CHAPTER-IV

LEVEL SUBSETS OF INTUITIONISTIC FUZZY SUBFIELD

4.1 Introduction: In this chapter, the basic definitions and properties of the level subsets of an intuitionistic fuzzy subfield are discussed. Using these concepts, some results are established.

4.1.1 Definition: Let $A$ be an intuitionistic fuzzy subset of $X$. For $\alpha$ in $[0, 1]$, the sets $U(\mu_A, \alpha) = \{ x \in X : \mu_A(x) \geq \alpha \}$ and $L(\nu_A, \alpha) = \{ x \in X : \nu_A(x) \leq \alpha \}$ are called $\mu$-level $\alpha$-cut and $\nu$-level $\alpha$-cut of $A$, respectively.

4.1.2 Definition: Let $A$ be an intuitionistic fuzzy subset of $X$. For $\alpha$ and $\beta$ in $[0, 1]$, the level subset of $A$ is the set $A(\alpha, \beta) = \{ x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta \}$. This is called an intuitionistic fuzzy level subset of $A$. 
4.2–PROPERTIES OF LEVEL SUBSETS OF
   INTUITIONISTIC FUZZY SUBFIELD:

4.2.1 Theorem: Let $A$ be an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$. Then for $\alpha$ and $\beta$ in $[0,1]$ such that $\alpha \leq \mu_A(0)$, $\alpha \leq \mu_A(1)$ and $\beta \geq \nu_A(0)$, $\beta \geq \nu_A(1)$, $A(\alpha, \beta)$ is a subfield of $F$, where 0 and 1 are identity elements of $F$.

Proof: For all $x$ and $y$ in $A(\alpha, \beta)$, we have, $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$ and $\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$.

Now, $\mu_A(x - y) \geq \min \{\mu_A(x), \mu_A(y)\} \geq \min \{\alpha, \alpha\} = \alpha$, which implies that, $\mu_A(x - y) \geq \alpha$.

Now, $\mu_A(xy^{-1}) \geq \min \{\mu_A(x), \mu_A(y)\} \geq \min \{\alpha, \alpha\} = \alpha$, which implies that, $\mu_A(xy^{-1}) \geq \alpha$.

And also, $\nu_A(x - y) \leq \max \{\nu_A(x), \nu_A(y)\} \leq \max \{\beta, \beta\} = \beta$, which implies that, $\nu_A(x - y) \leq \beta$.

And also, $\nu_A(xy^{-1}) \leq \max \{\nu_A(x), \nu_A(y)\} \leq \max \{\beta, \beta\} = \beta$, which implies that, $\nu_A(xy^{-1}) \leq \beta$.

Therefore, $\mu_A(x - y) \geq \alpha$, $\mu_A(xy^{-1}) \geq \alpha$ and $\nu_A(x - y) \leq \beta$, $\nu_A(xy^{-1}) \leq \beta$,

we get $x - y$ and $xy^{-1}$ in $A(\alpha, \beta)$.

Hence $A(\alpha, \beta)$ is a subfield of $F$.

4.2.1 Definition: Let $A$ be an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$. The level subfield $A(\alpha, \beta)$, for $\alpha$ and $\beta$ in $[0,1]$ such that
\( \alpha \leq \mu_A(0), \alpha \leq \mu_A(1) \) and \( \beta \geq \nu_A(0), \beta \geq \nu_A(1) \) is called intuitionistic fuzzy level subfield of \( A \).

### 4.2.2 Theorem:

Let \( A \) be an intuitionistic fuzzy subfield of a field \( (F, +, \cdot) \). Then two level subfields \( A(\alpha_1, \beta_1) \) and \( A(\alpha_2, \beta_2) \) and \( \alpha_1 \) and \( \alpha_2 \) in \([0,1]\) and \( \beta_1, \beta_2 \) in \([0,1]\) and \( \alpha_1 \leq \mu_A(0), \alpha_2 \leq \mu_A(0), \alpha_1 \leq \mu_A(1), \alpha_2 \leq \mu_A(1) \) and \( \beta_1 \geq \nu_A(0), \beta_2 \geq \nu_A(0), \beta_1 \geq \nu_A(1), \beta_2 \geq \nu_A(1) \) with \( \alpha_2 < \alpha_1 \) and \( \beta_1 < \beta_2 \) of \( A \) are equal if and only if there is no \( x \) in \( F \) such that \( \alpha_1 > \mu_A(x) > \alpha_2 \) and \( \beta_1 < \nu_A(x) < \beta_2 \), where 0 and 1 are identity elements of \( F \).

**Proof:** Assume that \( A(\alpha_1, \beta_1) = A(\alpha_2, \beta_2) \).

Suppose there exists \( x \in F \) such that \( \alpha_1 > \mu_A(x) > \alpha_2 \) and \( \beta_1 < \nu_A(x) < \beta_2 \).

Then \( A(\alpha_1, \beta_1) \subseteq A(\alpha_2, \beta_2) \), which implies that \( x \) belongs to \( A(\alpha_2, \beta_2) \), but not in \( A(\alpha_1, \beta_1) \).

This is contradiction to \( A(\alpha_1, \beta_1) = A(\alpha_2, \beta_2) \).

Therefore there is no \( x \in F \) such that \( \alpha_1 > \mu_A(x) > \alpha_2 \) and \( \beta_1 < \nu_A(x) < \beta_2 \).

Conversely, if there is no \( x \in F \) such that \( \alpha_1 > \mu_A(x) > \alpha_2 \) and \( \beta_1 < \nu_A(x) < \beta_2 \).

Then \( A(\alpha_1, \beta_1) = A(\alpha_2, \beta_2) \).

### 4.2.3 Theorem:

Let \( (F, +, \cdot) \) be a field and \( A \) be an intuitionistic fuzzy subset of \( F \) such that \( A(\alpha, \beta) \) be a subfield of \( F \). If \( \alpha \) and \( \beta \) in \([0,1]\) satisfying \( \alpha \leq \mu_A(0), \alpha \leq \mu_A(1) \) and \( \beta \geq \nu_A(0), \beta \geq \nu_A(1) \), then \( A \) is an intuitionistic fuzzy subfield of \( F \), where 0 and 1 are identity elements of \( F \).
Proof: Let \((F, +, \cdot)\) be a field. For \(x\) and \(y\) in \(F\).

Let \(\mu_A(x) = \alpha_1\) and \(\mu_A(y) = \alpha_2\), \(\nu_A(x) = \beta_1\) and \(\nu_A(y) = \beta_2\).

Case (i): If \(\alpha_1 < \alpha_2\) and \(\beta_1 > \beta_2\), then \(x\) and \(y\) in \(A(\alpha_1, \beta_1)\).

As \(A(\alpha_1, \beta_1)\) is a subfield of \(F\), \(x - y, xy^{-1}\) in \(A(\alpha_1, \beta_1)\).

Now, \(\mu_A( x - y ) \geq \alpha_1 = \min \{ \alpha_1, \alpha_2 \} = \min \{ \mu_A(x), \mu_A(y) \}\),

which implies that \(\mu_A( x - y ) \geq \min \{ \mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y\) in \(F\).

Now, \(\mu_A( xy^{-1} ) \geq \alpha_1 = \min \{ \alpha_1, \alpha_2 \} = \min \{ \mu_A(x), \mu_A(y) \}\), which implies that \(\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).

And, \(\nu_A(x - y) \leq \beta_1 = \max \{ \beta_1, \beta_2 \} = \max \{ \nu_A(x), \nu_A(y) \}\), which implies that \(\nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y\) in \(F\).

And, \(\nu_A(xy^{-1}) \leq \beta_1 = \max \{ \beta_1, \beta_2 \} = \max \{ \nu_A(x), \nu_A(y) \}\), which implies that \(\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).

Case (ii): If \(\alpha_1 < \alpha_2\) and \(\beta_1 < \beta_2\), then \(x\) and \(y\) in \(A(\alpha_1, \beta_2)\).

As \(A(\alpha_1, \beta_2)\) is a subfield of \(F\), \(x - y, xy^{-1}\) in \(A(\alpha_1, \beta_2)\).

Now, \(\mu_A( x - y ) \geq \alpha_1 = \min \{ \alpha_1, \alpha_2 \} = \min \{ \mu_A(x), \mu_A(y) \}\), which implies that \(\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y\) in \(F\).

Now, \(\mu_A( xy^{-1} ) \geq \alpha_1 = \min \{ \alpha_1, \alpha_2 \} = \min \{ \mu_A(x), \mu_A(y) \}\), which implies that \(\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).

And, \(\nu_A(x - y) \leq \beta_1 = \max \{ \beta_1, \beta_2 \} = \max \{ \nu_A(x), \nu_A(y) \}\), which implies that \(\nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y\) in \(F\).

And, \(\nu_A(xy^{-1}) \leq \beta_1 = \max \{ \beta_1, \beta_2 \} = \max \{ \nu_A(x), \nu_A(y) \}\), which implies that \(\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).
**Case (iii):** If $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$, then $x$ and $y$ in $A(\alpha_2, \beta_1)$.

As $A(\alpha_2, \beta_1)$ is a subfield of $F$, $x - y$, $xy^{-1}$ in $A(\alpha_2, \beta_1)$.

Now, $\mu_A(x - y) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} = \min\{\mu_A(x), \mu_A(y)\}$, which implies that $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$, for all $x$ and $y$ in $F$.

Now, $\mu_A(xy^{-1}) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} = \min\{\mu_A(x), \mu_A(y)\}$, which implies that $\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\}$, for all $x$ and $y \neq 0$ in $F$.

And, $v_A(x - y) \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{v_A(x), v_A(y)\}$, which implies that $v_A(x - y) \leq \max\{v_A(x), v_A(y)\}$, for all $x$ and $y$ in $F$.

And, $v_A(xy^{-1}) \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{v_A(x), v_A(y)\}$, which implies that $v_A(xy^{-1}) \leq \max\{v_A(x), v_A(y)\}$, for all $x$ and $y \neq 0$ in $F$.

**Case (iv):** If $\alpha_1 > \alpha_2$ and $\beta_1 < \beta_2$, then $x$ and $y$ in $A(\alpha_2, \beta_2)$.

As $A(\alpha_2, \beta_2)$ is a subfield of $F$, $x - y$ and $xy^{-1}$ in $A(\alpha_2, \beta_2)$.

Now, $\mu_A(x - y) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} = \min\{\mu_A(x), \mu_A(y)\}$, which implies that $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$, for all $x$ and $y$ in $F$.

Now, $\mu_A(xy^{-1}) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} = \min\{\mu_A(x), \mu_A(y)\}$, which implies that $\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\}$, for all $x$ and $y \neq 0$ in $F$.

And, $v_A(x - y) \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{v_A(x), v_A(y)\}$, which implies that $v_A(x - y) \leq \max\{v_A(x), v_A(y)\}$, for all $x$ and $y$ in $F$.

And, $v_A(xy^{-1}) \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{v_A(x), v_A(y)\}$, which implies that $v_A(xy^{-1}) \leq \max\{v_A(x), v_A(y)\}$, for all $x$ and $y \neq 0$ in $F$.

**Case (v):** If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

It is trivial.
In all the cases, A is an intuitionistic fuzzy subfield of a field F.

4.2.4 Theorem: Let A be an intuitionistic fuzzy subfield of a field (F, +, .). If any two level subfields of A belongs to F, then their intersection is also level subfield of A in F.

Proof: For \( \alpha_1 \) and \( \alpha_2 \) in \([0,1]\), \( \beta_1 \) and \( \beta_2 \) in \([0,1]\), \( \alpha_1 \leq \mu_A(0) \) and \( \alpha_2 \leq \mu_A(0) \), \( \alpha_1 \leq \mu_A(1) \) and \( \alpha_2 \leq \mu_A(1) \), \( \beta_1 \geq \nu_A(0) \) and \( \beta_2 \geq \nu_A(0) \), \( \beta_1 \geq \nu_A(1) \) and \( \beta_2 \geq \nu_A(1) \), where 0 and 1 are identity elements of F.

Case (i): If \( \alpha_1 < \mu_A(x) < \alpha_2 \) and \( \beta_1 > \nu_A(x) > \beta_2 \), then \( A(\alpha_2, \beta_2) \subseteq A(\alpha_1, \beta_1) \).

Therefore, \( A(\alpha_1, \beta_1) \cap A(\alpha_2, \beta_2) = A(\alpha_2, \beta_2) \), but \( A(\alpha_2, \beta_2) \) is a level subfield of A.

Case (ii): If \( \alpha_1 > \mu_A(x) > \alpha_2 \) and \( \beta_1 < \nu_A(x) < \beta_2 \), then \( A(\alpha_1, \beta_1) \subseteq A(\alpha_2, \beta_2) \).

Therefore, \( A(\alpha_1, \beta_1) \cap A(\alpha_2, \beta_2) = A(\alpha_1, \beta_1) \), but \( A(\alpha_1, \beta_1) \) is a level subfield of A.

Case (iii): If \( \alpha_1 < \mu_A(x) < \alpha_2 \) and \( \beta_1 < \nu_A(x) < \beta_2 \), then \( A(\alpha_2, \beta_1) \subseteq A(\alpha_1, \beta_2) \).

Therefore, \( A(\alpha_1, \beta_1) \cap A(\alpha_2, \beta_1) = A(\alpha_2, \beta_1) \), but \( A(\alpha_2, \beta_1) \) is a level subfield of A.

Case (iv): If \( \alpha_1 > \mu_A(x) > \alpha_2 \) and \( \beta_1 > \nu_A(x) > \beta_2 \), then \( A(\alpha_1, \beta_2) \subseteq A(\alpha_2, \beta_1) \).

Therefore, \( A(\alpha_1, \beta_2) \cap A(\alpha_2, \beta_1) = A(\alpha_2, \beta_1) \), but \( A(\alpha_2, \beta_1) \) is a level subfield of A.

Case (v): If \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \), then \( A(\alpha_1, \beta_1) = A(\alpha_2, \beta_2) \).

In all cases, intersection of any two level subfield is a level subfield of A.
4.2.5 Theorem: Let $A$ be an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$. If $\alpha_i$ and $\beta_j$ in $[0,1]$, $\alpha_i \leq \mu_A(0)$, $\alpha_i \leq \mu_A(1)$, $\beta_j \geq \nu_A(0)$, $\beta_j \geq \nu_A(1)$ and $A(\alpha_i, \beta_j)$, $i$ and $j$ in $I$, is a collection of level subfields of $A$, then their intersection is also a level subfield of $A$.

Proof: It is trivial.

4.2.6 Theorem: Let $A$ be an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$. If any two level subfields of $A$ belongs to $F$, then their union is also a level subfield of $A$ in $F$.

Proof: Let $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ in $[0,1]$, $\alpha_1 \leq \mu_A(0)$ and $\alpha_2 \leq \mu_A(0)$, $\alpha_1 \leq \mu_A(1)$ and $\alpha_2 \leq \mu_A(1)$, $\beta_1 \geq \nu_A(0)$ and $\beta_2 \geq \nu_A(0)$, $\beta_1 \geq \nu_A(1)$ and $\beta_2 \geq \nu_A(1)$, where 0 and 1 are identity elements of $F$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$ and $\beta_1 > \nu_A(x) > \beta_2$, then $A(\alpha_2, \beta_2) \subseteq A(\alpha_1, \beta_1)$.

Therefore, $A(\alpha_1, \beta_1) \cup A(\alpha_2, \beta_2) = A(\alpha_1, \beta_1)$, but $A(\alpha_1, \beta_1)$ is a level subfield of $A$.

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$ and $\beta_1 < \nu_A(x) < \beta_2$, then $A(\alpha_1, \beta_1) \subseteq A(\alpha_2, \beta_2)$.

Therefore, $A(\alpha_1, \beta_1) \cup A(\alpha_2, \beta_2) = A(\alpha_1, \beta_1)$, but $A(\alpha_1, \beta_1)$ is a level subfield of $A$.

Case (iii): If $\alpha_1 < \mu_A(x) < \alpha_2$ and $\beta_1 < \nu_A(x) < \beta_2$, then $A(\alpha_2, \beta_1) \subseteq A(\alpha_1, \beta_2)$.

Therefore, $A(\alpha_2, \beta_1) \cup A(\alpha_1, \beta_2) = A(\alpha_1, \beta_2)$, but $A(\alpha_1, \beta_2)$ is a level subfield of $A$.

Case (iv): If $\alpha_1 > \mu_A(x) > \alpha_2$ and $\beta_1 > \nu_A(x) > \beta_2$, then $A(\alpha_1, \beta_2) \subseteq A(\alpha_2, \beta_1)$.
Therefore, \( A(\alpha_1, \beta_2) \cup A(\alpha_2, \beta_1) = A(\alpha_2, \beta_1) \), but \( A(\alpha_2, \beta_1) \) is a level subfield of \( A \).

**Case (v):** If \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \), then \( A(\alpha_1, \beta_1) = A(\alpha_2, \beta_2) \).

In all cases, union of any two level subfield is also a level subfield of \( A \).

**4.2.7 Theorem:** Let \( A \) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\). If \( \alpha_i \) and \( \beta_j \) in \([0,1] \), \( \alpha_i \leq \mu_A(0) \), \( \alpha_i \leq \mu_A(1) \) and \( \beta_j \geq \nu_A(0) \), \( \beta_j \geq \nu_A(1) \) and \( A(\alpha_i, \beta_j) \), \( i \) and \( j \) in \( I \), is a collection of level subfields of \( A \), then their union is also a level subfield of \( A \).

**Proof:** It is trivial.

**4.2.8 Theorem:** Let \( A \) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\). If \( A \) is an intuitionistic fuzzy characteristic subfield of \( F \), then each level subfield of \( A \) is a characteristic subfield of \( F \).

**Proof:** Let \( A \) be an intuitionistic fuzzy characteristic subfield of a field \((F, +, \cdot)\). Let \( x \) and \( y \) in \( F \), \( \alpha \) in \( \text{Im} \mu_A \), \( \beta \) in \( \text{Im} \nu_A \); \( f \) in \( \text{Aut}(F) \) and \( x \) in \( A(\alpha, \beta) \).

Now, \( \mu_A(f(x)) = \mu_A(x) \geq \alpha \).

Therefore, \( \mu_A(f(x)) \geq \alpha \).

And, \( \nu_A(f(x)) = \nu_A(x) \leq \beta \).

Therefore, \( \nu_A(f(x)) \leq \beta \).

Therefore, \( f(x) \in A(\alpha, \beta) \).

Hence, \( f(A(\alpha, \beta)) \subseteq A(\alpha, \beta) \) \( \text{---------------------------(1).} \)

For the reverse inclusion, let \( x \) in \( f(A(\alpha, \beta)) \) and \( y \) in \( F \) such that \( f(y) = x \).
Then, \( \mu_A(y) = \mu_A(f(y)) = \mu_A(x) \geq \alpha \).

And, \( \nu_A(y) = \nu_A(f(y)) = \nu_A(x) \leq \beta \).

Therefore, \( \mu_A(y) \geq \alpha \) and \( \nu_A(y) \leq \beta \).

Hence, \( y \in A(\alpha, \beta) \), when \( x \in f(\alpha, \beta) \).

Hence, \( f(A(\alpha, \beta)) \subseteq A(\alpha, \beta) \) --------------------------- (2).

From (1) and (2), we get \( A(\alpha, \beta) \) is a characteristic subfield of a field \( F \).

4.2.9 Theorem: Any subfield \( H \) of a field \((F, +, \cdot)\) can be realized as a level subfield of some intuitionistic fuzzy subfield of \( F \).

Proof: Let \( A \) be the intuitionistic fuzzy subset of a field \((F, +, \cdot)\) defined by

\[
\mu_A(x) = \begin{cases} 
\alpha & \text{if } x \in H, 0 < \alpha < 1 \\
0 & \text{if } x \notin H \end{cases}
\]

and \( \nu_A(x) = \begin{cases} 
\beta & \text{if } x \in H, 0 < \beta < 1 \\
0 & \text{if } x \notin H \end{cases} \)

and \( \alpha + \beta \leq 1 \), where \( H \) is a subfield of \( F \).

We claim that \( A \) is an intuitionistic fuzzy subfield of \( F \).

Let \( x \) and \( y \) in \( F \).

If \( x \) and \( y \) in \( H \), then \( x-y, xy^{-1} \) in \( H \), since \( H \) is a subfield of \( F \),

\( \mu_A(x-y) = \alpha, \mu_A(xy^{-1}) = \alpha, \mu_A(x) = \alpha, \mu_A(y) = \alpha \).

So, \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \)

\( \mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \neq e \) in \( F \).
Also, $\nu_A(x - y) = \beta, \nu_A(xy^{-1}) = \beta, \nu_A(x) = \beta, \nu_A(y) = \beta$.

So, $\nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \in F$

$\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \neq e \in F.$

If $x \in H$, then $x - y, xy^{-1}$ not in $H$.

Then, $\mu_A(x - y) = 0, \mu_A(xy^{-1}) = 0, \mu_A(x) = \alpha, \mu_A(y) = 0.$

Therefore, $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}, \text{ for all } x \text{ and } y \in F,$

$\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}, \text{ for all } x \text{ and } y \neq e \in F.$

And $\nu_A(x - y) = 0, \nu_A(xy^{-1}) = 0, \nu_A(x) = \beta, \nu_A(y) = 0.$

Therefore, $\nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \in F,$

$\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \neq e \in F.$

If $x$ and $y$ not in $H$, then $x - y, xy^{-1}$ not belong to $H$.

Clearly $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}, \text{ for all } x \text{ and } y \in F,$

$\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}, \text{ for all } x \text{ and } y \neq e \in F.$

Also, $\nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \in F,$

$\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \neq e \in F.$

In any case, $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}, \text{ for all } x \text{ and } y \in F$ and

$\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}, \text{ for all } x \text{ and } y \neq e \in F$ and

$\nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \in F$ and

$\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}, \text{ for all } x \text{ and } y \neq e \in F.$

Thus in all the cases, $A$ is an intuitionistic fuzzy subfield of $F.$
**4.2.10 Theorem:** Let $f$ be any mapping from a field $F_1$ to $F_2$ and let $A$ be an intuitionistic fuzzy subfield of $F_1$. Then for $\alpha$ and $\beta$ in $[0,1]$, we have

$$f(A(\alpha, \beta)) = \bigcap_{\alpha > \varepsilon_0 > 0, \beta > \varepsilon_2 > 0} f(A(\alpha - \varepsilon_1, \beta + \varepsilon_2)).$$

**Proof:** Suppose that $\alpha$ and $\beta$ in $[0,1]$ and $y = f(x)$ in $F_2$.

If $y$ in $f(A(\alpha, \beta))$, then $f(\mu_A)(y) = \sup_{x \in f^{-1}(y)} \{\mu_A(x)\} \geq \alpha$ and $f(\nu_A)(y) = \inf_{x \in f^{-1}(y)} \{\nu_A(x)\} \leq \beta$.

Therefore, for every real number $\varepsilon_1, \varepsilon_2 > 0$, there exist $x_0 \in f^{-1}(y)$ such that $\mu_A(x_0) > \alpha - \varepsilon_1$ and $\nu_A(x_0) < \beta + \varepsilon_2$.

So, for every $\varepsilon_1, \varepsilon_2 > 0$, $y = f(x_0) \in f(A(\alpha - \varepsilon_1, \beta + \varepsilon_2))$ and hence

$$y \in \bigcap_{\alpha > \varepsilon_0 > 0, \beta > \varepsilon_2 > 0} f(A(\alpha - \varepsilon_1, \beta + \varepsilon_2)).$$

Therefore, $f(A(\alpha, \beta)) \subseteq \bigcap_{\alpha > \varepsilon_0 > 0, \beta > \varepsilon_2 > 0} f(A(\alpha - \varepsilon_1, \beta + \varepsilon_2)).$ ...............(1)

Conversely, $y \in \bigcap_{\alpha > \varepsilon_0 > 0, \beta > \varepsilon_2 > 0} f(A(\alpha - \varepsilon_1, \beta + \varepsilon_2))$, then for each $\varepsilon_1, \varepsilon_2 > 0$, we have

$$y \in f(A(\alpha - \varepsilon_1, \beta + \varepsilon_2))$$

and there exist $x_0 \in A(\alpha - \varepsilon_1, \beta + \varepsilon_2)$ such that $y = f(x_0)$.

Therefore for each $\varepsilon_1, \varepsilon_2 > 0$, there exist $x_0 \in f^{-1}(y)$ and $\mu_A(x_0) \geq \alpha - \varepsilon_1$ and $\nu_A(x_0) \leq \beta + \varepsilon_2$.

Hence, $f(\mu_A)(y) = \sup_{x \in f^{-1}(y)} \{\mu_A(x)\} \geq \sup_{\alpha > \varepsilon_0 > 0} \{\alpha - \varepsilon_1\} = \alpha$ and
\[ f(\nu_A)(y) = \inf_{x, y \in f^{-1}(y)} \{ \nu_A(x) \} \leq \inf_{\beta > \varepsilon > 0} \{ \beta + \varepsilon \} = \beta. \]

So, \( y \in f(A_{(\alpha, \beta)}) \).

Therefore, \[ \bigcap_{\alpha > \varepsilon > 0, \beta > \varepsilon > 0} f(A_{(\alpha-\varepsilon, \beta+\varepsilon)}) \subseteq f(A_{(\alpha, \beta)}) \quad \text{………(2).} \]

From (1) and (2) we get, \( f(A_{(\alpha, \beta)}) = \bigcap_{\alpha > \varepsilon > 0, \beta > \varepsilon > 0} f(A_{(\alpha-\varepsilon, \beta+\varepsilon)}) \).

**4.2.11 Theorem:** Let \( A \) be an intuitionistic fuzzy subset of a set \( X \). Then \( \mu_A(x) = \max \{ \alpha / x \in A_{(\alpha, \beta)} \} \) and \( \nu_A(x) = \min \{ \beta / x \in A_{(\alpha, \beta)} \} \), where \( x \in X \).

**Proof:** Let \( \alpha_1 = \max \{ \alpha / x \in A_{(\alpha, \beta)} \} \) and \( \varepsilon > 0 \) be arbitrary.

Then \( \alpha_1 - \varepsilon < \max \{ \alpha / x \in A_{(\alpha, \beta)} \} \),

which implies that \( \alpha_1 - \varepsilon < \alpha \), for some \( \alpha \) such that \( x \in A_{(\alpha, \beta)} \).

That is, \( \alpha_1 - \varepsilon < \mu_A(x) \), since \( \mu_A(x) \geq \alpha \).

Therefore, \( \alpha_1 \leq \mu_A(x) \), since \( \varepsilon > 0 \) is arbitrary \quad \text{------------------------ (1).} \)

Now, assume that \( \mu_A(x) = s \).

Then \( x \in A_{(s, s)} \) and so \( s \in \{ \alpha / x \in A_{(\alpha, \beta)} \} \).

Hence \( s \leq \max \{ \alpha / x \in A_{(\alpha, \beta)} \} \), where \( \mu_A(x) \leq \alpha_1 \quad \text{------------------------ (2).} \)

From (1) and (2), we get \( \mu_A(x) = \alpha_1 = \max \{ \alpha / x \in A_{(\alpha, \beta)} \} \).

And, let \( \alpha_2 = \min \{ \beta / x \in A_{(\alpha, \beta)} \} \) and \( \varepsilon > 0 \) be arbitrary.

Then \( \alpha_2 + \varepsilon > \min \{ \beta / x \in A_{(\alpha, \beta)} \} \),
which implies that $\alpha_2 + \varepsilon > \beta$, for some $\beta$ such that $x \in A_{(\alpha, \beta)}$.

That is, $\alpha_2 + \varepsilon > \nu_A(x)$, since $\nu_A(x) \leq \beta$.

Therefore, $\alpha_2 \geq \nu_A(x)$, since $\varepsilon > 0$ is arbitrary ------------------- (3).

Now, assume that $\nu_A(x) = t$.

Then $x \in A_{(\nu, t)}$, and so $t \in \{ \beta / x \in A_{(\alpha, \beta)} \}$.

Hence $t \leq \min \{ \beta / x \in A_{(\alpha, \beta)} \}$, implies $\nu_A(x) \geq \alpha_2$ --------------------- (4).

From (3) and (4), we get, $\nu_A(x) = \alpha_2 = \min \{ \beta / x \in A_{(\alpha, \beta)} \}$.

4.2.12 Theorem: Any two different intuitionistic fuzzy subfields of a field may have identical family of level subfields.

Proof: We consider the following example:

Consider the field $F = Z_5 = \{ 0, 1, 2, 3, 4 \}$ with addition modulo 5 and multiplication modulo 5 operations.

Define intuitionistic fuzzy subsets $A$ and $B$ of $F$ by

$A = \{ \langle 0, 0.7, 0.1 \rangle, \langle 1, 0.5, 0.4 \rangle, \langle 2, 0.5, 0.4 \rangle, \langle 3, 0.5, 0.4 \rangle, \langle 4, 0.5, 0.4 \rangle \}$ and $B = \{ \langle 0, 0.8, 0.2 \rangle, \langle 1, 0.6, 0.3 \rangle, \langle 2, 0.6, 0.3 \rangle, \langle 3, 0.6, 0.3 \rangle, \langle 4, 0.6, 0.3 \rangle \}$.

Clearly $A$ and $B$ are two different intuitionistic fuzzy subfields of $F$.

And, $\text{Im} \mu_A = \{ 0.7, 0.5 \}$, $\text{Im} \nu_A = \{ 0.1, 0.4 \}$.

The level subfields of $A$ are $A_{(0.7, 0.1)} = A_{(0.7, 0.4)} = A_{(0.5, 0.1)} = \{ 0 \}$,

$A_{(0.5, 0.4)} = \{ 0, 1, 2, 3, 4 \} = F$.

And, $\text{Im} \mu_B = \{ 0.8, 0.6 \}$, $\text{Im} \nu_B = \{ 0.2, 0.3 \}$. 
The level subfields of B are $B_{(0.8,0.2)} = B_{(0.8,0.3)} = B_{(0.6,0.2)} = \{0\}$, 
$B_{(0.6,0.3)} = \{0, 1, 2, 3, 4\} = F$.

Thus the two intuitionistic fuzzy subfields A and B have the same family of level subfields.

4.2.13 Theorem: Let I be the subset of [0,1] and let $(F, +, \cdot)$ be a field with subfields $\{H_i\}$, $i \in I$ such that $\cup H_i = F$ and $i < j$ implies that $H_i \subset H_j$. Then an intuitionistic fuzzy subset A of F defined by $\mu_A(x) = \max \{ i / x \in H_i \}$ and $\nu_A(x) = \min \{ i / x \in H_i \}$ is an intuitionistic fuzzy subfield of F.

Proof: Let A be an intuitionistic fuzzy subset of F defined by

$\mu_A(x) = \max \{ i / x \in H_i \}$ and $\nu_A(x) = \min \{ i / x \in H_i \}$, where $i \in I \subseteq [0,1]$.

Let $x$ and $y$ in F and $\mu_A(x) = m_1$ and $\mu_A(y) = n_1$.

If $\mu_A(x+y) = \max \{ i / x+y \in H_i \} < \min\{ m_1, n_1 \}$, then there exists j such that $x$ and $y$ are elements of $H_j$, but $x+y$ is not an element of $H_j$, since $H_j$ is a subfield of F, this is a contradiction.

Therefore, $\mu_A(x+y) \geq \min \{m_1, n_1\}$,

which implies that $\mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in F.

Clearly $\mu_A(-x) = \mu_A(x)$ for all $x$ in F.

If $\mu_A(xy) = \max \{ i / xy \in H_i \} < \min\{ m_1, n_1 \}$, then there exists j such that $x$ and $y$ are elements of $H_j$, but $xy$ is not an element of $H_j$, since $H_j$ is a subfield of F, this is a contradiction.

Therefore, $\mu_A(xy) \geq \min \{m_1, n_1\}$,

which implies that $\mu_A(xy) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in F.
Clearly $\mu_A(x^{-1}) = \mu_A(x)$, for all $x$ in $F\setminus \{e\}$.

Also, $\nu_A(x) = m_2$ and $\nu_A(y) = n_2$.

If $\nu_A(x+y) = \min \{ i \mid x+y \in H_i \} > \max \{ m_2, n_2 \}$, then there exists $j$ such that $x$ and $y$ are elements of $H_j$, but $x+y$ is not an element of $H_j$, since $H_j$ is a subfield of $F$, this is a contradiction.

Therefore, $\nu_A(x+y) \leq \max \{ m_2, n_2 \}$, which implies that $\nu_A(x+y) \leq \max \{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y$ in $F$.

Clearly $\nu_A(-x) = \nu_A(x)$, for all $x$ in $F$.

If $\nu_A(xy) = \min \{ i \mid xy \in H_i \} > \max \{ m_2, n_2 \}$, then there exists $j$ such that $x$ and $y$ are elements of $H_j$, but $xy$ is not an element of $H_j$, since $H_j$ is a subfield of $F$, this is a contradiction.

Therefore, $\nu_A(xy) \leq \max \{ m_2, n_2 \}$, which implies that $\nu_A(xy) \leq \max \{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y$ in $F$.

Clearly $\nu_A(x^{-1}) = \nu_A(x)$, for all $x$ in $F\setminus \{e\}$.

Hence $A$ is an intuitionistic fuzzy subfield of $F$.

4.2.14 Theorem: Let $A$ and $B$ be intuitionistic fuzzy subsets of the sets $G$ and $H$ respectively and $\alpha$ and $\beta$ in $[0,1]$. Then $(A \times B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \times B_{(\alpha,\beta)}$.

Proof: Let $\alpha$ and $\beta$ be in $[0,1]$ and $(x,y)$ be in $(A \times B)_{(\alpha,\beta)}$.

Now, $(x,y) \in (A \times B)_{(\alpha,\beta)}$

$\iff \mu_{AXB}(x,y) \geq \alpha$ and $\nu_{AXB}(x,y) \leq \beta$

$\iff \min \{ \mu_A(x), \mu_B(y) \} \geq \alpha$ and $\max \{ \nu_A(x), \nu_B(y) \} \leq \beta$

$\iff \{ \mu_A(x) \geq \alpha \text{ and } \mu_B(y) \geq \alpha \} \text{ and } \{ \nu_A(x) \leq \beta \text{ and } \nu_B(y) \leq \beta \}$
\[ \Leftrightarrow \{ \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta \} \text{ and } \{ \mu_B(y) \geq \alpha \text{ and } \nu_B(y) \leq \beta \} \]
\[ \Leftrightarrow x \in A_{(\alpha,\beta)} \text{ and } y \in B_{(\alpha,\beta)} \]
\[ \Leftrightarrow (x,y) \in A_{(\alpha,\beta)} \times B_{(\alpha,\beta)} \]

Therefore, \((A \times B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \times B_{(\alpha,\beta)}\).

**4.2.15 Theorem:** Let \((F, +, \cdot)\) be a finite field and \(A\) be an intuitionistic fuzzy subfield of \(F\). If \(\alpha, \gamma\) are elements of the image set of \(\mu_A\) of \(A\) and \(\beta, \delta\) are elements of the image set of \(\nu_A\) of \(A\) such that \(A_{(\alpha,\beta)} = A_{(\gamma,\delta)}\), then need not be \(\alpha = \gamma\) and \(\beta = \delta\).

**Proof:** We consider the following example:

Consider the field \(F = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}\) with addition modulo 5 and multiplication modulo 5 operations.

Define intuitionistic fuzzy subfield \(A\) by \(A = \{\langle 0, 0.7, 0.1 \rangle, \langle 1, 0.5, 0.4 \rangle, \langle 2, 0.5, 0.4 \rangle, \langle 3, 0.5, 0.4 \rangle, \langle 4, 0.5, 0.4 \rangle\}\).

If \(\alpha = 0.7, \gamma = 0.5\), are in \(\mu_A\) of \(A\) and \(\beta = 0.1, \delta = 0.1\) are in \(\nu_A\) of \(A\), then \(A_{(0.7,0.1)} = \{0\}\), \(A_{(0.5,0.1)} = \{0\}\) are the level subfields of \(F\).

Clearly \(\alpha \neq \gamma\).

**Result:** In a fuzzy subfield, if \(t_i, t_j\) are elements of the image set of \(A\) such that \(A_{t_i} = A_{t_j}\), then \(t_i = t_j\).

**4.2.16 Theorem:** Let \(A\) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\). Then for \(\alpha\) and \(\beta\) in \([0,1]\) such that \(\alpha \leq \mu_A(0), \alpha \leq \mu_A(1)\) and \(\beta \geq \nu_A(0), \beta \geq \nu_A(1)\), \(\mu\)-level \(\alpha\)-cut \(U(\mu_A, \alpha)\) is a subfield of \(F\), where 0 and 1 are identity elements of \(F\).
Proof: For all $x$ and $y$ in $A(\alpha, \beta)$, we have, $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$.

Now, $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \} \geq \min \{ \alpha, \alpha \} = \alpha$,

which implies that, $\mu_A(x - y) \geq \alpha$.

Now, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \} \geq \min \{ \alpha, \alpha \} = \alpha$,

which implies that, $\mu_A(xy^{-1}) \geq \alpha$.

Therefore, $\mu_A(x - y) \geq \alpha$ and $\mu_A(xy^{-1}) \geq \alpha$, we get $x - y$ and $xy^{-1}$ in $U(\mu_A, \alpha)$.

Hence $U(\mu_A, \alpha)$ is a subfield of $F$.

4.2.17 Theorem: Let $A$ be an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$. Then for $\alpha$ and $\beta$ in $[0,1]$ such that $\alpha \leq \mu_A(0)$, $\alpha \leq \mu_A(1)$ and $\beta \geq \nu_A(0)$, $\beta \geq \nu_A(1)$, $\nu$-level $\beta$-cut $L(\nu_A, \beta)$ is a subfield of $F$, where 0 and 1 are identity elements of $F$.

Proof: For all $x$ and $y$ in $A(\alpha, \beta)$, we have, $\nu_A(x) \leq \beta$ and $\nu_A(y) \leq \beta$.

Now, $\nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \} \leq \max \{ \beta, \beta \} = \beta$,

which implies that, $\nu_A(x - y) \leq \beta$.

And also, $\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \} \leq \max \{ \beta, \beta \} = \beta$,

which implies that, $\nu_A(xy^{-1}) \leq \beta$.

Therefore, $\nu_A(x - y) \leq \beta$ and $\nu_A(xy^{-1}) \leq \beta$, we get $x - y$ and $xy^{-1}$ in $L(\nu_A, \beta)$.

Hence $L(\nu_A, \beta)$ is a subfield of $F$. 

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4.3–HOMOMORPHISM AND ANTI-HOMOMORPHISM OF LEVEL SUBFIELDS OF INTUITIONISTIC FUZZY SUBFIELDS

4.3.1 Theorem: Let \((F, +, \cdot)\) and \((F', +, \cdot)\) be any two fields. If \(f : F \rightarrow F'\) is a homomorphism, then the homomorphic image of a level subfield of an intuitionistic fuzzy subfield of \(F\) is a level subfield of an intuitionistic fuzzy subfield of \(F'\).

Proof: Let \((F, +, \cdot)\) and \((F', +, \cdot)\) be any two fields and \(f : F \rightarrow F'\) be a homomorphism. That is, \(f(x+y) = f(x) + f(y)\), for all \(x\) and \(y\) in \(F\) and \(f(xy) = f(x)f(y)\), for all \(x\) and \(y\) in \(F\).

Let \(V = f(A)\), where \(A\) is an intuitionistic fuzzy subfield of \(F\).

Clearly \(V\) is an intuitionistic fuzzy subfield of \(F'\).

If \(x\) and \(y\) in \(F\), then \(f(x)\) and \(f(y)\) in \(F'\).

Let \(A_{(\alpha, \beta)}\) be a level subfield of \(A\).

Suppose \(x\) and \(y\) in \(A_{(\alpha, \beta)}\), then \(x-y\) and \(xy^{-1}\) in \(A_{(\alpha, \beta)}\).

That is, \(\mu_A(x) \geq \alpha\) and \(\nu_A(x) \leq \beta\), \(\mu_A(y) \geq \alpha\) and \(\nu_A(y) \leq \beta\),

\[\mu_A(x-y) \geq \alpha, \mu_A(xy^{-1}) \geq \alpha\) and \(\nu_A(x-y) \leq \beta, \nu_A(xy^{-1}) \leq \beta\).

We have to prove that \(f(A_{(\alpha, \beta)})\) is a level subfield of \(V\).

Now, \(\mu_V(f(x)) \geq \mu_A(x) \geq \alpha\), implies that \(\mu_V(f(x)) \geq \alpha\);

\[\mu_V(f(y)) \geq \mu_A(y) \geq \alpha\), implies that \(\mu_V(f(y)) \geq \alpha\),

\[\mu_V(f(x)-f(y)) = \mu_V(f(x)+f(-y))\), as \(f\) is a homomorphism.
\[ = \mu_V( f(x-y) ), \text{ as } f \text{ is a homomorphism} \]
\[ \geq \mu_A( x-y ) \geq \alpha, \]
which implies that \( \mu_V( f(x)-f(y) ) \geq \alpha, \) for all \( f(x) \) and \( f(y) \) in \( F^1 \).

\[ \mu_V( f(x)(f(y))^{-1} ) = \mu_V( f(x)f(y^{-1}) ) , \text{ as } f \text{ is a homomorphism} \]
\[ = \mu_V( f(xy^{-1}) ), \text{ as } f \text{ is a homomorphism} \]
\[ \geq \mu_A( xy^{-1} ) \geq \alpha, \]
which implies that \( \mu_V( f(x)(f(y))^{-1} ) \geq \alpha, \) for all \( f(x) \) and \( f(y) \neq 0^l \) in \( F^1 \).

And, \( v_V( f(x) ) \leq v_A(x) \leq \beta, \) implies that \( v_V( f(x) ) \leq \beta; \)
\[ v_V( f(y) ) \leq v_A(y) \leq \beta, \text{ implies that } v_V( f(y) ) \leq \beta, \]
\[ v_V( f(x)-f(y) ) = v_V( f(x)+f(-y) ), \text{ as } f \text{ is a homomorphism} \]
\[ = v_V( f(x-y) ), \text{ as } f \text{ is a homomorphism} \]
\[ \leq v_A( x-y ) \leq \beta, \]
which implies that \( v_V( f(x)-f(y) ) \leq \beta, \) for all \( f(x) \) and \( f(y) \) in \( F^1 \).

\[ v_V( f(x)(f(y))^{-1} ) = v_V( f(x)f(y^{-1}) ) , \text{ as } f \text{ is a homomorphism} \]
\[ = v_V( f(xy^{-1}) ), \text{ as } f \text{ is a homomorphism} \]
\[ \leq v_A( xy^{-1} ) \leq \beta, \]
which implies that \( v_V( f(x)(f(y))^{-1} ) \leq \beta, \) for all \( f(x) \) and \( f(y) \neq 0^l \) in \( F^1 \).

Therefore, \( \mu_V( f(x)-f(y) ) \geq \alpha, \) for all \( f(x) \) and \( f(y) \) in \( F^1 \) and
\( \mu_V( f(x)(f(y))^{-1} ) \geq \alpha, \) for all \( f(x) \) and \( f(y) \neq 0^l \) in \( F^1 \) and
\( v_V( f(x)-f(y) ) \leq \beta, \) for all \( f(x) \) and \( f(y) \) in \( F^1 \) and \( v_V( f(x)(f(y))^{-1} ) \leq \beta, \)
for all \( f(x) \) and \( f(y) \neq 0^l \) in \( F^1 \).
Hence \( f \left( A_{(\alpha,\beta)} \right) \) is a level subfield of an intuitionistic fuzzy subfield \( V \) of a field \( F^l \).

4.3.2 Theorem: Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) be any two fields. If \( f : F \to F^l \) is a homomorphism, then the homomorphic pre-image of a level subfield of an intuitionistic fuzzy subfield of \( F^l \) is a level subfield of an intuitionistic fuzzy subfield of \( F \).

Proof: Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) be any two fields and \( f : F \to F^l \) be a homomorphism. That is, \( f(x+y) = f(x)+f(y) \), for all \( x \) and \( y \) in \( F \) and \( f(xy) = f(x)f(y) \), for all \( x \) and \( y \) in \( F \).

Let \( V = f(A) \), where \( V \) is an intuitionistic fuzzy subfield of \( F^l \).

Clearly \( A \) is an intuitionistic fuzzy subfield of \( F \).

Let \( x \) and \( y \) in \( F \).

Let \( f(A_{(\alpha,\beta)}) \) be a level subfield of \( V \).

Suppose \( f(x) \) and \( f(y) \) in \( f(A_{(\alpha,\beta)}) \), then \( f(x) - f(y) \) and \( f(x)(f(y))^{-1} \) in \( f(A_{(\alpha,\beta)}) \).

That is, \( \mu_V( f(x) ) \geq \alpha \) and \( \nu_V( f(x) ) \leq \beta \); \( \mu_V( f(y) ) \geq \alpha \) and \( \nu_V( f(y) ) \leq \beta \);

\[ \mu_V( f(x)-f(y) ) \geq \alpha, \mu_V( f(x)(f(y))^{-1} ) \geq \alpha \quad \text{and} \quad \nu_V( f(x)-f(y) ) \leq \beta, \nu_V( f(x)(f(y))^{-1} ) \leq \beta. \]

We have to prove that \( A_{(\alpha,\beta)} \) is a level subfield of \( A \).

Now, \( \mu_A(x) = \mu_V( f(x) ) \geq \alpha \), implies that \( \mu_A(x) \geq \alpha \);

\( \mu_A(y) = \mu_V( f(y) ) \geq \alpha \), implies that \( \mu_A(y) \geq \alpha \),

we have \( \mu_A(x-y) = \mu_V( f(x-y) ) \),
\[ \mu_V( f(x)+f(-y) ) = \mu_V( f(x)-f(y) ) \], as \( f \) is a homomorphism
\[ \geq \alpha , \]
which implies that \( \mu_A(x-y) \geq \alpha \), for all \( x \) and \( y \) in \( F \).

And
\[ \mu_A( xy^{-1} ) = \mu_V( f(xy^{-1}) ) , \]
\[ = \mu_V( f(x)f(y^{-1}) ) , \] as \( f \) is a homomorphism
\[ = \mu_V( f(x)(f(y))^{-1} ) , \] as \( f \) is a homomorphism
\[ \geq \alpha , \]
which implies that \( \mu_A(xy^{-1}) \geq \alpha \), for all \( x \) and \( y \neq 0 \) in \( F \).

And, \( \nu_A(x) = \nu_V( f(x) ) \leq \beta \), implies that \( \nu_A(x) \leq \beta \);
\( \nu_A(y) = \nu_V( f(y) ) \leq \beta \), implies that \( \nu_A(y) \leq \beta \),
we have
\[ \nu_A(x-y) = \nu_V( f(x-y) ) , \]
\[ = \nu_V( f(x)+f(-y) ) , \] as \( f \) is a homomorphism
\[ = \nu_V( f(x)-f(y) ) , \] as \( f \) is a homomorphism
\[ \leq \beta \]
which implies that \( \nu_A(x-y) \leq \beta \), for all \( x \) and \( y \) in \( F \).

And
\[ \nu_A( xy^{-1} ) = \nu_V( f(xy^{-1}) ) , \]
\[ = \nu_V( f(x)f(y^{-1}) ) , \] as \( f \) is a homomorphism
\[ = \nu_V( f(x)(f(y))^{-1} ) , \] as \( f \) is a homomorphism
\[ \leq \beta \]
which implies that \( \nu_A(xy^{-1}) \leq \beta \), for all \( x \) and \( y \neq 0 \) in \( F \).
Therefore, $\mu_A(x-y) \geq \alpha$, for all $x$ and $y$ in $F$ and $\mu_A(xy^{-1}) \geq \alpha$, for all $x$ and $y \neq 0$ in $F$ and $\nu_A(x-y) \leq \beta$, for all $x$ and $y$ in $F$ and $\nu_A(xy^{-1}) \leq \beta$, for all $x$ and $y \neq 0$ in $F$.

Hence $A_{(\alpha,\beta)}$ is a level subfield of an intuitionistic fuzzy subfield $A$ of $F$.

4.3.3 Theorem: Let $(F, +, \cdot)$ and $(F^I, +, \cdot)$ be any two fields. If $f : F \rightarrow F^I$ is an anti-homomorphism, then the anti-homomorphic image of a level subfield of an intuitionistic fuzzy subfield of $F$ is a level subfield of an intuitionistic fuzzy subfield of $F^I$.

Proof: It is trivial.

4.3.4 Theorem: Let $(F, +, \cdot)$ and $(F^I, +, \cdot)$ be any two fields. If $f : F \rightarrow F^I$ is an anti-homomorphism, then the anti-homomorphic pre-image of a level subfield of an intuitionistic fuzzy subfield of $F^I$ is a level subfield of an intuitionistic fuzzy subfield of $F$.

Proof: It is trivial.