5.1 INTRODUCTION

In the fourth chapter, an EPQ model permitting two levels trade credit period without price discount. The model did consider the production cost. In this chapter price discount is allowed to maintain the demand. Since the price discount can be facilitated one if the production cost is known. Hence in this model a production cost is considered.

This work is based on the investigation by Chung K.J and Huang Y.F (2003) allowing permissible delay in payments and the model developed by Huang Y.F (2004) offering the price discount.

5.2 BASIC ASSUMPTIONS AND NOTATIONS

The notations used for the development of Economic Production Quantity (EPQ) models are listed below:
p : production rate per unit time.
d : actual demand for the product per unit time.
A : set up cost.
θ : a constant deterioration rate (unit/unit time).
h : inventory carrying cost per unit time.
r : price discount per unit time.
K : production cost per unit time.
I_e : interest earned per $ unit time.
I_p : interest charged per $ in stocks per unit time by the supplier.
M : the retailer’s trade credit period offered by supplier in time units.
N : the customer’s trade credit period offered by retailer in time units.
T : optimal cycle time.
T_1 : production period.
T_2 : time during which there is no production of the product i.e., T_2 = T - T_1.
I_1(t) : inventory level of the product during the production period, i.e., 0 ≤ t ≤ T_1.
I_2(t) : inventory level of the product during the period when there is no production i.e., T_1 ≤ t ≤ T_2.
I(M) : maximum inventory level of the product.
TVC (T) : total cost per unit time.
The following assumptions are used for the development of Economic Production Quantity models.

1. The demand rate for the product is known and is finite.
2. Shortage is not allowed.
3. An infinite planning horizon is assumed.
4. Once a unit of the product is produced, it is available to meet the demand.
5. Once the product starts deterioration, the production is terminated and the Price discount is provided.
6. The deterioration follows an exponential distribution.
7. There is no replacement or repair for a deteriorated item.
8. \( I_p \geq I_e, \ M \geq N \).
9. When \( T \geq M \), the account is settled at \( T = M \) and the retailer starts paying for the interest charges on the items in stock with rate \( I_p \). When \( T \leq M \), the account is settled at \( T=M \) and the retailer does not need to pay any interest charge.
10. The retailer can accumulate revenue and earn interest after its customer pays for the amount of purchasing cost to the retailer until the end of the trade credit period offered by the supplier. That is, the retailer can accumulate revenue and earn interest during the period \( N \) to \( M \) with rate \( I_e \) under the condition of trade credit.

5.3 MODEL DEVELOPMENT

The price discount and trade credit play an important role in this model. At time \( t = 0 \), the inventory level is zero. The production and supply
start simultaneously the inventory piles up at a rate of \( p-d \) in the interval \([0, T_1]\). There is no deterioration and the inventory reaches the maximum level \( I(M) \) at \( t= T_1 \). After the time \( T_1 \), the inventory starts deterioration and supply is continued at a discount rate. There is no fall in demand. When the inventory reduces to zero the production run begins. The inventory level of the product at time \( t \) over period \([0, T]\) has been represented by the differential equations:

\[
\frac{dI_1(t)}{dt} = p - d \quad \text{for } 0 \leq t \leq T_1 \tag{5.1}
\]

\[
\frac{dI_2(t)}{dt} + \theta I_2(t) = -d \quad \text{for } 0 \leq t \leq T_2 \tag{5.2}
\]

The boundary conditions associated with these equations are \( I_1(0) = 0 \) and \( I_2(T_2) = 0 \).

\[
I_1(t) = (p - d)t \quad \text{for } 0 \leq t \leq T_1 \tag{5.3}
\]

\[
I_2(t) = \frac{d}{\theta} \left( e^{\theta(T_2 - t)} - 1 \right) \quad \text{for } 0 \leq t \leq T_2 \tag{5.4}
\]

a) Production cost: The production cost per unit time is represented by the following equation:

\[
PC = pk \frac{T_1}{T} \tag{5.5}
\]
b) Setup cost: Setup cost is defined as the expenses incurred in setting up a machine, work center, or assembly line, to switch from one production job to the next. The setup cost per unit time is given by

\[ \text{SC} = \frac{A}{T} \]  

(5.6)

c) Holding cost (storage cost): The holding cost per unit time is given by

\[ \text{HC} = \frac{h}{T} \left[ \int_0^{T_1} I_1(t)dt + \int_0^{T_2} I_2(t)dt \right] \]

\[ = \frac{h}{T} \left[ \int_0^{T_1} (p-d)dt + \int_0^{T_2} \frac{d}{\theta} \left( e^{\theta(T_2-t)} - 1 \right)dt \right] \]

\[ = \frac{h}{T} \left[ (p-d) \frac{T_1^2}{2} + \frac{d}{\theta^2} (e^{\theta T_2} - \theta T_2 - 1) \right] \]  

(5.7)

d) Deterioration cost: The number of units that deteriorates in a cycle is the difference between the maximum inventory and the number of units used to meet the demand. Hence the deterioration cost per unit time is given as

\[ \text{DC} = \frac{k}{T} \left[ I_2(0) - \int_0^{T_2} d.d.t \right] \]

\[ = \frac{kd\theta T_2^2}{2T} \]  

(5.8)
e) Price discount: Price discount is offered as a fraction of production cost for the units in the period \([0,T_2]\). The equation associated with price discount represented as

\[
PD = \frac{kr}{T} \int_0^{T_2} d\cdot dt
\]

\[
= \frac{krdT_2}{T} \quad (5.9)
\]

The total cost per unit time is represented as

\[
TVC(T) = PC + SC + HC + DC + PD
\]

\[
TVC(T) = pk \frac{T_1}{T} + A + \frac{h}{T} \left[ (p - d) \frac{T_1^2}{2} + \frac{d}{\theta^2} \left( e^{\frac{MT}{T}} - \theta T_2 - 1 \right) \right] + \frac{k\theta T_1^2}{2T} + \frac{krdT_2}{T} \quad (5.10)
\]

From assumptions (8) and (9) there are three cases to discuss capital opportunity cost per unit time.

5.3.1 Case1 \((T_2 \geq M)\)

Capital opportunity cost per unit time=

\[
= \frac{klp}{T} \int_{M}^{T_1} I_2(t) dt - \frac{klc}{T} \int_{N}^{M} dt \cdot dt
\]

\[
= \frac{klp}{T \theta^2} \left( e^{\frac{p-d}{p} T-M} \right) - \theta \left( \frac{p-d}{p} T-M \right) - 1 \right) - \frac{klc}{2T} \left( M^2 - N^2 \right)
\]
Total cost per unit time, \( TVC_1(T) = p\frac{T_1}{T} + A\frac{T}{T} + kr\frac{T_2}{T} + \frac{kd\theta T^2}{2T} \)

\[
+ \frac{h}{T}\left[ (p - d)\frac{T_1^2}{2} + \frac{d}{\theta^2}\left(e^{\theta T_2} - \theta T_2 - 1\right)\right] + kr\frac{T_2}{T} + \\
\frac{kl_e d}{T\theta^2}\left(e^{\frac{p - d}{p - T - M}} - \theta\left[\frac{p - d}{p} T - M\right] - 1\right) - \frac{kl_e d}{2T}(M^2 - N^2) \text{ if } T > 0
\]

(5.11)

5.3.2 Case2 \( (N \leq T_2 \leq M) \)

Capital opportunity cost per unit time = \[-\frac{kl_e}{T}\left[ \int_{T_2}^{T} dt. dt + dT_2 (M - T_2) \right] \]

\[-= -\frac{kl_e d}{2T} \left[ 2T_2 M - N^2 - T_2^2 \right] \]

\[\therefore \text{ Total cost per unit time, } TVC_2(T) = p\frac{T_1}{T} + A\frac{T}{T} + \]

\[
+ \frac{h}{T}\left[ (p - d)\frac{T_1^2}{2} + \frac{d}{\theta^2}\left(e^{\theta T_2} - \theta T_2 - 1\right)\right] + \frac{kd\theta T^2}{2T} + kr\frac{T_2}{T} - \frac{kl_e d}{2T} \left[ 2T_2 M - N^2 - T_2^2 \right]
\]

(5.12)

5.3.3 Case3 \( (T_2 < N) \)

Capital opportunity cost per unit time = \[-\frac{kl_e}{T}\left[ \int_{T_2}^{M} dT_2. dt \right] \]

\[-= -\frac{kl_e d(p - d)}{p}[M - N] \]
Total cost per unit time, \( TVC_3(T) = \frac{pk}{T} + \frac{A}{T} + \)

\[
\frac{h}{T} \left[ (p-d) \frac{T_1^2}{2} + d \left( e^{\theta T_2} - \theta T_2 - 1 \right) \right] + \frac{k d \theta T_2^2}{2T} + k r d \left( \frac{T_2}{T} \right) - \frac{k l \epsilon (p-d)}{p} [M - N]
\]

(5.13)

According to the above arguments,

\[
TVC(T) = \begin{cases} 
TVC_1(T), & \text{if} \quad M \leq N \\
TVC_2(T), & \text{if} \quad N \leq T < M \\
TVC_3(T), & \text{if} \quad 0 < T < N 
\end{cases}
\]

\(TVC_1(M) = TVC_2(M)\) and \(TVC_2(N) = TVC_3(N)\), \(TVC(T)\) is continuous and well defined.

### 5.4 CONVEXITY

Here it has been shown that the three inventory functions derived in the above section are convex on their appropriate domains.

#### 5.4.1 Theorem

- \(TVC_1(T)\) is convex on \([M, \infty)\).
- \(TVC_2(T)\) is convex on \((0, \infty)\).
- \(TVC_3(T)\) is convex on \((0, \infty)\).
- \(TVC(T)\) is convex on \((0, \infty)\).

Before proving Theorem, the following required lemma is stated and proved.
5.4.2 Lemma

\[ e^{\frac{p-d}{p} T - m} - 1 - \theta \left( \frac{p-d}{p} \right) T e^{\frac{p-d}{p} T - m} + \left( \frac{p-d}{p} \right)^2 T^2 \theta^2 e^{\frac{p-d}{p} T - m} \]

if \( T_2 \geq M \)

\[ + \theta M - \frac{\theta^2 (M^2 - N^2)}{2} > 0 \]

Proof

Let \( g(T) = e^{\frac{p-d}{p} T - m} - 1 - \theta \left( \frac{p-d}{p} \right) T e^{\frac{p-d}{p} T - m} + \left( \frac{p-d}{p} \right)^2 T^2 \theta^2 e^{\frac{p-d}{p} T - m} \)

\[ + \theta M - \frac{\theta^2 (M^2 - N^2)}{2} > 0, \]

Then we have \( g'(T) = \left( \frac{p-d}{p} \right)^2 T^2 \theta^3 \frac{\theta^{(p-d)/p}}{2} \), so \( g(T) \) is increasing on \( [M, \infty) \)

and \( g(T) > g(M) = \frac{\theta^2 \left( \frac{p-d}{p} \right)^2 N^2}{2} > 0 \) if \( T \geq M \). This completes the proof.

Proof of Theorem

From equation (5.11), \( TVC_1(T) = -\frac{A}{T^2} + h(p-d) \frac{d^2}{2p^2} \)

\[ + \frac{d(k \theta + h)}{T^2 \theta^2} \left( \theta T \left[ \frac{p-d}{p} \right] e^{\frac{p-d}{p} T - m} - e^{\frac{p-d}{p} T - m} + 1 \right) \]

\[ + \frac{k l p d}{T^2 \theta^2} \left( \theta T \left[ \frac{p-d}{p} \right] e^{\theta (p-d) T - m} - e^{\theta (p-d) T - m} + 1 - \theta M \right) + \frac{k l d}{2T^2} (M^2 - N^2) \]
\[ \text{TVC}_1'(T) = \frac{2A}{T^3} + \frac{2d(k\theta + h)}{T^3\theta^2} \left( e^{\frac{p-d}{p}T} \left[ 1 - \theta \left( \frac{p-d}{p} \right) T + \left( \frac{p-d}{p} \right)^2 \frac{\theta^2T^2}{2} \right] - 1 \right) + \frac{2kl_p d}{T^3\theta^2} \left( \frac{\theta^2}{2} \left[ \frac{p-d}{p} \right]^2 T^2 e^{\frac{p-d}{p}T-M} + \theta M - \frac{\theta^2}{2}(M^2 - N^2) \right) \]

\[ \geq \frac{2A}{T^3} + \frac{2d(k\theta + h)}{T^3\theta^2} \left( e^{\frac{p-d}{p}T} \left[ 1 - e^{-\frac{p-d}{p}T-M} \right] - 1 \right) + \frac{2kl_p d}{T^3\theta^2} \left( \frac{\theta^2}{2} \left[ \frac{p-d}{p} \right]^2 T^2 e^{\frac{p-d}{p}T-M} + \theta M - \frac{\theta^2}{2}(M^2 - N^2) \right) \]

\[ = \frac{2A}{T^3} + \frac{2kl_p d}{T^3\theta^2} \left( e^{\frac{p-d}{p}T-M} \left[ 1 - \theta \left( \frac{p-d}{p} \right) T + \left( \frac{p-d}{p} \right)^2 \frac{\theta^2T^2}{2} \right] + \theta M - \frac{\theta^2}{2}(M^2 - N^2) \right) \]

Lemma imply that \( \frac{d^2\text{TVC}_1(T)}{dT^2} > 0 \) if \( T \geq M \), i.e., the second derivative is found to be positive. It is the basic requirement for \( T \) to be the minimum total cost in the EPQ model.

\[ \therefore \text{TVC}_1(T) \text{ is convex on } [M, \infty). \]
\[ TVC_2'(T) = -\frac{A}{T^2} + h(p-d) \frac{d^2}{2p^2} + d(k\theta + h) \left( \theta T \left[ \frac{p-d}{p} \right] e^{\left[ \frac{p-d}{p} \right]} - \theta \left[ \frac{p-d}{p} T \right] + 1 \right) \]

\[ -\frac{k l_c d}{2T^2} \left( N^2 - T^2 \right) \]

\[ TVC_2''(T) = \frac{2A}{T^3} + \frac{2d(k\theta + h)}{T^3 \theta^2} \left( e^{\left[ \frac{p-d}{p} \right]} \left( 1 - \theta \left( \frac{p-d}{p} \right) T + \left( \frac{p-d}{p} \right)^2 \theta^2 T^2 \right) - 1 \right) + \frac{k l_c d N^2}{T^3} \]

\[ \geq \frac{2A}{T^3} + \frac{2d(k\theta + h)}{T^3 \theta^2} \left( e^{\left[ \frac{p-d}{p} \right]} \left( 1 - \theta \left( \frac{p-d}{p} \right) T + \left( \frac{p-d}{p} \right)^2 \theta^2 T^2 \right) - 1 \right) + \frac{k l_c d N^2}{T^3} \]

\[ = \frac{2A}{T^3} + \frac{k l_c d N^2}{T^3} > 0. \]

and

\[ TVC_3'(T) = -\frac{A}{T^2} + h(p-d) \frac{d^2}{2p^2} + d(c\theta + h) \left( \theta T \left[ \frac{p-d}{p} \right] e^{\left[ \frac{p-d}{p} \right]} - \theta \left[ \frac{p-d}{p} T \right] + 1 \right). \]

\[ TVC_3''(T) = \frac{2A}{T^3} + \frac{2d(k\theta + h)}{T^3 \theta^2} \left( e^{\left[ \frac{p-d}{p} \right]} \left( 1 - \theta \left( \frac{p-d}{p} \right) T + \left( \frac{p-d}{p} \right)^2 \theta^2 T^2 \right) - 1 \right). \]

\[ \geq \frac{2A}{T^3} + \frac{2d(k\theta + h)}{T^3 \theta^2} \left( e^{\left[ \frac{p-d}{p} \right]} \left( 1 - \theta \left( \frac{p-d}{p} \right) T + \left( \frac{p-d}{p} \right)^2 \theta^2 T^2 \right) - 1 \right). \]

\[ = \frac{2A}{T^3} > 0. \]
i.e., the second derivative is found to be positive. It is the basic requirement for $T$ to be the minimum total cost in the EPQ model. Therefore, $TVC'_2(T)$ and $TVC'_3(T)$ is convex on $(0, \infty)$, respectively.

The case 1 implies that $TVC'_1(T)$ is increasing on $[M, \infty)$. Case 2 and case 3 implies that $TVC'_2(T)$ and $TVC'_3(T)$ is increasing on $(0, M]$. Since $TVC'_1(M) = TVC'_2(M)$ and $TVC'_2(N) = TVC'_3(N)$, then $TVC'(T)$ is increasing on $T > 0$. Consequently TVC (T) is convex on $T > 0$. Combining the above arguments completed the proof.

5.5 DETERMINATION OF THE OPTIMAL REPLENISHMENT CYCLE TIME $T$

Consider the following equations:

\[
TVC'_1(T) = 0 \quad (5.14)
\]
\[
TVC'_2(T) = 0 \quad (5.15)
\]
\[
TVC'_3(T) = 0 \quad (5.16)
\]

If the solution of Equation (5.14), (5.15) and (5.16) exists, then it is unique.

5.6 CONCLUSION

This study investigates EPQ model for retailer’s inventory system to minimize the cost under two level credit and price discount by determining the retailer’s optimal replenishment cycle time. Using theorem 1 the convexity
can also be proved. Theorem 5.4.1 establishes the convexity of the cost functions.

The supplier permits trade credit period enhancing the demand of the retailer. The supplier can also offer price discount on the products to promote the sales.