$, $A_2$, $B_1$ and $B_2$ as in Theorem 4.2.3 such that $A_1 = A_2 = B_1 = B_2 = A$. Then $X$ is a tight frame for $V_0$ with bound $2A$.

**Theorem 4.2.6 (General version of Theorem 4.2.3).** Let

$$X = \{ T_C^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z} \} \subset L^2(\mathbb{R})^n$$

and assume $\Phi_i(\gamma) = \sum_{k \in \mathbb{Z}} \left| \hat{\phi}_i(\gamma + 2\pi k) \right|^2$, $i = 1, \cdots, n$. Then $X$ has a well defined Bessel map $\mathcal{L}$ and

$$N(\mathcal{L}^*) = \{ c = (c_k) \in \ell^2(\mathbb{Z}) : \left( \hat{c}_k \hat{z}_1^k \right) = 0 \text{ on } [\Phi_i > 0], \ i = 1, \cdots, n. \}$$
4.2. Frames Generated by Translation of a Function in $L^2(\mathbb{R})^n$

Take $A_i = \inf \{ a : [\Phi_i \leq a] \cap [\Phi_i > 0] > 0 \}$ and $\text{esssup} \Phi_i = B_i < \infty$, $i = 1, \ldots, n$. Then $X$ is a frame for $V_0 = \overline{\text{span}} \{ T_k^c \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z} \}$ with lower frame bound greater than or equal to $A_1 + \cdots + A_n > 0$ and upper frame bound less than or equal to $B_1 + \cdots + B_n$.

**Corollary 4.2.7.** For $X$, $V_0$, $A_i$ and $B_i$ as in Theorem 4.2.6, if $A_1 + \cdots + A_n = B_1 + \cdots + B_n$, then $X$ is a tight frame for $V_0$ with bound $A_1 + \cdots + A_n$.

**Corollary 4.2.8.** For $X$, $V_0$, $A_i$ and $B_i$ as in Theorem 4.2.3 such that $A_1 = \cdots = A_n = B_1 = B_2 = A$, then $X$ is a tight frame for $V_0$ with bound $nA$.

In the next result we will see that if $\phi_1 \oplus \phi_2$ generates a frame for $V_0$, then elements of $V_0$ and $\phi_1 \oplus \phi_2$ are closely related by their Fourier transforms.

**Proposition 4.2.9.** Suppose $\{ T^k_c \phi_1 \oplus \phi_2 : k \in \mathbb{Z} \} \subseteq L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is a frame for its closed span $V_0$. Then

$$f_1 \oplus f_2 \in V_0 \Rightarrow \hat{f}_1 = F_1 \hat{\phi}_1 \text{ and } \hat{f}_2 = F_2 \hat{\phi}_2,$$

for some $F_1$, $F_2 \in L^2(\mathbb{T})$ depending on $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. In particular, for such an $f_1 \oplus f_2$, $\hat{f}_1 \oplus \hat{f}_2 = (0, 0)$ almost everywhere on the set $[\hat{\phi}_1 = 0] \cap [\hat{\phi}_2 = 0]$.

Conversely, if $\hat{f}_1 = F_1 \hat{\phi}_1$ and $\hat{f}_2 = F_2 \hat{\phi}_2$ with $F_1 = (c_k z_1^k)\hat{}$ and $F_2 = (c_k z_2^k)\hat{}$ for some sequence $(c_k) \in l^2(\mathbb{Z})$, then $f_1 \oplus f_2 \in V_0$.

**Proof.** Since $\{ T^k_c \phi_1 \oplus \phi_2 : k \in \mathbb{Z} \}$ is a frame for its closed span $V_0$, $f_1 \oplus f_2 \in V_0$ implies

$$f_1 \oplus f_2 = \sum_{k \in \mathbb{Z}} c_k T^k_c \phi_1 \oplus \phi_2,$$

for some sequence $c = (c_k) \in l^2(\mathbb{Z})$. That is,

$$f_1 \oplus f_2 = \sum_{k \in \mathbb{Z}} c_k z_1^k T^k_c \phi_1 \oplus z_2^k T^k \phi_2.$$
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Taking the Fourier transform of this equation gives

$$\hat{f}_1 \oplus \hat{f}_2 = \left( \sum_{k \in \mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1, \sum_{k \in \mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2 \right)$$

implies

$$\hat{f}_1 = F_1 \hat{\phi}_1 \text{ and } \hat{f}_2 = F_2 \hat{\phi}_2,$$

where $F_1 = \sum_{k \in \mathbb{Z}} c_k z_1^k e_k$ and $F_2 = \sum_{k \in \mathbb{Z}} c_k z_2^k e_k$. Here note that $F_1$ is the Fourier transform of the sequence $(c_k z_1^k)$ and $F_2$ that of $(c_k z_2^k)$. The fact that $F_1$ and $F_2 \in L^2(\mathbb{T})$ follows from Parseval’s theorem.

Conversely,

$$\hat{f}_1 = \sum_{k \in \mathbb{Z}} c_k z_1^k e_k \hat{\phi}_1 \text{ and } \hat{f}_2 = \sum_{k \in \mathbb{Z}} c_k z_2^k e_k \hat{\phi}_2$$

implies

$$\hat{f}_1 = \sum_{k \in \mathbb{Z}} c_k z_1^k (T^k \phi)_1 \text{ and } \hat{f}_2 = \sum_{k \in \mathbb{Z}} c_k z_2^k (T^k \phi)_2$$

implies

$$f_1 = \sum_{k \in \mathbb{Z}} c_k z_1^k T^k \phi_1 \text{ and } f_2 = \sum_{k \in \mathbb{Z}} c_k z_2^k T^k \phi_2$$

implies

$$f_1 \oplus f_2 = \sum_{k \in \mathbb{Z}} c_k T^k \phi_1 \oplus \phi_2$$

implies

$$f_1 \oplus f_2 \in V_0.$$

\[\square\]

**Proposition 4.2.10 (General version of Theorem 4.2.9).** Suppose

$$\{ T^k \phi_1 \oplus \cdots \oplus \phi_n : k \in \mathbb{Z} \} \subseteq L^2(\mathbb{R})^n$$

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is a frame for its closed span $V_0$. Then

$$f_1 \oplus \cdots \oplus f_n \in V_0 \Rightarrow \hat{f}_i = F_i \hat{\varphi}_i, i = 1, \cdots, n$$

for some $F_1, \cdots, F_n \in L^2(\mathbb{T})$ depending on $f_1 \oplus \cdots \oplus f_n \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Conversely, if $\hat{f}_i = F_i \hat{\varphi}_i$ with $F_i = (c_k z_i^k)^\sim$, $i = 1, \ldots, n$ for some sequence $(c_k) \in l^2(\mathbb{Z})$, then $f_1 \oplus \cdots \oplus f_2 \in V_0$. 