INTRODUCTION

Fuzzy subsets:

Fuzzy subsets were introduced by Zadeh in 1965 to represent/manipulate data and information possessing non-statistical uncertainties. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems.

The first publication in fuzzy subset theory by Zadeh (1965) and then by Goguen (1967, 1969) show the intention of the authors to generalize the classical set. In classical set theory, a subset \( A \) of a set \( X \) can be defined by its characteristic function \( \chi_A : X \rightarrow \{0, 1\} \)

is defined by \( \chi_A(x) = 0 \), if \( x \notin A \) and \( \chi_A(x) = 1 \), if \( x \in A \).

The mapping may be represented as a set of ordered pairs \( \{ (x, \chi_A(x)) \} \) with exactly one ordered pair present for each element of \( X \). The first element of the ordered pair is an element of the set \( X \) and the second is its value in \( \{0, 1\} \). The value ‘0’ is used to represent non-membership and the value ‘1’ is used to represent membership of the element \( A \). The truth or falsity of the statement “\( x \) is in \( A \)” is determined by the ordered pair. The statement is true, if the second element of the ordered pair is ‘1’, and the statement is false, if it is ‘0’
Similarly, a fuzzy subset $A$ of a set $X$ can be defined as a set of ordered pairs $\{ (x, \mu_A(x)) : x \in X \}$, each with the first element from $X$ and the second element from the interval $[0, 1]$ with exactly one ordered pair present for each element of $X$. This defines a mapping, $\mu_A$ between elements of the set $X$ and values in the interval $[0, 1]$. That is, $\mu_A : X \rightarrow [0, 1]$.

The value ‘0’ is used to represent complete non-membership, the value ‘1’ is used to represent complete membership and values in between are used to represent intermediate degrees of membership.

The set $X$ is referred to as the universe of discourse for the fuzzy subset $A$. Frequently, the mapping $\mu_A$ is described as a function, the membership function of $A$, the degree to which the statement “$x$ is in $A$” is true, is determined by finding the ordered pair $(x, \mu_A(x))$. The degree of truth of the statement is the second element of the ordered pair.

**Intuitionistic fuzzy subsets**:

Prof. K.T. Atanassov, a Bulgarian Engineer, introduced a new component which determines the degree of non-membership also in defining intuitionistic fuzzy subset (IFS) theory. In 1983, he came across A.Kauffmann’s book “Introduction to the theory of fuzzy subsets” Academic Press, New York, 1975, then he tried to introduce intuitionistic fuzzy subsets to study the properties of the new objects so defined. He defined ordinary operations as $\cap$, $\cup$, $+$ and $-$ over the new
sets, then defined operators similar to the operators ‘necessity’ and ‘possibility’.

George Gargov named new sets as the ‘‘Intuitionistic fuzzy subsets’’, as their fuzzification denies the law of the excluded middle, $A \cup A^c = X$. This has encouraged Prof. K.T.Atanassov to continue his work on intuitionistic fuzzy subsets.

An intuitionistic fuzzy subset $A$ of a set $X$ can be defined as a set of ordered pairs $\{ \langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \}$, each with the first element from $X$ and the second element from the interval $[0, 1]$ and the third element from the interval $[0, 1]$ with exactly one ordered pair present for each element of $X$ such that for every $x$ in $X$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. This defines the mapping, $\mu_A$ between elements of the set $X$ and values in the interval $[0, 1]$, is called the degree of membership and $\nu_A$ between elements of the set $X$ and values in the interval $[0, 1]$, is called the degree of non-membership of the element $x$ in $X$.

That is $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x$ in $X$ respectively and for every $x$ in $X$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. 
CHAPTER - I

PRELIMINARIES

1. Introduction: This chapter contains the basic concepts required to develop the thesis.

1.1 Definition: A non-empty set $G$ together with a binary operation $'*'$ that maps $G \times G$ into $G$ is called a group if the following conditions are satisfied:

(i) $'$ is associative,

(ii) there exists an element $e$ in $G$ such that $a \ast e = e \ast a = a$, for all $a$ in $G$, $e$ is called the identity element of $G$,

(iii) for any element $a$ in $G$ there exists an element $a'$ in $G$ such that $a \ast a' = a' \ast a = e$, $a'$ is called the inverse of $a$.

1.1 Example: If $Z$ is the set of integers, then $Z$ is a group under the usual addition.

1.2 Definition: A group $(G, \cdot)$ is said to be abelian if $a \cdot b = b \cdot a$, for all $a$ and $b$ in $G$.

1.3 Definition: A non-empty set $R$ together with two binary operations denoted by “+” and “.” and called addition and multiplication which satisfy the following conditions is called a ring:
(i) \((R, +)\) is an abelian group,

(ii) “\(\cdot\)” is an associative binary operation on \(R\),

(iii) \(a \cdot (b + c) = a \cdot b + a \cdot c\) and \((a + b) \cdot c = a \cdot c + b \cdot c\), for all \(a, b\) and \(c\) in \(R\).

**1.4 Definition:** A ring \((R, +, \cdot)\) is said to be commutative ring if \(ab = ba\), for all \(a\) and \(b\) in \(R\).

**1.5 Definition:** A field is a commutative ring with unit element in which every non-zero element has a multiplicative inverse.

**1.2 Example:** If \(R\) is the set of real numbers, then \(R\) is a field under the usual addition and multiplication operations.

**1.6 Definition:** Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) are any two fields. The function \(f : F \to F^l\) is called a field homomorphism if \(f(x+y) = f(x) + f(y)\), for all \(x\) and \(y\) in \(F\) and \(f(xy) = f(x)f(y)\), for all \(x\) and \(y\) in \(F\).

**1.7 Definition:** Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) are any two fields. The function \(f : F \to F^l\) is called a field anti-homomorphism if \(f(x+y) = f(y) + f(x)\), for all \(x\) and \(y\) in \(F\) and \(f(xy) = f(y)f(x)\), for all \(x\) and \(y\) in \(F\).

**Note:** In a field, homomorphism and anti-homomorphism are equal.

**1.8 Definition:** Let \((F, +, \cdot)\) and \((F^l, +, \cdot)\) be two fields. A homomorphism \(f : F \to F^l\) is called a field isomorphism if \(f\) is a bijection.
1.9 Definition: Let \(( F, +, . )\) and \(( F', +, . )\) be two fields. An anti-homomorphism \( f : F \to F'\) is called a **field anti-isomorphism** if \( f\) is a bijection.

1.10 Definition: An isomorphism of a field \(( F, +, . )\) to itself is called a **field automorphism** of \( F\). It is denoted by \( \text{Aut} F\).

1.11 Definition: An anti-isomorphism of a field \(( F, +, . )\) to itself is called a **field anti-automorphism** of \( F\). It is denoted by \( \text{anti-Aut} F\).

1.12 Definition: Let \( X\) be a non-empty set. A **fuzzy subset** \( A\) of \( X\) is a function \( A : X \to [0, 1]\).

1.3 Example: Let \( X = \{ a, b, c \}\) be a set. Then \( A = \{ \langle a, 0.54 \rangle, \langle b, 0.51 \rangle, \langle c, 0.3 \rangle \} \) is a fuzzy subset of \( X\).

1.13 Definition: The **union** of two fuzzy subsets \( A \) and \( B\) of a set \( X\) is defined by \((A \cup B)(x) = \max \{ A(x), B(x) \}\), for all \( x\) in \( X\).

1.4 Example: Let \( A = \{ \langle a, 0.34 \rangle, \langle b, 0.27 \rangle, \langle c, 0.63 \rangle \}\) and \( B = \{ \langle a, 0.52 \rangle, \langle b, 0.73 \rangle, \langle c, 0.43 \rangle \}\) be two fuzzy subsets of a set \( X = \{ a, b, c \}\). The union of the fuzzy subsets \( A \) and \( B\) is \( A \cup B = \{ \langle a, 0.52 \rangle, \langle b, 0.73 \rangle, \langle c, 0.63 \rangle \}\).

1.14 Definition: The **intersection** of two fuzzy subsets \( A \) and \( B\) of a set \( X\) is defined by \((A \cap B)(x) = \min \{ A(x), B(x) \}\), for all \( x\) in \( X\).

1.5 Example: Let \( A = \{ \langle a, 0.51 \rangle, \langle b, 0.57 \rangle, \langle c, 0.93 \rangle \}\) and \( B = \{ \langle a, 0.75 \rangle, \langle b, 0.63 \rangle, \langle c, 0.43 \rangle \}\) be two fuzzy subsets of a set \( X = \{ a, b, c \}\). The intersection of the fuzzy subsets \( A \) and \( B\) is \( A \cap B = \{ \langle a, 0.51 \rangle, \langle b, 0.57 \rangle, \langle c, 0.43 \rangle \}\).
1.15 Definition: If $A$ is a fuzzy subset of a set $X$, then the **complement** of $A$, denoted by $A^c$, is the fuzzy subset of $X$, is defined by $A^c(x) = 1 - A(x)$, for all $x$ in $X$.

1.6 Example: Let $A = \{ \langle a, 0.54 \rangle, \langle b, 0.72 \rangle, \langle c, 0.58 \rangle \}$ be a fuzzy subset of $X = \{ a, b, c \}$. The complement of $A$ is $A^c = \{ \langle a, 0.46 \rangle, \langle b, 0.28 \rangle, \langle c, 0.42 \rangle \}$.

1.16 Definition: Let $(F, +, \cdot)$ be a field. A fuzzy subset $A$ of $F$ is said to be a **fuzzy subfield** (FSF) of $F$ if the following conditions are satisfied:

- (i) $A(x-y) \geq \min \{A(x), A(y)\}$, for all $x$ and $y$ in $F$,
- (ii) $A(xy^{-1}) \geq \min \{A(x), A(y)\}$, for all $x$ and $y$ in $F-\{0\}$.

1.7 Example: Consider the field $Z_5 = \{0, 1, 2, 3, 4\}$ with addition modulo 5 and multiplication modulo 5 operations. Then $A = \{ \langle 0, 0.7 \rangle, \langle 1, 0.4 \rangle, \langle 2, 0.4 \rangle, \langle 3, 0.4 \rangle, \langle 4, 0.4 \rangle \}$ is a fuzzy subfield of $Z_5$.

1.17 Definition: An **intuitionistic fuzzy subset** (IFS) $A$ of a set $X$ is defined as an object of the form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x$ in $X$ respectively and for every $x$ in $X$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

1.8 Example: Let $X = \{ a, b, c \}$ be a set. Then $A = \{ \langle a, 0.52, 0.34 \rangle, \langle b, 0.14, 0.71 \rangle, \langle c, 0.25, 0.34 \rangle \}$ is an intuitionistic fuzzy subset of $X$. 
1.18 Definition: Let $A$ and $B$ be any two intuitionistic fuzzy subsets of a set $X$. We define the following relations and operations:

(i) $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x$ in $X$.

(ii) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x$ in $X$.

(iii) $\bar{A} = \{ \langle x, \nu_A(x), \mu_A(x) \rangle / x \in X \}$.

(iv) $A \cap B = \{ \langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\} \rangle / x \in X \}$.

(v) $A \cup B = \{ \langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\} \rangle / x \in X \}$.

(vi) $A + B = \{ \langle x, (\mu_A(x) + \mu_B(x)) - \mu_A(x), (\nu_A(x) - \nu_B(x)) \rangle \} / x \in X \}$.

(vii) $A \cdot B = \{ \langle x, (\mu_A(x) + \mu_B(x)), (\nu_A(x) + \nu_B(x))\rangle \} / x \in X \}$.

(viii) $A @ B = \{ \langle x, (\mu_A(x) - \mu_B(x)), (\nu_A(x) - \nu_B(x)) \rangle \} / x \in X \}$.

(ix) $A \$ B = \{ \langle x, \sqrt{\mu_A(x) \cdot \mu_B(x)}, \sqrt{\nu_A(x) \cdot \nu_B(x)} \rangle / x \in X \}$.

(x) $A * B = \{ \langle x, (\mu_A(x) + \mu_B(x)) / 2(\mu_A(x) \cdot \mu_B(x) + 1), (\nu_A(x) + \nu_B(x)) / 2(\nu_A(x) \cdot \nu_B(x) + 1) \rangle / x \in X \}$.

(xi) $A \land B = \{ \langle x, 2(\mu_A(x) + \mu_B(x)) / (\mu_A(x) + \mu_B(x)), 2(\nu_A(x) + \nu_B(x)) / (\nu_A(x) + \nu_B(x)) \rangle / x \in X \}$.

(xii) $A \leftrightarrow B = \{ \langle x, \max\{\nu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \nu_B(x)\} \rangle / x \in X \}$.

(xiii) $\square A = \{ \langle x, \mu_A(x), 1-\mu_A(x) \rangle \} / x \in X \}$.

(xiv) $\Diamond A = \{ \langle x, 1-\nu_A(x), \nu_A(x) \rangle \} / x \in X \}$, for all $x$ in $X$. 
1.1 **Theorem:** Let $F$ and $F^l$ be fields with identities. Let $f : F \rightarrow F^l$ be a homomorphism, then

(i) \[ f(0) = 0^l \text{ and } f(1) = 1^l, \] where $0, 1$ and $0^l, 1^l$ are identities of $F$ and $F^l$ respectively.

(ii) \[ f(-a) = -f(a), \text{ for all } a \in F \text{ and } f(a^{-1}) = (f(a))^{-1}, \text{ for all } a \in F - \{0\}. \]

**Proof:** It is trivial.

1.2 **Theorem:** Let $F$ and $F^l$ be fields with identities. Let $f : F \rightarrow F^l$ be an anti-homomorphism, then

(i) \[ f(0) = 0^l \text{ and } f(1) = 1^l, \] where $0, 1$ and $0^l, 1^l$ are identities of $F$ and $F^l$ respectively.

(ii) \[ f(-a) = -f(a), \text{ for all } a \in F \text{ and } f(a^{-1}) = (f(a))^{-1}, \text{ for all } a \in F - \{0\}. \]

**Proof:** It is trivial.