CHAPTER -III

INTUITIONISTIC FUZZY SUBFIELDS OF A FIELD

3.1 Introduction: This chapter contains some definitions and results in intuitionistic fuzzy subfield theory, which are required in the sequel. Some theorems are introduced in this chapter which have been used by homomorphism and anti-homomorphism of intuitionistic fuzzy subfields.

3.1.1 Definition: Let \(( F, +, \cdot )\) be a field. An intuitionistic fuzzy subset \(A\) of \(F\) is said to be an intuitionistic fuzzy subfield (IFSF) of \(F\) if the following conditions are satisfied:

(i) \(\mu_A( x+y ) \geq \min \{\mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y\) in \(F\),

(ii) \(\mu_A( -x ) \geq \mu_A( x )\), for all \(x\) in \(F\),

(iii) \(\mu_A( xy ) \geq \min \{\mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y\) in \(F\),

(iv) \(\mu_A( x^{-1} ) \geq \mu_A( x )\), for all \(x\) in \(F\)\(\backslash\{0\}\),

(v) \(\nu_A( x+y ) \leq \max \{\nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y\) in \(F\),

(vi) \(\nu_A( -x ) \leq \nu_A( x )\), for all \(x\) in \(F\),

(vii) \(\nu_A( xy ) \leq \max \{\nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y\) in \(F\),

(viii) \(\nu_A( x^{-1} ) \leq \nu_A( x )\), for all \(x\) in \(F\)\(\backslash\{0\}\).

3.1.1 Example: Consider the field \(Z_5 = \{ 0, 1, 2, 3, 4 \}\) with addition modulo 5 and multiplication modulo 5 operations. Then \(A = \{ (0, 0.7, 0.1), (1, 0.5, 0.4), (2, 0.5, 0.4), (3, 0.5, 0.4), (4, 0.5, 0.4) \}\) is an intuitionistic fuzzy subfield of \(Z_5\).
3.1.2 Definition: Let \(( F, +, \cdot )\) and \(( F^l, +, \cdot )\) be any two fields. Let \(f : F \to F^l\) be any function and \(A\) be an intuitionistic fuzzy subfield in \(F\), \(V\) be an intuitionistic fuzzy subfield in \(f(F) = F^l\), defined by 
\[
\mu_V(y) = \sup_{x \in f^{-1}(y)} \mu_A(x) \text{ and } v_V(y) = \inf_{x \in f^{-1}(y)} v_A(x), \text{ for all } x \text{ in } F \text{ and } y \text{ in } F^l.
\]
Then \(A\) is called a preimage of \(V\) under \(f\) and is denoted by \(f^{-1}(V)\).

**Note:** This definition is used throughout this thesis for image and preimage in functions.

3.1.3 Definition: Let \(A\) and \(B\) be any two intuitionistic fuzzy subsets of sets \(G\) and \(H\), respectively. The product of \(A\) and \(B\), denoted by \(A \times B\), is defined as 
\[
A \times B = \{ (x, y), \mu_{A \times B}(x, y), v_{A \times B}(x, y) \} / \text{for all } x \text{ in } G \text{ and } y \text{ in } H \}, \text{ where } 
\mu_{A \times B}(x, y) = \min \{ \mu_A(x), \mu_B(y) \} \text{ and } 
\nu_{A \times B}(x, y) = \max \{ \nu_A(x), \nu_B(y) \}, \text{ for all } x \text{ in } G \text{ and } y \text{ in } H.
\]

3.1.4 Definition: Let \(A\) be an intuitionistic fuzzy subset in a set \(S\), the **strongest intuitionistic fuzzy relation** on \(S\), that is an intuitionistic fuzzy relation on \(A\) is 
\[
V = \{ (x, y), \mu_V(x, y), v_V(x, y) \} / \text{for all } x \text{ and } y \text{ in } S \}
\]
given by 
\[
\mu_V(x, y) = \min \{ \mu_A(x), \mu_A(y) \} \text{ and } v_V(x, y) = \max \{ v_A(x), v_A(y) \}, \text{ for all } x \text{ and } y \text{ in } S.
\]

3.1.5 Definition: Let \(( F, +, \cdot )\) be a field. An intuitionistic fuzzy subfield \(A\) of \(F\) is said to be an intuitionistic fuzzy characteristic subfield(IFCSF) of \(F\) if the following conditions are satisfied:

\[
\begin{align*}
(i) & \quad \mu_A(x) = \mu_A( f(x) ), \\
(ii) & \quad v_A(x) = v_A( f(x) ), \text{ for all } x \text{ in } F \text{ and } f \text{ in } \text{Aut } F.
\end{align*}
\]
3.1.6 Definition: An intuitionistic fuzzy subset A of a set X is said to be normalized if there exist \( x \) in X such that \( \mu_A(x) = 1 \) and \( \nu_A(x) = 0 \).

3.1.7 Definition: Let A be an intuitionistic fuzzy subfield of a field \(( F, +, \cdot )\). For any \( a \) in F and \( b \) in \( F\setminus\{0\} \), \( aA_b \) defined by \( (a+\mu_A)(x) = \mu_A(-a+x) \), for all \( x \) in F and \( (b\mu_A)(x) = \mu_A(b^{-1}x) \), for all \( x \) in F and \( (a+\nu_A)(x) = \nu_A(-a+x) \), for all \( x \) in F, \( (b\nu_A)(x) = \nu_A(b^{-1}x) \), for all \( x \) in F, is called an intuitionistic fuzzy \((a,b)\)-coset of F.

3.1.8 Definition: Let A be an intuitionistic fuzzy subfield of a field \(( F, +, \cdot )\) and \( H = \{ x \in F / \mu_A(x)=\mu_A(0)=\mu_A(1) \) and \( \nu_A(x)=\nu_A(0)=\nu_A(1) \} \), then \( O(A) \), order of A is defined as \( O(A) = O(H) \), where 0 and 1 are additive and multiplicative identity elements.

3.1.9 Definition: Let A be an intuitionistic fuzzy subfield of a field \(( F, +, \cdot )\). Then for any \( a \) and \( b \) in F and \( c \) and \( d \) in \( F\setminus\{0\} \), an intuitionistic fuzzy \((a+b, cd)\)-coset \( a+bA_{cd} \) of F is defined by \( (a+\mu_A+b)(x) = \mu_A(-a+x-b) \), for all \( x \) in F and \( (c\mu_A)(x) = \mu_A(c^{-1}x^{-1}) \), for all \( x \) in F and \( (a+\nu_A+b)(x) = \nu_A(-a+x-b) \), for all \( x \) in F and \( (c\nu_A)(x) = \nu_A(c^{-1}x^{-1}) \), for all \( x \) in F.

3.1.10 Definition: Let A be an intuitionistic fuzzy subfield of a field \(( F, +, \cdot )\) and \( a \) in F. Then the pseudo intuitionistic fuzzy coset \((aA)^p\) is defined by \( ((a\mu_A)^p)(x) = p(a)\mu_A(x) \) and \( ((a\nu_A)^p)(x) = p(a)\nu_A(x) \), for every \( x \) in F and for some \( p \) in P.
3.2–PROPERTIES OF INTUITIONISTIC FUZZY SUBFIELDS:

3.2.1 Theorem: If A is an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\), then \(\mu_A(-x) = \mu_A(x)\), for all \(x\) in \(F\) and \(\mu_A(x^{-1}) = \mu_A(x)\), for all \(x\) in \(F-\{0\}\) and \(\nu_A(-x) = \nu_A(x)\), for all \(x\) in \(F\) and \(\nu_A(x^{-1}) = \nu_A(x)\), for all \(x\) in \(F-\{0\}\) and \(\mu_A(x) \leq \mu_A(0)\), for all \(x\) in \(F\) and \(\mu_A(x) \leq \mu_A(1)\), for all \(x\) in \(F\) and \(\nu_A(x) \geq \nu_A(0)\), for all \(x\) in \(F\) and \(\nu_A(x) \geq \nu_A(1)\), for all \(x\) in \(F\), where 0 and 1 are identity elements in \(F\).

Proof: For \(x\) in \(F\) and 0 , 1 are identity elements in \(F\).

Now, \(\mu_A(x) = \mu_A(-(-x))\)

\[\geq \mu_A(-x) \geq \mu_A(x).\]

Therefore, \(\mu_A(-x) = \mu_A(x)\), for all \(x\) in \(F\).

Now, \(\mu_A(x) = \mu_A((x^{-1})^{-1})\)

\[\geq \mu_A(x^{-1}) \geq \mu_A(x).\]

Therefore, \(\mu_A(x^{-1}) = \mu_A(x)\), for all \(x\) in \(F-\{0\}\).

And, \(\nu_A(x) = \nu_A(-(-x))\)

\[\leq \nu_A(-x) \leq \nu_A(x).\]

Therefore, \(\nu_A(-x) = \nu_A(x)\), for all \(x\) in \(F\).

And, \(\nu_A(x) = \nu_A((x^{-1})^{-1})\)

\[\leq \nu_A(x^{-1}) \leq \nu_A(x).\]

Therefore, \(\nu_A(x^{-1}) = \nu_A(x)\), for all \(x\) in \(F-\{0\}\).
Now, \( \mu_A(0) = \mu_A(x-x) \)
\[ \geq \min \{ \mu_A(x), \mu_A(-x) \} = \mu_A(x). \]
Therefore, \( \mu_A(0) \geq \mu_A(x) \), for all \( x \) in \( F \).

Now, \( \mu_A(1) = \mu_A(xx^{-1}) \)
\[ \geq \min \{ \mu_A(x), \mu_A(x^{-1}) \} = \mu_A(x). \]
Therefore, \( \mu_A(1) \geq \mu_A(x) \), for all \( x \) in \( F \).

And, \( \nu_A(0) = \nu_A(x-x) \)
\[ \leq \max \{ \nu_A(x), \nu_A(-x) \} = \nu_A(x). \]
Therefore, \( \nu_A(0) \leq \nu_A(x) \), for all \( x \) in \( F \).

And, \( \nu_A(1) = \nu_A(xx^{-1}) \)
\[ \leq \max \{ \nu_A(x), \nu_A(x^{-1}) \} = \nu_A(x). \]
Therefore, \( \nu_A(1) \leq \nu_A(x) \), for all \( x \) in \( F \).

3.2.2 Theorem: If \( A \) is an intuitionistic fuzzy subfield of a field \( (F, +, \cdot) \), then (i) \( \mu_A(x-y) = \mu_A(0) \) gives \( \mu_A(x) = \mu_A(y) \), for all \( x \) and \( y \) in \( F \), (ii) \( \mu_A(xy^{-1}) = \mu_A(1) \) gives \( \mu_A(x) = \mu_A(y) \), for all \( x \) and \( y \neq 0 \) in \( F \), (iii) \( \nu_A(x-y) = \nu_A(0) \) gives \( \nu_A(x) = \nu_A(y) \), for all \( x \) and \( y \) in \( F \) and (iv) \( \nu_A(xy^{-1}) = \nu_A(1) \) gives \( \nu_A(x) = \nu_A(y) \), for all \( x \) and \( y \neq 0 \) in \( F \), where 0 and 1 are identity elements in \( F \).

Proof: Let \( x \) and \( y \) in \( F \) and 0, 1 are identity elements in \( F \).

(i) Now, \( \mu_A(x) = \mu_A(x-y+y) \)
\[ \geq \min \{ \mu_A(x-y), \mu_A(y) \} \]
\[ = \min \{ \mu_A(0), \mu_A(y) \} \]
\[= \mu_A(y)\]
\[= \mu_A(\ x-(x-y)\ )\]
\[\geq \min \{\mu_A(x-y), \mu_A(x)\}\]
\[= \min \{\mu_A(0), \mu_A(x)\}\]
\[= \mu_A(x).\]

Therefore, \(\mu_A(x) = \mu_A(y)\), for all \(x\) and \(y\) in \(F\).

(ii) Now, \(\mu_A(x) = \mu_A(\ xy^{-1}y\ )\)
\[\geq \min \{\mu_A(xy^{-1}), \mu_A(y)\}\]
\[= \min \{\mu_A(1), \mu_A(y)\}\]
\[= \mu_A(y)\]
\[= \mu_A((xy^{-1})^{-1}x\ )\]
\[\geq \min \{\mu_A(xy^{-1}), \mu_A(x)\}\]
\[= \min \{\mu_A(1), \mu_A(x)\}\]
\[= \mu_A(x).\]

Therefore, \(\mu_A(x) = \mu_A(y)\), for all \(x\) and \(y \neq 0\) in \(F\).

(iii) Now, \(\nu_A(x) = \nu_A(\ x-y+y\ )\)
\[\leq \max \{\nu_A(\ x-y), \nu_A(y)\}\]
\[= \max \{\nu_A(0), \nu_A(y)\}\]
\[= \nu_A(y)\]
\[= \nu_A(\ x-(x-y)\ )\]
\[\leq \max \{\nu_A(x-y), \nu_A(x)\}\]
\[= \max \{\nu_A(0), \nu_A(x)\}\]
Therefore, \( \nu_A(x) = \nu_A(y) \), for all \( x \) and \( y \) in \( F \).

(iv) Now, \( \nu_A(x) = \nu_A(y^{-1}y) \)

\[
\leq \max \{ \nu_A(\ y^{-1}), \nu_A(y) \} \\
= \max \{ \nu_A(1), \nu_A(y) \} \\
= \nu_A(y) \\
= \nu_A(1^{-1}x) \\
\leq \max \{ \nu_A(\ y^{-1}), \nu_A(x) \} \\
= \max \{ \nu_A(1), \nu_A(x) \} \\
= \nu_A(x).
\]

Therefore, \( \nu_A(x) = \nu_A(y) \), for all \( x \) and \( y \neq 0 \) in \( F \).

3.2.3 **Theorem:** \( A \) is an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\) if and only if \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \) and \( \mu_A(x^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \) and \( \nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \) and \( \nu_A(x^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \).

**Proof:** Let \( A \) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\) and \( x \) and \( y \) in \( F \) and \( 0, 1 \) are identity elements of \( F \).

Then, \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(-y) \} \)

\[
\geq \min \{ \mu_A(x), \mu_A(y) \}.
\]

Therefore, \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

Then, \( \mu_A(x^{-1}) \geq \min \{ \mu_A(x), \mu_A(y^{-1}) \} \)
Therefore, \( \mu_A(\, xy^{-1}\, ) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \).

And, \( v_A(x-y) \leq \max \{ v_A(x), v_A(-y) \} \)

\[ \leq \max \{ v_A(x), v_A(y) \}. \]

Therefore, \( v_A(x-y) \leq \max \{ v_A(x), v_A(y) \} \), for all \( x \) and \( y \) in \( F \).

And, \( v_A(xy^{-1}) \leq \max \{ v_A(x), v_A(y^{-1}) \} \)

\[ \leq \max \{ v_A(x), v_A(y) \}. \]

Therefore, \( v_A(xy^{-1}) \leq \max \{ v_A(x), v_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \).

Conversely, if \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \) and

\( \mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \) and

\( v_A(x-y) \leq \max \{ v_A(x), v_A(y) \} \), for all \( x \) and \( y \) in \( F \) and

\( v_A(xy^{-1}) \leq \max \{ v_A(x), v_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \), replace \( y \) by \( x \),

then,

\( \mu_A(x) \leq \mu_A(0) \), for all \( x \) in \( F \) and \( \mu_A(x) \leq \mu_A(1) \), for all \( x \) in \( F \) and

\( v_A(x) \geq v_A(0) \), for all \( x \) in \( F \) and \( v_A(x) \geq v_A(1) \), for all \( x \) in \( F \).

Now, \( \mu_A(-x) = \mu_A(0-x) \)

\[ \geq \min \{ \mu_A(0), \mu_A(x) \} \]

\[ = \mu_A(x). \]

Therefore, \( \mu_A(-x) \geq \mu_A(x) \), for all \( x \) in \( F \).

Now, \( \mu_A(x^{-1}) = \mu_A(\, 1.x^{-1}\, ) \)

\[ \geq \min \{ \mu_A(1), \mu_A(x) \} \]

\[ = \mu_A(x). \]
Therefore, $\mu_A(x^{-1}) \geq \mu_A(x)$, for all $x$ in $F-\{0\}$.

It follows that, $\mu_A(x+y) = \mu_A(x - (-y))$

\[
\geq \min \{ \mu_A(x), \mu_A(-y) \}
\]

\[
\geq \min \{ \mu_A(x), \mu_A(y) \}.
\]

Therefore, $\mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$.

It follows that, $\mu_A(xy) = \mu_A( x(y^{-1})^{-1} )$

\[
\geq \min \{ \mu_A(x), \mu_A(y^{-1}) \}
\]

\[
\geq \min \{ \mu_A(x), \mu_A(y) \}.
\]

Therefore, $\mu_A(xy) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$.

And, $\nu_A(-x) = \nu_A(0-x)$

\[
\leq \max \{ \nu_A(0), \nu_A(x) \}
\]

\[
= \nu_A(x).
\]

Therefore, $\nu_A(-x) \leq \nu_A(x)$, for all $x$ in $F$.

And, $\nu_A(x^{-1}) = \nu_A(1.x^{-1})$

\[
\leq \max \{ \nu_A(1), \nu_A(x) \}
\]

\[
= \nu_A(x).
\]

Therefore, $\nu_A(x^{-1}) \leq \nu_A(x)$, for all $x$ in $F-\{0\}$.

Then, $\nu_A(x+y) = \nu_A(x - (-y))$

\[
\leq \max \{ \nu_A(x), \nu_A(-y) \}
\]

\[
\leq \max \{ \nu_A(x), \nu_A(y) \}.
\]

Therefore, $\nu_A(x+y) \leq \max \{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y$ in $F$.

Then, $\nu_A(xy) = \nu_A( x(y^{-1})^{-1} )$
\[ \leq \max \{ \nu_A(x), \nu_A(y^{-1}) \} \]

\[ \leq \max \{ \nu_A(x), \nu_A(y) \}. \]

Therefore, \( \nu_A(xy) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

Hence \( A \) is an intuitionistic fuzzy subfield of \( F \).

### 3.2.4 Theorem:

Let \( A \) be an intuitionistic fuzzy subset of a field \( (F, +, \cdot) \).

If \( \mu_A(e) = \mu_A(e^1) = 1 \) and \( \nu_A(e) = \nu_A(e^1) = 0 \) and \( \mu_A(\cdot - y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \) and \( \mu_A(x \cdot y^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \neq e \) in \( F \) and \( \nu_A(x - y) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \) and \( \nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \neq e \) in \( F \), then \( A \) is an intuitionistic fuzzy subfield of \( F \), where \( e \) and \( e^1 \) are identity elements of \( F \).

**Proof:** Let \( x \) and \( y \) in \( F \) and \( e, e^1 \) are identity elements of \( F \).

Now \( \mu_A(-x) = \mu_A(e-x) \)

\[ \geq \min \{ \mu_A(e), \mu_A(x) \} \]

\[ = \min \{ 1, \mu_A(x) \} \]

\[ = \mu_A(x). \]

Therefore, \( \mu_A(-x) \geq \mu_A(x) \), for all \( x \) in \( F \).

Now \( \mu_A(x^{-1}) = \mu_A(e^1 \cdot x^{-1}) \)

\[ \geq \min \{ \mu_A(e^1), \mu_A(x) \} \]

\[ = \min \{ 1, \mu_A(x) \} \]

\[ = \mu_A(x). \]

Therefore, \( \mu_A(x^{-1}) \geq \mu_A(x) \), for all \( x \) in \( F-\{e\} \).
And \( v_A(-x) = v_A(e-x) \)
\[
\leq \max \{ v_A(e), v_A(x) \}
\]
\[
= \max \{ 0, v_A(x) \}
\]
\[
= v_A(x).
\]
Therefore, \( v_A(-x) \leq v_A(x) \), for all \( x \) in \( F \).

And \( v_A(x^{-1}) = v_A(e^1 x^{-1}) \)
\[
\leq \max \{ v_A(e^1), v_A(x) \}
\]
\[
= \max \{ 0, v_A(x) \}
\]
\[
= v_A(x).
\]
Therefore, \( v_A(x^{-1}) \leq v_A(x) \), for all \( x \) in \( F^{-\{e\}} \).

Now, \( \mu_A(x+y) = \mu_A(x-(-y)) \)
\[
\geq \min \{ \mu_A(x), \mu_A(-y) \}
\]
\[
\geq \min \{ \mu_A(x), \mu_A(y) \}.
\]
Therefore, \( \mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

Now, \( \mu_A(xy) = \mu_A(x(y^{-1})^{-1}) \)
\[
\geq \min \{ \mu_A(x), \mu_A(y^{-1}) \}
\]
\[
\geq \min \{ \mu_A(x), \mu_A(y) \}.
\]
Therefore, \( \mu_A(xy) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

And \( v_A(x+y) = v_A(x-(-y)) \)
\[
\leq \max \{ v_A(x), v_A(-y) \}
\]
\[
\leq \max \{ v_A(x), v_A(y) \}.
\]
Therefore, \( v_A(x+y) \leq \max \{ v_A(x), v_A(y) \} \), for all \( x \) and \( y \) in \( F \).
And \( \nu_A(xy) = \nu_A(x(y^{-1})^{-1}) \)

\[ \leq \max \{ \nu_A(x), \nu_A(y^{-1}) \} \]

\[ \leq \max \{ \nu_A(x), \nu_A(y) \} . \]

Therefore, \( \nu_A(xy) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

Hence \( A \) is an intuitionistic fuzzy subfield of \( F \).

**3.2.5 Theorem:** If \( A \) is an intuitionistic fuzzy subfield of a field \( (F, +, \cdot) \), then \( H = \{ x / x \in F: \mu_A(x) = 1, \nu_A(x) = 0 \} \) is either empty or a subfield of \( F \).

**Proof:** If no element satisfies this condition, then \( H \) is empty.

If \( x \) and \( y \) in \( H \), then

\[ \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(-y) \} \]

\[ \geq \min \{ \mu_A(x), \mu_A(y) \} \]

\[ = \min \{ 1, 1 \} = 1. \]

Therefore, \( \mu_A(x-y) = 1 \), for all \( x \) and \( y \) in \( H \).

\[ \mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y^{-1}) \} \]

\[ \geq \min \{ \mu_A(x), \mu_A(y) \} \]

\[ = \min \{ 1, 1 \} = 1. \]

Therefore, \( \mu_A(xy^{-1}) = 1 \), for all \( x \) and \( y \neq e \) in \( H \).

And, \( \nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(-y) \} \)

\[ \leq \max \{ \nu_A(x), \nu_A(y) \} \]

\[ = \max \{ 0, 0 \} = 0. \]

Therefore, \( \nu_A(x-y) = 0 \), for all \( x \) and \( y \) in \( H \).
And, \( \nu_A( xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y^{-1}) \} \)

\[ \leq \max \{ \nu_A(x), \nu_A(y) \} \]

\[ = \max \{0, 0\} = 0. \]

Therefore, \( \nu_A(xy^{-1}) = 0 \), for all \( x \) and \( y \neq e \) in \( H \).

We get \( x-y, xy^{-1} \) in \( H \).

Therefore, \( H \) is a subfield of \( F \).

Hence \( H \) is either empty or a subfield of \( F \).

3.2.6 Theorem: If \( A \) is an intuitionistic fuzzy subfield of a field \( (F, +, \cdot) \),
then \( H = \{ \langle x, \mu_A(x) \rangle : 0 < \mu_A(x) \leq 1 \text{ and } \nu_A(x) = 0 \} \) is either empty or
a fuzzy subfield of \( F \).

Proof: If no element satisfies this condition, then \( H \) is empty.

If \( x \) and \( y \) satisfies this condition, then

\( \nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(-y) \} \)

\[ \leq \max \{ \nu_A(x), \nu_A(y) \} \]

\[ = \max \{ 0, 0 \} = 0. \]

Therefore, \( \nu_A(x-y) = 0 \), for all \( x \) and \( y \) in \( F \).

And, \( \nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y^{-1}) \} \)

\[ \leq \max \{ \nu_A(x), \nu_A(y) \} \]

\[ = \max \{ 0, 0 \} = 0. \]

Therefore, \( \nu_A(xy^{-1}) = 0 \), for all \( x \) and \( y \neq e \) in \( F \).

And, \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(-y) \} \)

\[ \geq \min \{ \mu_A(x), \mu_A(y) \}. \]
Therefore, $\mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$.

And, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y^{-1}) \}$

\[ \geq \min \{ \mu_A(x), \mu_A(y) \}. \]

Therefore, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y \neq e$ in $F$.

Hence $H$ is a fuzzy subfield of $F$.

Therefore, $H$ is either empty or a fuzzy subfield of $F$.

3.2.7 Theorem: If $A$ is an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$
then $H = \{ \langle x, \mu_A(x) \rangle : 0 < \mu_A(x) \leq 1 \}$ is either empty or a fuzzy subfield of $F$.

**Proof:** If no element satisfies this condition, then $H$ is empty.

If $x$ and $y$ satisfies this condition, then

\[ \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(-y) \} \]

\[ \geq \min \{ \mu_A(x), \mu_A(y) \}. \]

Therefore, $\mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$.

And $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y^{-1}) \}$

\[ \geq \min \{ \mu_A(x), \mu_A(y) \}. \]

Therefore, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y \neq e$ in $F$.

Therefore, $H$ is either empty or a fuzzy subfield of $F$.

3.2.8 Theorem: If $A$ is an intuitionistic fuzzy subfield of a field
$(F, +, \cdot)$, then $H = \{ \langle x, \nu_A(x) \rangle : 0 < \nu_A(x) \leq 1 \}$ is either empty or an anti-fuzzy subfield of $F$.

**Proof:** If no element satisfies this condition, then $H$ is empty.

If $x$ and $y$ satisfies this condition, then
\[ \nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(-y) \} \]

\[ \leq \max \{ \nu_A(x), \nu_A(y) \}. \]

Therefore, \( \nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

And \( \nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y^{-1}) \} \)

\[ \leq \max \{ \nu_A(x), \nu_A(y) \}. \]

Therefore, \( \nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \neq e \) in \( F \).

Hence \( H \) is either empty or an anti-fuzzy subfield of \( F \).

**3.2.9 Theorem:** If \( A \) is an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\),

then \( H = \{ x \in F : \mu_A(x) = \mu_A(e) = \mu_A(e^1) \text{ and } \nu_A(x) = \nu_A(e) = \nu_A(e^1) \} \) is either empty or a subfield of \( F \), where \( e \) and \( e^1 \) are identity elements of \( F \).

**Proof:** If no element satisfies this condition, then \( H \) is empty.

If \( x \) and \( y \) satisfies this condition, then

\[ \mu_A(-x) = \mu_A(x) = \mu_A(e), \text{ for all } x \text{ in } H \text{ and } \mu_A(x^{-1}) = \mu_A(x) = \mu_A(e^1), \]

for all \( x \) in \( H - \{ e \} \) and \( \nu_A(-x) = \nu_A(x) = \nu_A(e), \text{ for all } x \text{ in } H \text{ and } \nu_A(x^{-1}) = \nu_A(x) = \nu_A(e^1), \) for all \( x \) in \( H - \{ e \}, \) by Theorem 3.2.1.

Therefore, \( \mu_A(-x) = \mu_A(e), \text{ for all } x \text{ in } H \text{ and } \mu_A(x^{-1}) = \mu_A(e^1), \text{ for all } x \)

in \( H - \{ e \} \) and \( \nu_A(-x) = \nu_A(e), \text{ for all } x \text{ in } H \text{ and } \nu_A(x^{-1}) = \nu_A(e^1), \text{ for all } x \)

in \( H - \{ e \}. \)

Hence \( -x, x^{-1} \) in \( H \).

Now, \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(-y) \} \)

\[ \geq \min \{ \mu_A(x), \mu_A(y) \} \]

\[ = \min \{ \mu_A(e), \mu_A(e) \} \]
Therefore, \( \mu_A(x-y) \geq \mu_A(e) \) ---------------------- (1).

And, \( \mu_A(e) = \mu_A((x-y) - (x-y)) \)

\[
\geq \min \{ \mu_A(x-y), \mu_A(- (x-y)) \}
\geq \min \{ \mu_A(x-y), \mu_A(x-y) \}
= \mu_A(x-y).
\]

Therefore, \( \mu_A(e) \geq \mu_A(x-y) \) ---------------------- (2).

From (1) and (2), we get \( \mu_A(e) = \mu_A(x-y) \), for all \( x \) and \( y \) in \( H \).

Now, \( \mu_A(x^{-1}) \geq \min \{ \mu_A(x), \mu_A(y^{-1}) \} \)

\[
\geq \min \{ \mu_A(x), \mu_A(y) \}
= \min \{ \mu_A(e_1), \mu_A(e_1) \}
= \mu_A(e_1).
\]

Therefore, \( \mu_A(x^{-1}) \geq \mu_A(e_1) \) ---------------------- (3).

And, \( \mu_A(e_1) = \mu_A((x^{-1})(x^{-1})^{-1}) \)

\[
\geq \min \{ \mu_A(x^{-1}), \mu_A((x^{-1})^{-1}) \}
\geq \min \{ \mu_A(x^{-1}), \mu_A(x^{-1}) \}
= \mu_A(x^{-1}).
\]

Therefore, \( \mu_A(e_1) \geq \mu_A(x^{-1}) \) ---------------------- (4).

From (3) and (4), we get \( \mu_A(e_1) = \mu_A(x^{-1}) \), for all \( x \) and \( y \neq e \) in \( H \).

Now, \( \nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(-y) \} \)

\[
\leq \max \{ \nu_A(x), \nu_A(y) \}
= \max \{ \nu_A(e), \nu_A(e) \}
\]

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Therefore, $\nu_A(x-y) \leq \nu_A(e)$--------------------------- (5).

And, $\nu_A(e) = \nu_A((x-y)-(x-y))$

$\leq \max \{ \nu_A(x-y), \nu_A(-(x-y)) \}$

$\leq \max \{ \nu_A(x-y), \nu_A(x-y) \}$

$= \nu_A(x-y)$.

Therefore, $\nu_A(e) \leq \nu_A(x-y)$ --------------------------- (6).

From (5) and (6), we get $\nu_A(e) = \nu_A(x-y)$, for all $x$ and $y$ in $H$.

Now, $\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y^{-1}) \}$

$\leq \max \{ \nu_A(x), \nu_A(y) \}$

$= \max \{ \nu_A(e^1), \nu_A(e^1) \}$

$= \nu_A(e^1)$.

Therefore, $\nu_A(xy^{-1}) \leq \nu_A(e^1)$ ------------------------ (7).

And, $\nu_A(e^1) = \nu_A((xy^{-1})(xy^{-1})^{-1})$

$\leq \max \{ \nu_A(xy^{-1}), \nu_A((xy^{-1})^{-1}) \}$

$\leq \max \{ \nu_A(xy^{-1}), \nu_A(xy^{-1}) \}$

$= \nu_A(xy^{-1})$.

Therefore, $\nu_A(e^1) \leq \nu_A(xy^{-1})$ ------------------------------- (8).

From (7) and (8), we get $\nu_A(e^1) = \nu_A(xy^{-1})$, for all $x$ and $y \neq e$ in $H$.

Hence $\mu_A(e) = \mu_A(x-y)$, for all $x$ and $y$ in $H$ and $\mu_A(e^1) = \mu_A(xy^{-1})$, for all $x$ and $y$ in $H$ and $\nu_A(e^1) = \nu_A(xy^{-1})$, for all $x$ and $y \neq e$ in $H$. 

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Therefore, $x - y$ and $xy^{-1}$ in $H$.

Hence $H$ is either empty or a subfield of $F$.

**3.2.10 Theorem:** If $A$ is an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$, then $H = \{ (x, \mu_A(x)) : \mu_A(x) = \mu_A(e) = \mu_A(e^1) \}$ and $\nu_A(x) = \nu_A(e) = \nu_A(e^1) \}$ is either empty or a fuzzy subfield of $F$, where $e$ and $e^1$ are identity elements of $F$.

**Proof:** If no element satisfies this condition, then $H$ is empty.

If $x$ and $y$ satisfies this condition, by Theorem 3.2.9, $H$ is a subfield of $F$, we get $x - y$ and $xy^{-1}$ in $H$.

Now, $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(-y) \}$

$\geq \min \{ \mu_A(x), \mu_A(y) \}$.

Therefore, $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $H$.

Now, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y^{-1}) \}$

$\geq \min \{ \mu_A(x), \mu_A(y) \}$.

Therefore, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y \neq e$ in $H$.

Hence $H$ is either empty or a fuzzy subfield of $F$.

**3.2.11 Theorem:** If $A$ is an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$, then $H = \{ (x, \nu_A(x)) : \mu_A(x) = \mu_A(e) = \mu_A(e^1) \}$ and $\nu_A(x) = \nu_A(e) = \nu_A(e^1) \}$ is either empty or an anti-fuzzy subfield of $F$, where $e$ and $e^1$ are identity elements of $F$.

**Proof:** If no element satisfies this condition, then $H$ is empty.

If $x$ and $y$ satisfies this condition, by Theorem 3.2.9, $H$ is a subfield of $F$, we get $x - y$ and $xy^{-1}$ in $H$. 

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Now, $\nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(-y) \}$
\[
\leq \max \{ \nu_A(x), \nu_A(y) \}.
\]
Therefore, $\nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y$ in $H$.

Now, $\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y^{-1}) \}$
\[
\leq \max \{ \nu_A(x), \nu_A(y) \}.
\]
Therefore, $\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y \neq e$ in $H$.

Hence $H$ is either empty or an anti-fuzzy subfield of $F$.

**3.2.12 Theorem:** Let $A$ be an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$. Then (i) if $\mu_A(x-y) = 1$, then $\mu_A(x) = \mu_A(y)$, for all $x$ and $y$ in $F$ and if $\mu_A(xy^{-1}) = 1$, then $\mu_A(x) = \mu_A(y)$, for all $x$ and $y \neq e$ in $F$,
(ii) if $\nu_A(x-y) = 0$, then $\nu_A(x) = \nu_A(y)$, for all $x$ and $y$ in $F$ and if $\nu_A(xy^{-1}) = 0$, then $\nu_A(x) = \nu_A(y)$, for all $x$ and $y \neq e$ in $F$, where $e$ and $e'$ are identity elements of $F$.

**Proof:** Let $x$ and $y$ in $F$.

(i) Now, $\mu_A(x) = \mu_A(x-y+y)$
\[
\geq \min \{ \mu_A(x-y), \mu_A(y) \}
\]
\[
= \min \{ 1, \mu_A(y) \} = \mu_A(y)
\]
\[
= \mu_A(-y)
\]
\[
= \mu_A(-x+x-y)
\]
\[
\geq \min \{ \mu_A(-x), \mu_A(x-y) \}
\]
\[
= \min \{ \mu_A(-x), 1 \}
\]
\[
= \mu_A(-x) = \mu_A(x).
\]
Therefore, \( \mu_A(x) = \mu_A(y) \), for all \( x \) and \( y \) in \( F \).

And, \( \mu_A(x) = \mu_A(xy^{-1}y) \)

\[
\geq \min \{ \mu_A(xy^{-1}), \mu_A(y) \}
\]

\[
= \min \{ 1, \mu_A(y) \}
\]

\[
= \mu_A(y)
\]

\[
= \mu_A(y^{-1})
\]

\[
= \mu_A(x^{-1}y)
\]

\[
\geq \min \{ \mu_A(x^{-1}), \mu_A(xy^{-1}) \}
\]

\[
= \min \{ \mu_A(x^{-1}), 1 \}
\]

\[
= \mu_A(x^{-1})
\]

\[
= \mu_A(x).
\]

Therefore, \( \mu_A(x) = \mu_A(y) \), for all \( x \) and \( y \neq e \) in \( F \).

(ii) Now, \( \nu_A(x) = \nu_A(x-y+y) \)

\[
\leq \max \{ \nu_A(x-y), \nu_A(y) \}
\]

\[
= \max \{ 0, \nu_A(y) \} = \nu_A(y)
\]

\[
= \nu_A(-y)
\]

\[
= \nu_A(-x+x-y)
\]

\[
\leq \max \{ \nu_A(-x), \nu_A(x-y) \}
\]

\[
= \max \{ \nu_A(-x), 0 \}
\]

\[
= \nu_A(-x) = \nu_A(x).
\]

Therefore, \( \nu_A(x) = \nu_A(y) \), for all \( x \) and \( y \) in \( F \).
And, \( \nu_A(x) = \nu_A( x y^{-1} y ) \)
\[
\leq \max \{ \nu_A(xy^{-1}), \nu_A(y) \}
\]
\[
= \max \{ 0, \nu_A(y) \} = \nu_A(y)
\]
\[
= \nu_A(y^{-1})
\]
\[
= \nu_A(x^{-1} y y^{-1})
\]
\[
\leq \max \{ \nu_A(x^{-1}), \nu_A(xy^{-1}) \}
\]
\[
= \max \{ \nu_A(x^{-1}), 0 \}
\]
\[
= \nu_A(x^{-1}) = \nu_A(x).
\]

Therefore, \( \nu_A(x) = \nu_A(y) \), for all \( x \) and \( y \neq e \) in \( F \).

3.2.13 Theorem: If \( A \) be an intuitionistic fuzzy subfield of a field \(( F, +, \cdot )\), then (i) if \( \mu_A(x-y) = 0 \), then either \( \mu_A(x) = 0 \) or \( \mu_A(y) = 0 \), for all \( x \) and \( y \) in \( F \) and if \( \mu_A(xy^{-1}) = 0 \), then either \( \mu_A(x) = 0 \) or \( \mu_A(y) = 0 \), for all \( x \) and \( y \neq e \) in \( F \), (ii) if \( \nu_A(x-y) = 1 \), then either \( \nu_A(x) = 1 \) or \( \nu_A(y) = 1 \), for all \( x \) and \( y \) in \( F \) and if \( \nu_A(xy^{-1}) = 1 \), then either \( \nu_A(x) = 1 \) or \( \nu_A(y) = 1 \), for all \( x \) and \( y \neq e \) in \( F \), where \( e \) and \( e^1 \) are identity elements of \( F \).

Proof: Let \( x \) and \( y \) in \( F \).

(i) By the definition \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \} \),

which implies that \( 0 \geq \min \{ \mu_A(x), \mu_A(y) \} \).

Therefore, either \( \mu_A(x) = 0 \) or \( \mu_A(y) = 0 \), for all \( x \) and \( y \) in \( F \).

And, by the definition \( \mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \} \),

which implies that \( 0 \geq \min \{ \mu_A(x), \mu_A(y) \} \).

Therefore, either \( \mu_A(x) = 0 \) or \( \mu_A(y) = 0 \), for all \( x \) and \( y \neq e \) in \( F \).
(ii) By the definition $v_A(x - y) \leq \max \{ v_A(x), v_A(y) \}$,
which implies that $1 \leq \max \{ v_A(x), v_A(y) \}$.
Therefore, either $v_A(x) = 1$ or $v_A(y) = 1$, for all $x$ and $y$ in $F$.
And by the definition $v_A(xy^{-1}) \leq \max \{ v_A(x), v_A(y) \}$,
which implies that $1 \leq \max \{ v_A(x), v_A(y) \}$.
Therefore, either $v_A(x) = 1$ or $v_A(y) = 1$, for all $x$ and $y \neq e$ in $F$.

3.2.14 Theorem: Let $(F, +, \cdot)$ be a field. If $A$ is an intuitionistic fuzzy subfield of $F$, then $\mu_A(x+y) = \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$ and $\mu_A(xy) = \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$ and $v_A(x+y) = \max \{ v_A(x), v_A(y) \}$, for all $x$ and $y$ in $F$ and $v_A(xy) = \max \{ v_A(x), v_A(y) \}$, for all $x$ and $y$ in $F$ with $\mu_A(x) \neq \mu_A(y)$ and $v_A(x) \neq v_A(y)$, where 0 and 1 are identity elements of $F$.

Proof: Let $x$ and $y$ belongs to $F$.
Assume that $\mu_A(x) > \mu_A(y)$ and $v_A(x) < v_A(y)$.

Now, $\mu_A(y) = \mu_A(-x+x+y)$
\[
\geq \min \{ \mu_A(-x), \mu_A(x+y) \}
\]
\[
\geq \min \{ \mu_A(x), \mu_A(x+y) \}
\]
\[
= \mu_A(x+y)
\]
\[
\geq \min \{ \mu_A(x), \mu_A(y) \}
\]
\[
= \mu_A(y).
\]
Therefore, $\mu_A(x+y) = \mu_A(y) = \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$.
Now, $\mu_A(y) = \mu_A( x^{-1}xy)$
\[
\begin{align*}
\geq \min \{ \mu_A(x^{-1}), \mu_A(xy) \} \\
\geq \min \{ \mu_A(x), \mu_A(xy) \} \\
= \mu_A(xy) \\
\geq \min \{ \mu_A(x), \mu_A(y) \} \\
= \mu_A(y).
\end{align*}
\]

Therefore, \( \mu_A(xy) = \mu_A(y) = \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

And,
\[
\begin{align*}
\nu_A(y) &= \nu_A(-x + x + y) \\
&\leq \max \{ \nu_A(-x), \nu_A(x+y) \} \\
&\leq \max \{ \nu_A(x), \nu_A(x+y) \} \\
&= \nu_A(x+y) \\
&\leq \max \{ \nu_A(x), \nu_A(y) \} \\
&= \nu_A(y).
\end{align*}
\]

Therefore, \( \nu_A(x+y) = \nu_A(y) = \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

And,
\[
\begin{align*}
\nu_A(y) &= \nu_A(x^{-1}xy) \\
&\leq \max \{ \nu_A(x^{-1}), \nu_A(xy) \} \\
&\leq \max \{ \nu_A(x), \nu_A(xy) \} \\
&= \nu_A(xy) \\
&\leq \max \{ \nu_A(x), \nu_A(y) \} \\
&= \nu_A(y).
\end{align*}
\]

Therefore, \( \nu_A(xy) = \nu_A(y) = \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \).
3.2.15 Theorem: If A and B are any two intuitionistic fuzzy subfields of a field \((F, +, \cdot )\), then their intersection \(A \cap B\) is an intuitionistic fuzzy subfield of \(F\).

Proof: Let \(x\) and \(y\) belong to \(F\), \(A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle / x \in F \}\) and 
\[ B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle / x \in F \}. \]

Let \(C = A \cap B\) and \(C = \{ \langle x, \mu_C(x), \nu_C(x) \rangle / x \in F \}\), where 
\(\mu_C(x) = \min \{ \mu_A(x), \mu_B(x) \}\) and \(\nu_C(x) = \max \{ \nu_A(x), \nu_B(x) \}\).

(i) \[ \mu_C(x-y) = \min \{ \mu_A(x-y), \mu_B(x-y) \} \]
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_C(x), \mu_C(y) \}. \]

Therefore, \(\mu_C(x-y) \geq \min \{ \mu_C(x), \mu_C(y) \}\), for all \(x\) and \(y\) in \(F\).

(ii) \[ \mu_C(xy^{-1}) = \min \{ \mu_A(xy^{-1}), \mu_B(xy^{-1}) \} \]
\[ \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \}, \min \{ \mu_B(x), \mu_B(y) \} \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(y), \mu_B(y) \} \} \]
\[ = \min \{ \mu_C(x), \mu_C(y) \}. \]

Therefore, \(\mu_C(xy^{-1}) \geq \min \{ \mu_C(x), \mu_C(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).

(iii) \[ \nu_C(x-y) = \max \{ \nu_A(x-y), \nu_B(x-y) \} \]
\[ \leq \max \{ \max \{ \nu_A(x), \nu_A(y) \}, \max \{ \nu_B(x), \nu_B(y) \} \} \]
\[ = \max \{ \max \{ \nu_A(x), \nu_B(x) \}, \max \{ \nu_A(y), \nu_B(y) \} \} \]
\[ = \max \{ \nu_C(x), \nu_C(y) \}. \]

Therefore, \(\nu_C(x-y) \leq \max \{ \nu_C(x), \nu_C(y) \}\), for all \(x\) and \(y\) in \(F\).
(iv) $v_C(xy^{-1}) = \max \{ v_A(xy^{-1}), v_B(xy^{-1}) \}$

\[ \leq \max \{ \max \{ v_A(x), v_A(y) \}, \max \{ v_B(x), v_B(y) \} \} \]

\[ = \max \{ \max \{ v_A(x), v_B(x) \}, \max \{ v_A(y), v_B(y) \} \} \]

\[ = \max \{ v_C(x), v_C(y) \}. \]

Therefore, $v_C(xy^{-1}) \leq \max \{ v_C(x), v_C(y) \}$, for all $x$ and $y \neq 0$ in $F$.

Hence $A \cap B$ is an intuitionistic fuzzy subfield of a field $F$.

### 3.2.16 Theorem:

The intersection of a family of intuitionistic fuzzy subfields of a field $(F, +, \cdot)$ is an intuitionistic fuzzy subfield of $F$.

**Proof:** Let $\{ A_i \}_{i=1}^n$ be a family of intuitionistic fuzzy subfields of a field $F$ and $A = \bigcap_{i=1}^n A_i$, where $\mu_A(x) = \inf_{i=1}^n \mu_{A_i}(x)$ and $v_A(x) = \sup_{i=1}^n v_{A_i}(x)$.

Then for $x$ and $y$ belongs to $F$, we have

(i) $\mu_A(x-y) = \inf_{i=1}^n \mu_{A_i}(x-y)$

\[ \geq \inf_{i=1}^n \min \{ \mu_{A_i}(x), \mu_{A_i}(y) \} \]

\[ = \min \{ \inf_{i=1}^n (\mu_{A_i}(x)), \inf_{i=1}^n (\mu_{A_i}(y)) \} \]

\[ = \min \{ \mu_A(x), \mu_A(y) \}. \]

Therefore, $\mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$.

(ii) $\mu_A(xy^{-1}) = \inf_{i=1}^n \mu_{A_i}(xy^{-1})$

\[ \geq \inf_{i=1}^n \min \{ \mu_{A_i}(x), \mu_{A_i}(y) \} \]

\[ = \min \{ \inf_{i=1}^n (\mu_{A_i}(x)), \inf_{i=1}^n (\mu_{A_i}(y)) \} \]

\[ = \min \{ \mu_A(x), \mu_A(y) \}. \]
Therefore, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y \neq 0$ in $F$.

(iii) $\nu_A(x-y) = \sup_{i \in I} \nu_A(x-y)$

$\leq \sup_{i \in I} \max \{ \nu_A(x), \nu_A(y) \}$

$= \max \{ \sup_{i \in I} (\nu_A(x)), \sup_{i \in I} (\nu_A(y)) \}$

$= \max \{ \nu_A(x), \nu_A(y) \}.$

Therefore, $\nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y$ in $F$.

(iv) $\nu_A(xy^{-1}) = \sup_{i \in I} \nu_A(xy^{-1})$

$\leq \sup_{i \in I} \max \{ \nu_A(x), \nu_A(y) \}$

$= \max \{ \sup_{i \in I} (\nu_A(x)), \sup_{i \in I} (\nu_A(y)) \}$

$= \max \{ \nu_A(x), \nu_A(y) \}.$

Therefore, $\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y \neq 0$ in $F$.

Hence the intersection of a family of intuitionistic fuzzy subfields of a field $F$ is an intuitionistic fuzzy subfield of $F$.

3.2.17 Theorem: Let $A$ be an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$. If $\mu_A(x) < \mu_A(y)$ and $\nu_A(x) > \nu_A(y)$, for some $x$ and $y$ in $F$, then

(i) $\mu_A(x+y) = \mu_A(x) = \mu_A(y+x)$, for all $x$ and $y$ in $F$ and $\mu_A(xy) = \mu_A(x) = \mu_A(yx)$, for all $x$ and $y$ in $F$, (ii) $\nu_A(x+y) = \nu_A(x) = \nu_A(y+x)$, for all $x$ and $y$ in $F$ and $\nu_A(xy) = \nu_A(x) = \nu_A(yx)$, for all $x$ and $y$ in $F$.

proof: Let $A$ be an intuitionistic fuzzy subfield of a field $F$.

Also we have $\mu_A(x) < \mu_A(y)$ and $\nu_A(x) > \nu_A(y)$, for some $x$ and $y$ in $F$,
\[ \mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y) \} \]
\[ = \mu_A(x) \; \text{and} \]
\[ \mu_A(x) = \mu_A(x+y-y) \]
\[ \geq \min \{ \mu_A(x+y), \mu_A(y) \} \]
\[ \geq \min \{ \mu_A(x+y), \mu_A( -y ) \} \]
\[ = \mu_A(x+y). \]

Therefore, \( \mu_A(x+y) = \mu_A(x) \), for all \( x \) and \( y \) in \( F \).

Hence \( \mu_A(x+y) = \mu_A(x) = \mu_A(y+x) \), for all \( x \) and \( y \) in \( F \).

And, \( \mu_A(xy) \geq \min \{ \mu_A(x), \mu_A(y) \} \),
\[ = \mu_A(x) \; \text{and} \]
\[ \mu_A(x) = \mu_A(xyy^{-1}) \]
\[ \geq \min \{ \mu_A(xy), \mu_A(y^{-1}) \} \]
\[ \geq \min \{ \mu_A(xy), \mu_A(y) \} \]
\[ = \mu_A(xy). \]

Therefore, \( \mu_A(xy) = \mu_A(x) \), for all \( x \) and \( y \) in \( F \).

Hence \( \mu_A(xy) = \mu_A(x) = \mu_A(yx) \), for all \( x \) and \( y \) in \( F \).

Thus (i) is proved.

Now, \( \nu_A(x+y) \leq \max \{ \nu_A(x), \nu_A(y) \} \)
\[ = \nu_A(x) \; \text{and} \]
\[ \nu_A(x) = \nu_A(x+y-y) \]
\[ \leq \max \{ \nu_A(x+y), \nu_A(-y) \} \]
\[ \leq \max \{ \nu_A(x+y), \nu_A(y) \} , \]

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Therefore, \( \nu_A(x+y) = \nu_A(x) \), for all \( x \) and \( y \) in \( F \).

Hence \( \nu_A(x+y) = \nu_A(x) = \nu_A(y+x) \), for all \( x \) and \( y \) in \( F \).

Now, \( \nu_A(xy) \leq \max \{ \nu_A(x), \nu_A(y) \} \)

\[ \nu_A(x) = \nu_A(xyy^{-1}) \]

\[ \leq \max \{ \nu_A(xy), \nu_A(y^{-1}) \} \]

\[ \leq \max \{ \nu_A(xy), \nu_A(y) \} \]

\[ = \nu_A(xy). \]

Therefore, \( \nu_A(xy) = \nu_A(x) \), for all \( x \) and \( y \) in \( F \).

Hence \( \nu_A(xy) = \nu_A(x) = \nu_A(yx) \), for all \( x \) and \( y \) in \( F \).

Thus (ii) is proved.

**3.2.18 Theorem:** Let \( A \) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\). If \( \mu_A(x) < \mu_A(y) \) and \( \nu_A(x) < \nu_A(y) \), for some \( x \) and \( y \) in \( F \), then (i) \( \mu_A(x+y) = \mu_A(x) = \mu_A(y+x) \), for all \( x \) and \( y \) in \( F \) and \( \mu_A(xy) = \mu_A(x) = \mu_A(yx) \), for all \( x \) and \( y \) in \( F \) (ii) \( \nu_A(x+y) = \nu_A(x) = \nu_A(y+x) \), for all \( x \) and \( y \) in \( F \) and \( \nu_A(xy) = \nu_A(y) = \nu_A(yx) \), for all \( x \) and \( y \) in \( F \).

**proof:** It is trivial.

**3.2.19 Theorem:** Let \( A \) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\). If \( \mu_A(x) > \mu_A(y) \) and \( \nu_A(x) > \nu_A(y) \), for some \( x \) and \( y \) in \( F \), then (i) \( \mu_A(x+y) = \mu_A(y) = \mu_A(y+x) \), for all \( x \) and \( y \) in \( F \) and \( \mu_A(xy) = \mu_A(xy) = \mu_A(xy) \).
\[ \mu_A(y) = \mu_A(yx), \text{ for all } x \text{ and } y \text{ in } F, \]
\[ (ii) \nu_A(x+y) = \nu_A(x) = \nu_A(y+x), \text{ for all } x \text{ and } y \text{ in } F. \]

**Proof:** It is trivial.

**3.2.20 Theorem:** Let \( A \) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\). If \( \mu_A(x) > \mu_A(y) \) and \( \nu_A(x) < \nu_A(y) \), for some \( x \) and \( y \) in \( F \), then (i) \( \mu_A(x+y) = \mu_A(y) = \mu_A(y+x) \), for all \( x \) and \( y \) in \( F \) and \( \mu_A(xy) = \mu_A(y) = \mu_A(yx), \) for all \( x \) and \( y \) in \( F \) (ii) \( \nu_A(x+y) = \nu_A(y) = \nu_A(y+x) \), for all \( x \) and \( y \) in \( F \) and \( \nu_A(xy) = \nu_A(y) = \nu_A(yx) \), for all \( x \) and \( y \) in \( F \).

**Proof:** It is trivial.

**3.2.21 Theorem:** Let \( A \) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\) such that \( \text{Im} \ \mu_A = \{\alpha\} \) and \( \text{Im} \ \nu_A = \{\beta\} \), where \( \alpha \) and \( \beta \) in \([0, 1]\).

If \( A = B \cup C \), where \( B \) and \( C \) are intuitionistic fuzzy subfields of \( F \), then either \( B \subseteq C \) or \( C \subseteq B \).

**Proof:** Let \( A = B \cup C = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in F \} \),
\( B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \mid x \in F \} \) and \( C = \{ \langle x, \mu_C(x), \nu_C(x) \rangle \mid x \in F \} \).

**Case (i):** Assume that \( \mu_B(x) > \mu_C(x) \) and \( \mu_B(y) < \mu_C(y) \), for some \( x \) and \( y \) in \( F \).

Then, \[ \alpha = \mu_A(x) \]
\[ = \mu_{B \cup C}(x) \]
\[ = \max \{ \mu_B(x), \mu_C(x) \} \]
\[ = \mu_B(x) > \mu_C(x). \]

Therefore, \( \alpha > \mu_C(x) \).
And, \[ \alpha = \mu_A(y) \]
\[ = \mu_{B \cup C}(y) \]
\[ = \max \{ \mu_B(y), \mu_C(y) \} \]
\[ = \mu_C(y) > \mu_B(y). \]

Therefore, \( \alpha > \mu_B(y). \)

So that, \( \mu_C(y) > \mu_C(x) \) and \( \mu_B(x) > \mu_B(y). \)

Hence, \( \mu_B(x+y) = \mu_B(y) \), for all \( x \) and \( y \) in \( F \) and \( \mu_B(xy) = \mu_B(y) \), for all \( x \) and \( y \) in \( F \) and \( \mu_C(x+y) = \mu_C(x) \), for all \( x \) and \( y \) in \( F \) and \( \mu_C(xy) = \mu_C(x) \), for all \( x \) and \( y \) in \( F \), by Theorem 3.2.18 and 3.2.19.

But then, \( \alpha = \mu_A(x+y) = \mu_{B \cup C}(x+y) \)
\[ = \max \{ \mu_B(x+y), \mu_C(x+y) \} \]
\[ = \max \{ \mu_B(y), \mu_C(x) \} \]
\[ < \alpha \] \hspace{1cm} (1).

And \( \alpha = \mu_A(xy) = \mu_{B \cup C}(xy) \)
\[ = \max \{ \mu_B(xy), \mu_C(xy) \} \]
\[ = \max \{ \mu_B(y), \mu_C(x) \} \]
\[ < \alpha \] \hspace{1cm} (2).

**Case (ii):** Assume that \( \nu_B(x) < \nu_C(x) \) and \( \nu_B(y) > \nu_C(y) \), for some \( x \) and \( y \) in \( F \). Then, \( \beta = \nu_A(x) = \nu_{B \cup C}(x) \)
\[ = \min \{ \nu_B(x), \nu_C(x) \} \]
\[ = \nu_B(x) < \nu_C(x). \]

Therefore, \( \beta < \nu_C(x). \)
And, \[ \beta = \nu_A(y) = \nu_{B \cup C}(y) \]
\[ = \min \{ \nu_B(y), \nu_C(y) \} \]
\[ = \nu_C(y) < \nu_B(y). \]

Therefore, \( \beta < \nu_B(y) \).

So that, \( \nu_C(y) < \nu_C(x) \) and \( \nu_B(x) < \nu_B(y) \).

Hence \( \nu_B(x+y) = \nu_B(y) \), \( \nu_B(xy) = \nu_B(y) \) and \( \nu_C(x+y) = \nu_C(x) \), \( \nu_C(xy) = \nu_C(x) \), by Theorem 3.2.18 and 3.2.19.

But then, \( \beta = \nu_A(x+y) = \nu_{B \cup C}(x+y) \)
\[ = \min \{ \nu_B(x+y), \nu_C(x+y) \} \]
\[ = \min \{ \nu_B(y), \nu_C(x) \} \]
\[ > \beta \] \---------(3).

And \( \beta = \nu_A(xy) = \nu_{B \cup C}(xy) \)
\[ = \min \{ \nu_B(xy), \nu_C(xy) \} \]
\[ = \min \{ \nu_B(y), \nu_C(x) \} \]
\[ > \beta \] \---------(4).

It is a contradiction by (1), (2), (3) and (4).

Therefore, either \( B \subseteq C \) or \( C \subseteq B \) is true.

\textbf{3.2.22 Theorem:} If \( A \) is an intuitionistic fuzzy subfield of a field \( (F, +, \cdot) \), then \( \Box A \) is an intuitionistic fuzzy subfield of \( F \).

\textbf{Proof:} Let \( A \) be an intuitionistic fuzzy subfield of a field \( F \).

Consider \( A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \} \), for all \( x \) in \( F \), we take

\( \Box A = B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \} \), where \( \mu_B(x) = \mu_A(x) \), \( \nu_B(x) = 1 - \mu_A(x) \).
Clearly, $\mu_B(x-y) \geq \min \{ \mu_B(x), \mu_B(y) \}$, for all $x$ and $y$ in $F$ and $\mu_B(xy^{-1}) \geq \min \{ \mu_B(x), \mu_B(y) \}$, for all $x$ and $y \neq 0$ in $F$.

Since $A$ is an intuitionistic fuzzy subfield of $F$, we have $\mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $F$, which implies that $1-\nu_B(x-y) \geq \min \{ (1-\nu_B(x)), (1-\nu_B(y)) \}$,

which implies that $\nu_B(x-y) \leq 1-\min \{ (1-\nu_B(x)), (1-\nu_B(y)) \}$

\[ = \max \{ \nu_B(x), \nu_B(y) \}. \]

Therefore, $\nu_B(x-y) \leq \max \{ \nu_B(x), \nu_B(y) \}$, for all $x$ and $y$ in $F$.

And $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y \neq 0$ in $F$,

which implies that $1-\nu_B(xy^{-1}) \geq \min \{ (1-\nu_B(x)), (1-\nu_B(y)) \}$

which implies that $\nu_B(xy^{-1}) \leq 1-\min \{ (1-\nu_B(x)), (1-\nu_B(y)) \}$

\[ = \max \{ \nu_B(x), \nu_B(y) \}. \]

Therefore, $\nu_B(xy^{-1}) \leq \max \{ \nu_B(x), \nu_B(y) \}$, for all $x$ and $y \neq 0$ in $F$.

Hence $B = \Box A$ is an intuitionistic fuzzy subfield of a field $F$.

**Remark:** The converse of the above theorem is not true. It is shown by the following example:

Consider the field $Z_5 = \{ 0, 1, 2, 3, 4 \}$ with addition modulo 5 and multiplication modulo 5 operations. Then $A = \{ \langle 0, 0.7, 0.2 \rangle, \langle 1, 0.5, 0.1 \rangle, \langle 2, 0.5, 0.4 \rangle, \langle 3, 0.5, 0.1 \rangle, \langle 4, 0.5, 0.4 \rangle \}$ is not an intuitionistic fuzzy subfield of $Z_5$, but $\Box A = \{ \langle 0, 0.7, 0.3 \rangle, \langle 1, 0.5, 0.5 \rangle, \langle 2, 0.5, 0.5 \rangle, \langle 3, 0.5, 0.5 \rangle, \langle 4, 0.5, 0.5 \rangle \}$ is an intuitionistic fuzzy subfield of $Z_5$. 
3.2.23 Theorem: If $A$ is an intuitionistic fuzzy subfield of a field $(F, +, \cdot)$, then $\Diamond A$ is an intuitionistic fuzzy subfield of $F$.

Proof: Let $A$ be an intuitionistic fuzzy subfield of a field $F$.

That is $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \}$, for all $x$ in $F$.

Let $\Diamond A = B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \}$, where $\mu_B(x) = 1 - \nu_A(x)$, $\nu_B(x) = \nu_A(x)$.

Clearly, $\nu_B(x-y) \leq \max\{ \nu_B(x), \nu_B(y) \}$, for all $x$ and $y$ in $F$ and

$\nu_B(xy^{-1}) \leq \max\{ \nu_B(x), \nu_B(y) \}$, for all $x$ and $y \neq 0$ in $F$.

Since $A$ is an intuitionistic fuzzy subfield of $F$, we have

$\nu_A(x-y) \leq \max\{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y$ in $F$,

which implies that $1 - \mu_B(x-y) \leq \max\{ (1 - \mu_B(x)), (1 - \mu_B(y)) \}$

which implies that $\mu_B(x-y) \geq 1 - \max\{ (1 - \mu_B(x)), (1 - \mu_B(y)) \}$

$= \min\{ \mu_B(x), \mu_B(y) \}$.

Therefore, $\mu_B(x-y) \geq \min\{ \mu_B(x), \mu_B(y) \}$, for all $x$ and $y$ in $F$.

And $\nu_A(xy^{-1}) \leq \max\{ \nu_A(x), \nu_A(y) \}$, for all $x$ and $y \neq 0$ in $F$,

which implies that $1 - \mu_B(xy^{-1}) \leq \max\{ (1 - \mu_B(x)), (1 - \mu_B(y)) \}$,

which implies that $\mu_B(xy^{-1}) \geq 1 - \max\{ (1 - \mu_B(x)), (1 - \mu_B(y)) \}$

$= \min\{ \mu_B(x), \mu_B(y) \}$.

Therefore, $\mu_B(xy^{-1}) \geq \min\{ \mu_B(x), \mu_B(y) \}$, for all $x$ and $y \neq 0$ in $F$.

Hence $B = \Diamond A$ is an intuitionistic fuzzy subfield of a field $F$.

Remark: The converse of the above theorem is not true. It is shown by the following example:
Consider the field \( Z_5 = \{ 0, 1, 2, 3, 4 \} \) with addition modulo 5 and multiplication modulo 5 operations. Then \( A = \{ \langle 0, 0.5, 0.1 \rangle, \langle 1, 0.6, 0.4 \rangle, \langle 2, 0.5, 0.4 \rangle, \langle 3, 0.6, 0.4 \rangle, \langle 4, 0.5, 0.4 \rangle \} \) is not an intuitionistic fuzzy subfield of \( Z_5 \), but \( \hat{A} = \{ \langle 0, 0.9, 0.1 \rangle, \langle 1, 0.6, 0.4 \rangle, \langle 2, 0.6, 0.4 \rangle, \langle 3, 0.6, 0.4 \rangle, \langle 4, 0.6, 0.4 \rangle \} \) is an intuitionistic fuzzy subfield of \( Z_5 \).

### 3.2.24 Theorem:
If \( A \) is an intuitionistic fuzzy subfield of a field \(( F, +, \cdot )\) and if there is a sequence \((x_n)\) in \( F \) such that

\[
\lim_{n \to \infty} \min\{ \mu_A(x_n), \mu_A(x_n) \} = 1 \quad \text{and} \quad \lim_{n \to \infty} \max\{ \nu_A(x_n), \nu_A(x_n) \} = 0,
\]

then \( \mu_A(e) = \mu_A(e^1) = 1 \) and \( \nu_A(e) = \nu_A(e^1) = 0 \), where \( e \) and \( e^1 \) are the identity elements in \( F \).

**Proof:** Let \( A \) be an intuitionistic fuzzy subfield of a field \( F \) and \( e \) and \( e^1 \) be the identity elements of \( F \) and \( x \) be an arbitrary element of \( F \).

We have \( x \) in \( F \) implies \( x-x = e \) and \( x \) in \( F-e \) implies \( xx^{-1} = e^1 \).

Then, we have

\[
\mu_A(e) = \mu_A(x-x) \\
\geq \min \{ \mu_A(x), \mu_A(x) \}.
\]

For each \( x \) in \( F \), we have \( \mu_A(e) \geq \min \{ \mu_A(x), \mu_A(x) \} \).

Since \( \mu_A(e) \geq \lim_{n \to \infty} \min \{ \mu_A(x_n), \mu_A(x_n) \} = 1 \),

Therefore \( \mu_A(e) = 1 \).

And \( \mu_A(e^1) = \mu_A(xx^{-1}) \)

\[
\geq \min \{ \mu_A(x), \mu_A(x) \}.
\]
For each x in \( F - \{ e \} \), we have \( \mu_A(e^1) \geq \min \{ \mu_A(x), \mu_A(x) \} \).

Since \( \mu_A(e^1) \geq \lim_{n \to \infty} \min \{ \mu_A(x_n), \mu_A(x_n) \} = 1 \).

Therefore \( \mu_A(e^1) = 1 \).

And, \( \nu_A(e) = \nu_A(x-x) \leq \max \{ \nu_A(x), \nu_A(x) \} \).

For each x in \( F \), we have \( \nu_A(e) \leq \max \{ \nu_A(x), \nu_A(x) \} \).

Since \( \nu_A(e) \leq \lim_{n \to \infty} \max \{ \nu_A(x_n), \nu_A(x_n) \} = 0 \).

Therefore, \( \nu_A(e) = 0 \).

And \( \nu_A(e^1) = \nu_A(xx^{-1}) \leq \max \{ \nu_A(x), \nu_A(x) \} \).

For each x in \( F - \{ e \} \), we have \( \nu_A(e^1) \leq \max \{ \nu_A(x), \nu_A(x) \} \).

Since \( \nu_A(e^1) \leq \lim_{n \to \infty} \max \{ \nu_A(x_n), \nu_A(x_n) \} = 0 \).

Therefore, \( \nu_A(e^1) = 0 \).

3.2.25 Theorem: If A and B are intuitionistic fuzzy subfields of the fields G and H respectively, then \( A \times B \) is an intuitionistic fuzzy subfield of \( G \times H \).

Proof: Let A and B be intuitionistic fuzzy subfields of the fields G and H respectively.

Let \( x_1 \) and \( x_2 \) be in \( G \), \( y_1 \) and \( y_2 \) be in \( H \).

Then \( (x_1, y_1) \) and \( (x_2, y_2) \) are in \( G \times H \).

Now, \( \mu_{A \times B} \left[ (x_1, y_1) - (x_2, y_2) \right] = \mu_{A \times B} (x_1 - x_2, y_1 - y_2) \)
\[= \min \{ \mu_A(x_1 - x_2), \mu_B(y_1 - y_2) \}\]
\[\geq \min\{ \min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\} \}\]
\[= \min\{ \min\{\mu_A(x_1), \mu_B(y_1)\}, \min\{\mu_A(x_2), \mu_B(y_2)\} \}\]
\[= \min\{ \mu_{AXB}(x_1, y_1), \mu_{AXB}(x_2, y_2) \}\].

Therefore, \( \mu_{AXB}[(x_1, y_1) - (x_2, y_2)] \geq \min \{ \mu_{AXB}(x_1, y_1), \mu_{AXB}(x_2, y_2) \} \), for all \( x_1 \) and \( x_2 \) in \( G \) and \( y_1 \) and \( y_2 \) in \( H \).

Now, \( \mu_{AXB}[(x_1, y_1)(x_2, y_2)^{-1}] = \mu_{AXB}(x_1x_2^{-1}, y_1y_2^{-1}) \)
\[= \min \{ \mu_A(x_1x_2^{-1}), \mu_B(y_1y_2^{-1}) \}\]
\[\geq \min\{ \min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_B(y_1), \mu_B(y_2)\} \}\]
\[= \min\{ \min\{\mu_A(x_1), \mu_B(y_1)\}, \min\{\mu_A(x_2), \mu_B(y_2)\} \}\]
\[= \min\{ \mu_{AXB}(x_1, y_1), \mu_{AXB}(x_2, y_2) \}\].

Therefore, \( \mu_{AXB}[(x_1, y_1)(x_2, y_2)^{-1}] \geq \min \{ \mu_{AXB}(x_1, y_1), \mu_{AXB}(x_2, y_2) \} \), for all \( x_1 \) and \( x_2 \neq 0 \) in \( G \) and \( y_1 \) and \( y_2 \neq 0 \) in \( H \).

And, \( \nu_{AXB}[(x_1, y_1) - (x_2, y_2)] = \nu_{AXB}(x_1 - x_2, y_1 - y_2) \)
\[= \max \{ \nu_A(x_1 - x_2), \nu_B(y_1 - y_2) \}\]
\[\leq \max\{ \max\{\nu_A(x_1), \nu_A(x_2)\}, \max\{\nu_B(y_1), \nu_B(y_2)\} \}\]
\[= \max\{ \max\{\nu_A(x_1), \nu_B(y_1)\}, \max\{\nu_A(x_2), \nu_B(y_2)\} \}\]
\[= \max\{ \nu_{AXB}(x_1, y_1), \nu_{AXB}(x_2, y_2) \}\].

Therefore, \( \nu_{AXB}[(x_1, y_1) - (x_2, y_2)] \leq \max \{ \nu_{AXB}(x_1, y_1), \nu_{AXB}(x_2, y_2) \} \), for all \( x_1 \) and \( x_2 \) in \( G \) and \( y_1 \) and \( y_2 \) in \( H \).

And, \( \nu_{AXB}[(x_1, y_1)(x_2, y_2)^{-1}] = \nu_{AXB}(x_1x_2^{-1}, y_1y_2^{-1}) \)
\[= \max\{\nu_A(x_1x_2^{-1}), \nu_B(y_1y_2^{-1})\}\]
\[ \leq \max \left\{ \max \{ \nu_A(x_1), \nu_A(x_2) \}, \max \{ \nu_B(y_1), \nu_B(y_2) \} \right\} \]
\[ = \max \left\{ \max \{ \nu_A(x_1), \nu_B(y_1) \}, \max \{ \nu_A(x_2), \nu_B(y_2) \} \right\} \]
\[ = \max \left\{ \nu_{AXB}(x_1, y_1), \nu_{AXB}(x_2, y_2) \right\}. \]

Therefore, \( \nu_{AXB}[ (x_1, y_1)(x_2, y_2)^{-1}] \leq \max \left\{ \nu_{AXB}(x_1, y_1), \nu_{AXB}(x_2, y_2) \right\} \), for all \( x_1 \) and \( x_2 \neq 0 \) in \( G \) and \( y_1 \) and \( y_2 \neq 0^{\prime} \) in \( H \).

Hence \( AxB \) is an intuitionistic fuzzy subfield of \( GxH \).

**3.2.26 Theorem:** Let \( A \) and \( B \) be intuitionistic fuzzy subsets of the fields \( G \) and \( H \) respectively. Suppose that \( 0, 1 \) and \( 0^1, 1^1 \) are the identity elements of \( G \) and \( H \) respectively. If \( AxB \) is an intuitionistic fuzzy subfield of \( GxH \), then at least one of the following two statements must hold.

(i) \( \mu_B(0^1) \geq \mu_A(x) \), for all \( x \) in \( G \) and \( \mu_B(1^1) \geq \mu_A(x) \), for all \( x \) in \( G \) and \( \nu_B(0^1) \leq \nu_A(x) \), for all \( x \) in \( G \) and \( \nu_B(1^1) \leq \nu_A(x) \), for all \( x \) in \( G \),

(ii) \( \mu_A(0) \geq \mu_B(y) \), for all \( y \) in \( H \) and \( \mu_A(1) \geq \mu_B(y) \), for all \( y \) in \( H \) and \( \nu_A(0) \leq \nu_B(y) \), for all \( y \) in \( H \) and \( \nu_A(1) \leq \nu_B(y) \), for all \( y \) in \( H \).

**Proof:** Let \( AxB \) is an intuitionistic fuzzy subfield of \( GxH \).

By contraposition, suppose that none of the statements (i) and (ii) holds.

Then we can find \( a \) in \( G \) and \( b \) in \( H \) such that \( \mu_A(a) > \mu_B(0^1) \), \( \nu_A(a) < \nu_B(0^1) \) and \( \mu_B(b) > \mu_A(0) \), \( \nu_B(b) < \nu_A(0) \) and we can find \( a \) in \( G \) and \( b \) in \( H \) such that \( \mu_A(a) > \mu_B(1^1) \), \( \nu_A(a) < \nu_B(1^1) \) and \( \mu_B(b) > \mu_A(1) \), \( \nu_B(b) < \nu_A(1) \).

We have, \( \mu_{AXB}(a, b) = \min \{ \mu_A(a), \mu_B(b) \} \)
\[ > \min \{ \mu_A(0), \mu_B(0^1) \} \]
\[ = \mu_{A \times B}(0, 0^1). \]
And, \( \mu_{A \times B}(a, b) = \min \{ \mu_A(a), \mu_B(b) \} \)
\[ > \min \{ \mu_A(1), \mu_B(1^1) \} \]
\[ = \mu_{A \times B}(1, 1^1). \]
We have, \( \nu_{A \times B}(a, b) = \max \{ \nu_A(a), \nu_B(b) \} \)
\[ < \max \{ \nu_A(0), \nu_B(0^1) \} \]
\[ = \nu_{A \times B}(0, 0^1). \]
And, \( \nu_{A \times B}(a, b) = \max \{ \nu_A(a), \nu_B(b) \} \)
\[ < \max \{ \nu_A(1), \nu_B(1^1) \} \]
\[ = \nu_{A \times B}(1, 1^1). \]
Thus AxB is not an intuitionistic fuzzy subfield of GxH.

Hence either \( \mu_B(0^1) \geq \mu_A(x) \), for all x in G and \( \mu_B(1^1) \geq \mu_A(x) \), for all x in G and \( \nu_B(0^1) \leq \nu_A(x) \), for all x in G and \( \nu_B(1^1) \leq \nu_A(x) \), for all x in G or \( \mu_A(0) \geq \mu_B(y) \), for all y in H and \( \mu_A(1) \geq \mu_B(y) \), for all y in H and \( \nu_A(0) \leq \nu_B(y) \), for all y in H and \( \nu_A(1) \leq \nu_B(y) \), for all y in H.

3.2.27 Theorem: Let A and B be intuitionistic fuzzy subsets of the fields G and H, respectively and AxB is an intuitionistic fuzzy subfield of GxH. Then the following are true:

(i) if \( \mu_A(x) \leq \mu_B(0^1) \), for all x in G and \( \mu_A(x) \leq \mu_B(1^1) \), for all x in G and \( \nu_A(x) \geq \nu_B(0^1) \), for all x in G and \( \nu_A(x) \geq \nu_B(1^1) \), for all x in G, then A is an intuitionistic fuzzy subfield of G, where \( 0^1, 1^1 \) are identity elements of H.
(ii) if $\mu_B(x) \leq \mu_A(0)$ for all $x$ in $H$ and $\mu_B(x) \leq \mu_A(1)$, for all $x$ in $H$ and $\nu_B(x) \geq \nu_A(0)$, for all $x$ in $H$ and $\nu_B(x) \geq \nu_A(1)$, for all $x$ in $H$, then $B$ is an intuitionistic fuzzy subfield of $H$, where $0$, $1$ are identity elements of $G$.

(iii) either $A$ is an intuitionistic fuzzy subfield of $G$ or $B$ is an intuitionistic fuzzy subfield of $H$, where $0$, $1$ and $0^i$, $1^i$ are the identity elements of $G$ and $H$ respectively.

**Proof:** Let $AxB$ be an intuitionistic fuzzy subfield of $GxH$ and $x$ and $y$ in $G$. Then $(x, 0^i)$, $(x, 1^i)$ and $(y, 0^i)$, $(y, 1^i)$ are in $GxH$.

Now, using the property if $\mu_A(x) \leq \mu_B(0^i)$, for all $x$ in $G$ and $\mu_A(x) \leq \mu_B(1^i)$, for all $x$ in $G$ and $\nu_A(x) \geq \nu_B(0^i)$, for all $x$ in $G$ and $\nu_A(x) \geq \nu_B(1^i)$, for all $x$ in $G$, where $0$ and $1$ are identity elements of $G$ and $0^i$ and $1^i$ are identity elements of $H$,

we get, 

$$\mu_A(x-y) = \min \left\{ \mu_A(x-y), \mu_B(0^i+0^j) \right\}$$

$$= \mu_{AxB}(x-y, (0^i+0^j))$$

$$= \mu_{AxB}(x, 0^i) + (-y, 0^j)$$

$$\geq \min \left\{ \mu_{AxB}(x, 0^i), \mu_{AxB}(-y, 0^j) \right\}$$

$$= \min \{ \min \{ \mu_A(x), \mu_B(0^j) \}, \min \{ \mu_A(-y), \mu_B(0^j) \} \}$$

$$= \min \{ \mu_A(x), \mu_A(-y) \}$$

$$\geq \min \{ \mu_A(x), \mu_A(y) \}.$$ 

Therefore, $\mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \}$, for all $x$ and $y$ in $G$.

And 

$$\mu_A(xy^{-1}) = \min \{ \mu_A(xy^{-1}), \mu_B(1^i1^j) \}$$

$$= \mu_{AxB}(xy^{-1}, (1^i1^j))$$

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\[ \mu_{AB} [ (x, 1^i)(y^{-1}, 1^i) ] \]
\[ \geq \min \{ \mu_{AB}(x, 1^i), \mu_{AB}(y^{-1}, 1^i) \} \]
\[ = \min \{ \min \{ \mu_A(x), \mu_B(1^i) \}, \min \{ \mu_A(y^{-1}), \mu_B(1^i) \} \} \]
\[ = \min \{ \mu_A(x), \mu_A(y^{-1}) \} \]
\[ \geq \min \{ \mu_A(x), \mu_A(y) \}. \]

Therefore, \( \mu_A( xy^{-1} ) \geq \min \{ \mu_A(x), \mu_A(y) \}, \) for all \( x \) and \( y \neq 0 \) in \( G \).

And,
\[ \nu_A( x-y ) = \max \{ \nu_A(x-y), \nu_B(0^i+0^i) \} \]
\[ = \nu_{AB} [ (x-y), (0^i+0^i) ] \]
\[ = \nu_{AB} [ (x, 0^i)+(-y, 0^i) ] \]
\[ \leq \max \{ \nu_{AB}(x, 0^i), \nu_{AB}(-y, 0^i) \} \]
\[ = \max \{ \max \{ \nu_A(x), \nu_B(0^i) \}, \max \{ \nu_A(-y), \nu_B(0^i) \} \} \]
\[ = \max \{ \nu_A(x), \nu_A(-y) \} \]
\[ \leq \max \{ \nu_A(x), \nu_A(y) \}. \]

Therefore, \( \nu_A( x-y ) \leq \max \{ \nu_A(x), \nu_A(y) \}, \) for all \( x \) and \( y \) in \( G \).

And,
\[ \nu_A( xy^{-1} ) = \max \{ \nu_A(xy^{-1}), \nu_B(1^i1^i) \} \]
\[ = \nu_{AB} [ (xy^{-1}), (1^i1^i) ] \]
\[ = \nu_{AB} [ (x, 1^i)(y^{-1}, 1^i) ] \]
\[ \leq \max \{ \nu_{AB}(x, 1^i), \nu_{AB}(y^{-1}, 1^i) \} \]
\[ = \max \{ \max \{ \nu_A(x), \nu_B(1^i) \}, \max \{ \nu_A(y^{-1}), \nu_B(1^i) \} \} \]
\[ = \max \{ \nu_A(x), \nu_A(y^{-1}) \} \]
\[ \leq \max \{ \nu_A(x), \nu_A(y) \}. \]

Therefore, \( \nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}, \) for all \( x \) and \( y \neq 0 \) in \( G \).
Hence A is an intuitionistic fuzzy subfield of G.

Thus (i) is proved.

Now, using the property $\mu_B(x) \leq \mu_A(0)$ for all $x$ in $H$ and $\mu_B(x) \leq \mu_A(1)$, for all $x$ in $H$ and $\nu_B(x) \geq \nu_A(0)$, for all $x$ in $H$ and $\nu_B(x) \geq \nu_A(1)$, for all $x$ in $H$,
we get,

$\mu_B(x-y) = \min \{ \mu_B(x-y), \mu_A(0) \}$

$= \mu_{AXB}(0+0, x-y)$

$= \mu_{AXB}(x+y, -y)$

$\geq \min \{ \mu_{AXB}(0, x), \mu_{AXB}(0, -y) \}$

$= \min \{ \min \{ \mu_A(0), \mu_B(x) \}, \min \{ \mu_A(0), \mu_B(-y) \} \}$

$= \min \{ \mu_B(x), \mu_B(-y) \}$

$\geq \min \{ \mu_B(x), \mu_B(y) \}.$

Therefore, $\mu_B(x-y) \geq \min \{ \mu_B(x), \mu_B(y) \}$, for all $x$ and $y$ in $H$.

And $\mu_B(xy^{-1}) = \min \{ \mu_B(xy^{-1}), \mu_A(1) \}$

$= \mu_{AXB}(1.1, xy^{-1})$

$= \mu_{AXB}(1, x)(1, y^{-1})$

$\geq \min \{ \mu_{AXB}(1, x), \mu_{AXB}(1, y^{-1}) \}$

$= \min \{ \min \{ \mu_A(1), \mu_B(x) \}, \min \{ \mu_A(1), \mu_B(y^{-1}) \} \}$

$\geq \min \{ \mu_B(x), \mu_B(y^{-1}) \}$.

Therefore, $\mu_B(xy^{-1}) \geq \min \{ \mu_B(x), \mu_B(y) \}$, for all $x$ and $y \neq 0$ in $H$.

And, $\nu_B(x-y) = \max \{ \nu_B(x-y), \nu_A(0+0) \}$
\[ \nu_{A \times B}((0+0), (x-y)) \]
\[ \nu_{A \times B}((0, x)+(0, -y)) \]
\[ \leq \max\{\nu_{A \times B}(0, x), \nu_{A \times B}(0, -y)\} \]
\[ = \max\{\min\{\nu_A(0), \nu_B(x)\}, \min\{\nu_A(0), \nu_B(-y)\}\} \]
\[ = \max\{\nu_B(x), \nu_B(-y)\} \]
\[ \leq \max\{\nu_B(x), \nu_B(y)\}. \]

Therefore, \( \nu_B(x-y) \leq \max\{\nu_B(x), \nu_B(y)\} \), for all \( x \) and \( y \) in \( H \).

And, \( \nu_B(xy^{-1}) = \max\{\nu_B(xy^{-1}), \nu_A(1,1)\} \)
\[ \nu_{A \times B}((1,1), (xy^{-1})) \]
\[ \nu_{A \times B}((1, x)(1, y^{-1})) \]
\[ \leq \max\{\nu_{A \times B}(1, x), \nu_{A \times B}(1, y^{-1})\} \]
\[ = \max\{\min\{\nu_A(1), \nu_B(x)\}, \min\{\nu_A(1), \nu_B(y^{-1})\}\} \]
\[ = \max\{\nu_B(x), \nu_B(y^{-1})\} \]
\[ \leq \max\{\nu_B(x), \nu_B(y)\}. \]

Therefore, \( \nu_B(xy^{-1}) \leq \max\{\nu_B(x), \nu_B(y)\} \), for all \( x \) and \( y \neq 0 \) in \( H \).

Hence \( B \) is an intuitionistic fuzzy subfield of \( H \).

Thus (ii) is proved.

Hence (iii) is clear.

3.2.28 Theorem: Let \( A \) be an intuitionistic fuzzy subset of a field \((F, +, \cdot)\) and \( V \) be the strongest intuitionistic fuzzy relation of \( F \). Then \( A \) is an intuitionistic fuzzy subfield of \( F \) if and only if \( V \) is an intuitionistic fuzzy subfield of \( F \times F \).
**Proof:** Suppose that $A$ is an intuitionistic fuzzy subfield of $F$.

Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $FxF$.

We have, $\mu_V(x - y) = \mu_V[(x_1, x_2) - (y_1, y_2)]$

$$= \mu_V(x_1 - y_1, x_2 - y_2)$$

$$= \min \{\mu_A(x_1 - y_1), \mu_A(x_2 - y_2)\}$$

$$\geq \min \{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_A(x_2), \mu_A(y_2)\}\}$$

$$= \min \{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_A(y_1), \mu_A(y_2)\}\}$$

$$= \min \{\mu_V(x_1, x_2), \mu_V(y_1, y_2)\} = \min \{\mu_V(x), \mu_V(y)\}.$$

Therefore, $\mu_V(x - y) \geq \min \{\mu_V(x), \mu_V(y)\}$, for all $x$ and $y$ in $FxF$.

And, $\mu_V(xy^{-1}) = \mu_V[(x_1, x_2)(y_1, y_2)^{-1}]$

$$= \mu_V(x_1y_1^{-1}, x_2y_2^{-1})$$

$$= \min \{\mu_A(x_1y_1^{-1}), \mu_A(x_2y_2^{-1})\}$$

$$\geq \min \{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_A(x_2), \mu_A(y_2)\}\}$$

$$= \min \{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_A(y_1), \mu_A(y_2)\}\}$$

$$= \min \{\mu_V(x_1, x_2), \mu_V(y_1, y_2)\} = \min \{\mu_V(x), \mu_V(y)\}.$$

Therefore, $\mu_V(xy^{-1}) \geq \min \{\mu_V(x), \mu_V(y)\}$, for all $x$ and $y \neq (0,0)$ in $FxF$.

Also we have, $\nu_V(x - y) = \nu_V[(x_1, x_2) - (y_1, y_2)]$

$$= \nu_V(x_1 - y_1, x_2 - y_2)$$

$$= \max \{\nu_A(x_1 - y_1), \nu_A(x_2 - y_2)\}$$

$$\leq \max \{\max\{\nu_A(x_1), \nu_A(y_1)\}, \max\{\nu_A(x_2), \nu_A(y_2)\}\}$$

$$= \max \{\max\{\nu_A(x_1), \nu_A(x_2)\}, \max\{\nu_A(y_1), \nu_A(y_2)\}\}$$

$$= \max \{\nu_V(x_1, x_2), \nu_V(y_1, y_2)\} = \max \{\nu_V(x), \nu_V(y)\}.$$
Therefore, \( \nu_V(x - y) \leq \max\{\nu_V(x), \nu_V(y)\} \), for all \( x \) and \( y \) in \( FxF \).

And \( \nu_V(xy^{-1}) = \nu_V[(x_1, x_2)(y_1, y_2)^{-1}] \)

\[ = \nu_V(x_1y_1^{-1}, x_2y_2^{-1}) \]

\[ = \max\{\nu_A(x_1y_1^{-1}), \nu_A(x_2y_2^{-1})\} \]

\[ \leq \max\{\max\{\nu_A(x_1), \nu_A(y_1)\}, \max\{\nu_A(x_2), \nu_A(y_2)\}\} \]

\[ = \max\{\max\{\nu_A(x_1), \nu_A(x_2)\}, \max\{\nu_A(y_1), \nu_A(y_2)\}\} \]

\[ = \max\{\nu_V(x_1, x_2), \nu_V(y_1, y_2)\} = \max\{\nu_V(x), \nu_V(y)\}. \]

Therefore, \( \nu_V(xy^{-1}) \leq \max\{\nu_V(x), \nu_V(y)\} \), for all \( x \) and \( y \neq (0,0) \) in \( FxF \).

This proves that \( V \) is an intuitionistic fuzzy subfield of \( FxF \).

Conversely, assume that \( V \) is an intuitionistic fuzzy subfield of \( FxF \), then for any \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) are in \( FxF \), we have

\[ \min\{\mu_A(x_1 - y_1), \mu_A(x_2 - y_2)\} = \mu_V(x_1 - y_1, x_2 - y_2) \]

\[ = \mu_V[(x_1, x_2) - (y_1, y_2)] \]

\[ = \mu_V(x - y) \]

\[ \geq \min\{\mu_V(x), \mu_V(y)\} \]

\[ = \min\{\mu_V(x_1, x_2), \mu_V(y_1, y_2)\} \]

\[ = \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_A(y_1), \mu_A(y_2)\}\}. \]

If we put \( x_2 = y_2 = 0 \), we get, \( \mu_A(x_1 - y_1) \geq \min\{\mu_A(x_1), \mu_A(y_1)\} \), for all \( x_1 \) and \( y_1 \) in \( F \).

And, \( \min\{\mu_A(x_1y_1^{-1}), \mu_A(x_2y_2^{-1})\} = \mu_V(x_1y_1^{-1}, x_2y_2^{-1}) \)

\[ = \mu_V[(x_1, x_2)(y_1, y_2)^{-1}] \]

\[ = \mu_V(xy^{-1}) \]
\[ \geq \min \{ \mu_{V}(x), \mu_{V}(y) \} \]
\[ = \min \{ \mu_{V}(x_{1}, x_{2}), \mu_{V}(y_{1}, y_{2}) \} \]
\[ = \min \{ \min \{ \mu_{A}(x_{1}), \mu_{A}(x_{2}) \}, \min \{ \mu_{A}(y_{1}), \mu_{A}(y_{2}) \} \}. \]

If we put \( x_{2} = y_{2} = 1 \), we get, \( \mu_{A}(x_{1}y_{1}^{-1}) \geq \min \{ \mu_{A}(x_{1}), \mu_{A}(y_{1}) \} \), for all \( x_{1} \) and \( y_{1} \neq 0 \) in \( F \).

Also we have, \( \max \{ \nu_{A}(x_{1} - y_{1}), \nu_{A}(x_{2} - y_{2}) \} = \nu_{V}(x_{1} - y_{1}, x_{2} - y_{2}) \)
\[ = \nu_{V}(x_{1}, x_{2}) - (y_{1}, y_{2}) \]
\[ = \nu_{V}(x - y) \]
\[ \leq \max \{ \nu_{V}(x), \nu_{V}(y) \} \]
\[ = \max \{ \nu_{V}(x_{1}, x_{2}), \nu_{V}(y_{1}, y_{2}) \} \]
\[ = \max \{ \max \{ \nu_{A}(x_{1}), \nu_{A}(x_{2}) \}, \max \{ \nu_{A}(y_{1}), \nu_{A}(y_{2}) \} \}. \]

If we put \( x_{2} = y_{2} = 0 \), we get, \( \nu_{A}(x_{1} - y_{1}) \leq \max \{ \nu_{A}(x_{1}), \nu_{A}(y_{1}) \} \), for all \( x_{1} \) and \( y_{1} \) in \( F \).

\[ \max \{ \nu_{A}(x_{1}y_{1}^{-1}), \nu_{A}(x_{2}y_{2}^{-1}) \} = \nu_{V}(x_{1}y_{1}^{-1}, x_{2}y_{2}^{-1}) \]
\[ = \nu_{V}(x_{1}, x_{2})(y_{1}, y_{2})^{-1} \]
\[ = \nu_{V}(xy^{-1}) \]
\[ \leq \max \{ \nu_{V}(x), \nu_{V}(y) \} \]
\[ = \max \{ \nu_{V}(x_{1}, x_{2}), \nu_{V}(y_{1}, y_{2}) \} \]
\[ = \max \{ \max \{ \nu_{A}(x_{1}), \nu_{A}(x_{2}) \}, \max \{ \nu_{A}(y_{1}), \nu_{A}(y_{2}) \} \}. \]

If we put \( x_{2} = y_{2} = 1 \), we get, \( \nu_{A}(x_{1}y_{1}^{-1}) \leq \max \{ \nu_{A}(x_{1}), \nu_{A}(y_{1}) \} \), for all \( x_{1} \) and \( y_{1} \neq 0 \) in \( F \).

Hence \( A \) is an intuitionistic fuzzy subfield of \( F \).
3.3 INTUITIONISTIC FUZZY SUBFIELDS OF A FIELD

UNDER HOMOMORPHISM AND ANTI-HOMOMORPHISM

3.3.1 Theorem: Let \((F, +, \cdot)\) and \((F', +, \cdot)\) be any two fields. The homomorphic image of an intuitionistic fuzzy subfield of \(F\) is an intuitionistic fuzzy subfield of \(F'\).

Proof: Let \((F, +, \cdot)\) and \((F', +, \cdot)\) be any two fields and \(f : F \rightarrow F'\) be a homomorphism. That is \(f(x+y) = f(x)+f(y)\) for all \(x\) and \(y\) in \(F\),
\(f(xy) = f(x)f(y)\), for all \(x\) and \(y\) in \(F\).

Let \(V = f(A)\), where \(A\) is an intuitionistic fuzzy subfield of \(F\).

We have to prove that \(V\) is an intuitionistic fuzzy subfield of \(F'\).

Now, for \(f(x)\) and \(f(y)\) in \(F'\), we have
\[
\mu_V(f(x)-f(y)) = \mu_V(f(x)-y), \quad \text{as } f \text{ is a homomorphism}
\]
\[
\geq \mu_A(x-y)
\]
\[
\geq \min \{ \mu_A(x), \mu_A(y) \}
\]
which implies that
\[
\mu_V(f(x)-f(y)) \geq \min \{ \mu_V(f(x)), \mu_V(f(y)) \}
\]
for all \(f(x)\) and \(f(y)\) in \(F'\).

And \(\mu_V(f(x)(f(y))^{-1}) = \mu_V(f(xy^{-1}))\), as \(f\) is a homomorphism
\[
\geq \mu_A(xy^{-1})
\]
\[
\geq \min \{ \mu_A(x), \mu_A(y) \}
\]
which implies that
\[
\mu_V(f(x)(f(y))^{-1}) \geq \min \{ \mu_V(f(x)), \mu_V(f(y)) \}
\]
for all \(f(x)\) and \(f(y)\) \(\neq 0'\) in \(F'\).

We have \(\nu_V(f(x)-f(y)) = \nu_V(f(x-y))\), as \(f\) is a homomorphism
\[ \leq \nu_A(x-y) \]
\[ \leq \max \{ \nu_A(x), \nu_A(y) \}, \]

which implies that \( \nu_V( f(x)-f(y) ) \leq \max \{ \nu_V(f(x)), \nu_V(f(y)) \} \), for all \( f(x) \) and \( f(y) \) in \( F^l \).

\[ \nu_V( f(x)(f(y))^{-1} ) = \nu_V( f(xy^{-1}) ) \], as \( f \) is a homomorphism

\[ \leq \nu_A(xy^{-1}) \]
\[ \leq \max \{ \nu_A(x), \nu_A(y) \}, \]

which implies that \( \nu_V( f(x)(f(y))^{-1} ) \leq \max\{ \nu_V(f(x)), \nu_V( f(y) ) \} \), for all \( f(x) \) and \( f(y) \neq 0^l \) in \( F^l \).

Hence \( V \) is an intuitionistic fuzzy subfield of a field \( F^l \).

3.3.2 Theorem: Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) be any two fields. The homomorphic pre-image of an intuitionistic fuzzy subfield of \( F^l \) is an intuitionistic fuzzy subfield of \( F \).

Proof: Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) be any two fields and \( f : F \rightarrow F^l \) be a homomorphism. That is \( f(x+y) = f(x)+f(y) \), for all \( x \) and \( y \) in \( F \) and \( f(xy) = f(x)f(y) \), for all \( x \) and \( y \) in \( F \).

Let \( V=f(A) \), where \( V \) is an intuitionistic fuzzy subfield of \( F^l \).

We have to prove that \( A \) is an intuitionistic fuzzy subfield of \( F \).

Let \( x \) and \( y \) in \( F \). Then,

\[ \mu_A(x-y) = \mu_V( f(x-y) ) \], since \( \mu_A(x) = \mu_V( f(x) ) \)

\[ = \mu_V( f(x)-f(y) ) \], as \( f \) is a homomorphism

\[ \geq \min \{ \mu_V(f(x)), \mu_V(f(y)) \} \],

\[ = \min \{ \mu_A(x), \mu_A(y) \} \], since \( \mu_A(x) = \mu_V( f(x) ) \),
which implies that \( \mu_A(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

And, \( \mu_A(xy^{-1}) = \mu_V( f(xy^{-1}) ) \), since \( \mu_A(x) = \mu_V( f(x) ) \)

\[
= \mu_V(f(x))(f(y^{-1})), \text{ as } f \text{ is a homomorphism}
\]

\[
= \mu_V(f(x)(f(y)^{-1}), \text{ as } f \text{ is a homomorphism}
\]

\[
\geq \min \{ \mu_V(f(x)), \mu_V(f(y)) \},
\]

\[
= \min \{ \mu_A(x), \mu_A(y) \}, \text{ since } \mu_A(x) = \mu_V(f(x)),
\]

which implies that \( \mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \).

And, \( \nu_A(x-y) = \nu_V( f(x-y) ) \), since \( \nu_A(x) = \nu_V( f(x) ) \)

\[
= \nu_V( f(x) - f(y) ), \text{ as } f \text{ is a homomorphism}
\]

\[
\leq \max \{ \nu_V(f(x)), \nu_V(f(y)) \}, \text{ as } V \text{ is an IFSF of } F
\]

\[
= \max \{ \nu_A(x), \nu_A(y) \},
\]

which implies that \( \nu_A(x-y) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \) in \( F \).

And, \( \nu_A(xy^{-1}) = \nu_V( f(xy^{-1}) ) \), since \( \nu_A(x) = \nu_V( f(x) ) \)

\[
= \nu_V(f(x)f(y^{-1})), \text{ as } f \text{ is a homomorphism}
\]

\[
= \nu_V(f(x)(f(y)^{-1}), \text{ as } f \text{ is a homomorphism}
\]

\[
\leq \max \{ \nu_V(f(x)), \nu_V(f(y)) \}, \text{ as } V \text{ is an IFSF of } F^l
\]

\[
= \max \{ \nu_A(x), \nu_A(y) \},
\]

which implies \( \nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \} \), for all \( x \) and \( y \neq 0 \) in \( F \).

Hence \( A \) is an intuitionistic fuzzy subfield of a field \( F \).

**3.3.3 Theorem:** Let \( (F, +, \cdot) \) and \( (F^l, +, \cdot) \) be any two fields. The anti-homomorphic image of an intuitionistic fuzzy subfield of \( F \) is an intuitionistic fuzzy subfield of \( F^l \).
Proof: It is trivial.

3.3.4 Theorem: Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) be any two fields. The anti-homomorphic pre-image of an intuitionistic fuzzy subfield of \(F^l\) is an intuitionistic fuzzy subfield of \(F\).

Proof: It is trivial.

3.3.5 Theorem: An intuitionistic fuzzy subfield \(A\) of a field \((F, +, \cdot)\) is normalized if and only if \(\mu_A(e) = \mu_A(e^l) = 1\) and \(\nu_A(e) = \nu_A(e^l) = 0\), where \(e\) and \(e^l\) are identity elements of the field \(F\).

Proof: If \(A\) is normalized, then there exists \(x \in F\) such that \(\mu_A(x) = 1\) and \(\nu_A(x) = 0\), but by properties of an intuitionistic fuzzy subfield \(A\) of \(F\), \(\mu_A(x) \leq \mu_A(e)\), for all \(x \in F\) and \(\mu_A(x) \leq \mu_A(e^l)\), for all \(x \in F\) and \(\nu_A(x) \geq \nu_A(e)\), for all \(x \in F\) and \(\nu_A(x) \geq \nu_A(e^l)\), for all \(x \in F\), where \(e\) and \(e^l\) are identity elements of the field \(F\).

Since \(\mu_A(x) = 1\) and \(\nu_A(x) = 0\) and \(\mu_A(x) \leq \mu_A(e)\), for all \(x \in F\) and \(\mu_A(x) \leq \mu_A(e^l)\), for all \(x \in F\) and \(\nu_A(x) \geq \nu_A(e)\), for all \(x \in F\) and \(\nu_A(x) \geq \nu_A(e^l)\), for all \(x \in F\).

Therefore \(1 \leq \mu_A(e)\), \(1 \leq \mu_A(e^l)\) and \(0 \geq \nu_A(e)\), \(0 \geq \nu_A(e^l)\).

But \(1 \geq \mu_A(e)\), \(1 \geq \mu_A(e^l)\) and \(0 \leq \nu_A(e)\), \(0 \leq \nu_A(e^l)\).

Hence \(\mu_A(e) = \mu_A(e^l) = 1\) and \(\nu_A(e) = \nu_A(e^l) = 0\).

Conversely, if \(\mu_A(e) = \mu_A(e^l) = 1\) and \(\nu_A(e) = \nu_A(e^l) = 0\), then by the definition of normalized intuitionistic fuzzy subset, \(A\) is normalized.
In the following Theorem \( \circ \) is the composition operation of functions:

3.3.6 Theorem: Let \( A \) be an intuitionistic fuzzy subfield of a field \( H \) and \( f \) is an isomorphism from a field \( F \) onto \( H \). Then \( A \circ f \) is an intuitionistic fuzzy subfield of \( F \).

Proof: Let \( x \) and \( y \) in \( F \) and \( A \) be an intuitionistic fuzzy subfield of a field \( H \).

Then we have,

\[
(\mu_{A \circ f})(x-y) = \mu_A(f(x-y)) \\
= \mu_A(f(x)+f(-y)), \text{ as } f \text{ is an isomorphism} \\
= \mu_A(f(x)-f(y)), \\
\geq \min \{\mu_A(f(x)), \mu_A(f(y))\}, \text{ as } A \text{ is an IFSF of } H \\
\geq \min \{(\mu_{A \circ f})(x), (\mu_{A \circ f})(y)\},
\]

which implies that \( (\mu_{A \circ f})(x-y) \geq \min \{(\mu_{A \circ f})(x), (\mu_{A \circ f})(y)\} \), for all \( x \) and \( y \) in \( F \).

And \( (\mu_{A \circ f})(xy^{-1}) = \mu_A(f(xy^{-1})) \)

\[
= \mu_A(f(x)f(y^{-1})), \text{ as } f \text{ is an isomorphism} \\
= \mu_A(f(x)(f(y))^{-1}), \\
\geq \min \{\mu_A(f(x)), \mu_A(f(y))\}, \text{ as } A \text{ is an IFSF of } H \\
\geq \min \{(\mu_{A \circ f})(x), (\mu_{A \circ f})(y)\},
\]

which implies that \( (\mu_{A \circ f})(xy^{-1}) \geq \min \{(\mu_{A \circ f})(x), (\mu_{A \circ f})(y)\} \), for all \( x \) and \( y \neq 0 \) in \( F \).

We have, \( (v_{A \circ f})(x-y) = v_A(f(x-y)) \)
\[ v_A(f(x) + f(-y)), \text{ as } f \text{ is an isomorphism} \]
\[ v_A(f(x) - f(y)), \]
\[ \leq \max \{ v_A(f(x)), v_A(f(y)) \} \]
\[ \leq \max \{ (v_A \circ f)(x), (v_A \circ f)(y) \}, \]

which implies that \((v_A \circ f)(x - y) \leq \max \{ (v_A \circ f)(x), (v_A \circ f)(y) \}, \) for all \(x\) and \(y\) in \(F\).

And,
\[ (v_A \circ f)(xy^{-1}) = v_A(f(xy^{-1})) \]
\[ = v_A(f(x)f(y^{-1})), \text{ as } f \text{ is an isomorphism} \]
\[ = v_A(f(x)(f(y))^{-1}), \]
\[ \leq \max \{ v_A(f(x)), v_A(f(y)) \} \]
\[ \leq \max \{ (v_A \circ f)(x), (v_A \circ f)(y) \}, \]

which implies that \((v_A \circ f)(xy^{-1}) \leq \max \{ (v_A \circ f)(x), (v_A \circ f)(y) \}, \) for all \(x\) and \(y \neq 0\) in \(F\).

Therefore \((A \circ f)\) is an intuitionistic fuzzy subfield of a field \(F\).

**3.3.7 Theorem:** Let \(A\) be an intuitionistic fuzzy subfield of a field \(H\) and \(f\) is an anti-isomorphism from a field \(F\) onto \(H\). Then \(A \circ f\) is an intuitionistic fuzzy subfield of \(F\).

**Proof:** It is trivial.

**3.3.8 Theorem:** Let \(A\) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\), then the pseudo intuitionistic fuzzy coset \((aA)^p\) is an intuitionistic fuzzy subfield of a field \(F\), for every \(a \in F\).

**Proof:** Let \(A\) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\).

For every \(x\) and \(y\) in \(F\), we have,
\[( (a\mu_A)^p)(x-y) = p(a)\mu_A(x-y) \]
\[\geq p(a) \min \{ \mu_A(x), \mu_A(y) \} \]
\[= \min \{ p(a)\mu_A(x), p(a)\mu_A(y) \} \]
\[= \min \{ ( (a\mu_A)^p )(x), ( (a\mu_A)^p )(y) \} . \]
Therefore, \(( (a\mu_A)^p )(x-y) \geq \min \{ ( (a\mu_A)^p )(x) , ( (a\mu_A)^p )(y) \} , \) for all \(x\) and \(y\) in \(F\).

And for every \(x\) and \(y \neq 0\) in \(F\),
\[( (a\mu_A)^p )(xy^{-1}) = p(a)\mu_A(xy^{-1}) \]
\[\geq p(a) \min \{ \mu_A(x), \mu_A(y) \} \]
\[= \min \{ p(a)\mu_A(x), p(a)\mu_A(y) \} \]
\[= \min \{ ( (a\mu_A)^p )(x), ( (a\mu_A)^p )(y) \} . \]
Therefore, \( ((a\mu_A)^p)(xy^{-1}) \geq \min \{ ((a\mu_A)^p)(x), ((a\mu_A)^p)(y) \} , \) for all \(x\) and \(y \neq 0\) in \(F\).

We have, \( ( (a\nu_A)^p ) (x-y) = p(a)\nu_A(x-y) \)
\[\leq p(a) \max \{ \nu_A(x), \nu_A(y) \} \]
\[= \max \{ p(a)\nu_A(x), p(a)\nu_A(y) \} \]
\[= \max \{ ( (a\nu_A)^p )(x), ( (a\nu_A)^p )(y) \} . \]
Therefore, \( ( (a\nu_A)^p ) (x-y) \leq \max \{ ( (a\nu_A)^p )(x), ( (a\nu_A)^p )(y) \} , \) for all \(x\) and \(y\) in \(F\).

And for every \(x\) and \(y \neq 0\) in \(F\),
\[( (a\nu_A)^p )(xy^{-1}) = p(a)\nu_A(xy^{-1}) \]
\[\leq p(a) \max \{ \nu_A(x), \nu_A(y) \} \]
= \max \{ p(a)v_A(x), p(a)v_A(y) \}
= \max \{ (av_A)^p(x), (av_A)^p(y) \}.

Therefore, \((av_A)^p(xy^{-1}) \leq \max \{ (av_A)^p(x), (av_A)^p(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).

Hence \((aA)^p\) is an intuitionistic fuzzy subfield of a field \(F\).

3.3.9 Theorem: Let \(A\) be an intuitionistic fuzzy subfield of a field \((F, +, . )\), then the intuitionistic fuzzy \((0, 1 )\)-coset \(_0A_1\) is an intuitionistic fuzzy subfield of a field \(F\), where \(0\) and \(1\) are identity elements of \(F\).

Proof: Let \(A\) be an intuitionistic fuzzy subfield of a field \((F, +, . )\).

For every \(x\) and \(y\) in \(F\), we have,
\[(0+\mu_A)(x-y) = \mu_A(0+x-y)\]
\[= \mu_A(x-y)\]
\[\geq \min \{ \mu_A(x), \mu_A(y) \}.

Therefore \((0+\mu_A)(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y\) in \(F\).

And, we have
\[(1\mu_A)(xy^{-1}) = \mu_A(1.xy^{-1})\]
\[= \mu_A(xy^{-1})\]
\[\geq \min \{ \mu_A(x), \mu_A(y) \}.

Therefore \((1\mu_A)(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).

We have \((0+v_A)(x-y) = v_A(0+x-y)\)
\[= v_A(x-y)\]
\[\leq \max \{ v_A(x), v_A(y) \}.

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Therefore \((0+\nu_A)(x-y) \leq \max \{ \nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y\) in \(F\).

And, we have
\[
(1\nu_A)(xy^{-1}) = \nu_A(1 \cdot xy^{-1}) \\
= \nu_A(xy^{-1}) \\
\leq \max \{ \nu_A(x), \nu_A(y) \}.
\]

Therefore \((1\nu_A)(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}\), for all \(x\) and \(y \neq 0\) in \(F\).

Hence the intuitionistic fuzzy \((0, 1)\)-coset \(_0A_1^1\) is an intuitionistic fuzzy subfield of a field \(F\).

3.3.10 Theorem: Let \(A\) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\), then the intuitionistic fuzzy \((a-a, \cdot b^{-1})\)-coset \(_{a-a}^1A_{b^{-1}}^1\) is an intuitionistic fuzzy subfield of a field \(F\), for \(a\) and \(b\) in \(F\).

**Proof:** Let \(A\) be an intuitionistic fuzzy subfield of a field \((F, +, \cdot)\).

For every \(x\) and \(y\) in \(F\) and \(a\) in \(F\), we have,
\[
(a+\mu_A-a)(x-y) = \mu_A((-a+x-y)+a) \\
= \mu_A((-a+a+x-y)) \\
= \mu_A(x-y) \\
\geq \min \{ \mu_A(x), \mu_A(y) \}.
\]

Therefore \((a+\mu_A-a)(x-y) \geq \min \{ \mu_A(x), \mu_A(y) \}\), for all \(x\) and \(y\) in \(F\).

And for every \(x, y \neq 0\) and \(b \neq 0\) in \(F\), we have
\[
(b\mu_A b^{-1})(xy^{-1}) = \mu_A(b^{-1}xy^{-1}b) \\
= \mu_A(b^{-1}bxy^{-1})
\]
\[= \mu_A(xy^{-1})\]
\[\geq \min\{\mu_A(x), \mu_A(y)\}.\]

Therefore \((b\mu_A b^{-1})(xy^{-1}) \geq \min \{\mu_A(x), \mu_A(y)\}\), for all \(x \text{ and } y \neq 0\) in \(F\).

We have \((a+v_A-a)(x-y) = v_A(-a+x-y+a)\)
\[= v_A(-a+a+x-y)\]
\[= v_A(x-y)\]
\[\leq \max\{v_A(x), v_A(y)\}.\]

Therefore \((a+v_A-a)(x-y) \leq \max \{v_A(x), v_A(y)\}\), for all \(x \text{ and } y \) in \(F\).

And for every \(x, y \neq 0\) and \(b \neq 0\) in \(F\),
we have
\[\quad (bv_A b^{-1})(xy^{-1}) = v_A(b^{-1}xy^{-1}b)\]
\[= v_A(b^{-1}bxy^{-1})\]
\[= v_A(xy^{-1})\]
\[\leq \max\{v_A(x), v_A(y)\}.\]

Therefore \((bv_A b^{-1})(xy^{-1}) \leq \max \{v_A(x), v_A(y)\}\), for all \(x \text{ and } y \neq 0\) in \(F\).

Hence the intuitionistic fuzzy \((a-a, bb^{-1})\)-coset \(A^{-a} A_{bb^{-1}}\) is an intuitionistic fuzzy subfield of a field \(F\).

### Theorem 3.3.11

Let \((F, +, \cdot)\) be a field and \(A\) be a non empty subset of \(F\). Then \(A\) is a subfield of \(F\) if and only if \(B = <\chi_A, \chi_A>\) is an intuitionistic fuzzy subfield of \(F\), where \(\chi_A\) is the characteristic function.
**Proof:** Let \((F, +, \cdot)\) be a field and \(A\) be a non-empty subset of \(F\). First let \(A\) be a subfield of \(F\). Take \(x, y\) in \(F\).

Case (i): If \(x\) and \(y\) in \(A\), then \(x - y, xy^{-1}\) in \(A\), since \(A\) is a subfield of \(F\),
\[
\chi_A(x) = \chi_A(y) = \chi_A(x - y) = \chi_A(xy^{-1}) = 1 \quad \text{and} \quad \chi_A(x) = \chi_A(y) = \chi_A(x - y) = \chi_A(xy^{-1}) = 0.
\]
So, \(\chi_A(x - y) \geq \min \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y\) in \(F\),
\[
\chi_A(xy^{-1}) \geq \min \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y \neq e\) in \(F\)
and \(e\) is the additive identity element of \(F\).

So, \(\chi_A(x - y) \leq \max \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y\) in \(F\),
\[
\chi_A(xy^{-1}) \leq \max \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y \neq e\) in \(F\).

Case (ii): If \(x\) in \(A\), \(y\) not in \(A\) (or \(x\) not in \(A\), \(y\) in \(A\)), then \(x - y, xy^{-1}\) may or may not be in \(A\), \(\chi_A(x) = 1, \chi_A(y) = 0\) (or \(\chi_A(x) = 0, \chi_A(y) = 1\)),
\[
\chi_A(x - y) = \chi_A(xy^{-1}) = 1 \quad \text{(or 0)} \quad \text{and} \quad \chi_A(x) = 0, \chi_A(y) = 1 \quad \text{(or} \quad \chi_A(x) = 1, \chi_A(y) = 0\), \(\chi_A(x - y) = \chi_A(xy^{-1}) = 0\) (or 1).

Clearly \(\chi_A(x - y) \geq \min \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y\) in \(F\),
\[
\chi_A(xy^{-1}) \geq \min \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y \neq e\) in \(F\)
and \(\chi_A(x - y) \leq \max \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y \neq e\) in \(F\),
\[
\chi_A(xy^{-1}) \leq \max \{ \chi_A(x), \chi_A(y) \}\), for all \(x, y \neq e\) in \(F\).

Case (iii): If \(x\) and \(y\) not in \(A\), then \(x - y, xy^{-1}\) may or may not be in \(A\),
\[
\chi_A(x) = \chi_A(y) = 0, \chi_A(x - y) = \chi_A(xy^{-1}) = 1 \quad \text{or} \quad 0 \quad \text{and} \quad \chi_A(x) = \chi_A(y) = 1, \chi_A(x - y) = \chi_A(xy^{-1}) = 0 \quad \text{or} \quad 1.
\]
Clearly \( \chi_a(x - y) \geq \min \{ \chi_a(x), \chi_a(y) \} \), for all \( x \) and \( y \) in \( F \)

\[ \chi_a(xy^{-1}) \geq \min \{ \chi_a(x), \chi_a(y) \}, \] for all \( x \) and \( y \neq e \) in \( F \),

and \( \overline{\chi_a}(x - y) \leq \max \{ \overline{\chi_a}(x), \overline{\chi_a}(y) \} \), for all \( x \) and \( y \) in \( F \)

\[ \overline{\chi_a}(xy^{-1}) \leq \max \{ \overline{\chi_a}(x), \overline{\chi_a}(y) \}, \] for all \( x \) and \( y \neq e \) in \( F \).

So in all the three cases, we have \( B \) is an intuitionistic fuzzy subfield of a field \( F \).

Conversely, let \( x \) and \( y \) in \( A \), since \( A \) is a non empty subset of \( F \),

so, \( \chi_a(x) = \chi_a(y) = 1, \overline{\chi_a}(x) = \overline{\chi_a}(y) = 0. \)

Since \( B = <\chi_a, \overline{\chi_a}> \) is an intuitionistic fuzzy subfield of \( F \),

we have \( \chi_a(x - y) \geq \min \{ \chi_a(x), \chi_a(y) \} = \min \{1, 1 \} = 1, \)

\[ \chi_a(xy^{-1}) \geq \min \{ \chi_a(x), \chi_a(y) \} = \min \{1, 1 \} = 1. \]

Therefore \( \chi_a(x - y) = \chi_a(xy^{-1}) = 1. \)

And, \( \overline{\chi_a}(x - y) \leq \max \{ \overline{\chi_a}(x), \overline{\chi_a}(y) \} = \max \{0, 0 \} = 0, \)

\[ \overline{\chi_a}(xy^{-1}) \leq \max \{ \overline{\chi_a}(x), \overline{\chi_a}(y) \} = \max \{0, 0 \} = 0. \]

Therefore \( \overline{\chi_a}(x - y) = \overline{\chi_a}(xy^{-1}) = 0. \)

Hence \( x - y \) and \( xy^{-1} \) in \( A \), so \( A \) is a subfield of \( F \).