CHAPTER ONE

INTRODUCTION

1.1. ORDER STATISTICS

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a specified or unspecified population arranged in nondecreasing order, $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Then $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are collectively called the order statistics of the sample and $X_{r:n}$ ($r=1, 2, \ldots, n$) is called the $r$-th order statistic of the sample. Order statistics is that branch of the subject of statistics which deals with the mathematical properties of order statistics and with statistical methods based upon them.

dealing in detail with its different aspects. Asymptotic theory of extremes and related development of order statistics are well described in an applausive work of Galamboas(1978) and on the more applied side Gumbel's(1958) account continues to valuable. Recurrence relations for the moments of order statistics of interest have been obtained by Khan et al(1983a,b) for continuous distributions. Also references may be made to Barnett and Lewis(1984), Balakrishnan and Cohen(1991), Arnold et al.(1992) and the references therein.

Order statistics and functions of order statistics play an important role in statistical theory and methodology. Floods and droughts, longevity, breaking strength, aeronautics, oceanography, duration of humans, organisms, components and devices of various kinds can all be studied by the theory of extreme values. The range is widely used, particularly in statistical quality control, as an estimate of $\sigma$. Many short-cut test have been based on the range and other order statistics. In dealing with small samples the studentized range is useful in a variety of ways. Apart from supplying the basis of many of quick tests, it plays a key role in procedures for ranking "treatment" means in the analysis of variance situation. The studentized range is also used in the detection of outliers.

By applying the Gauss-Markoff theorem of least squares, it is possible to use linear functions of order statistics for estimating the parameters of distribution functions. This
application is very useful particularly when some of the
observations in the sample have been "censored" since in that
case standard methods of estimation tend to become laborious or
otherwise unsatisfactory (Lloyd, 1952). Life tests provide an
ideal illustration of the advantage of order statistics in
censored data. Since such an experiment may take a long time to
complete, it is often advantageous to stop after failure of the
first r out of n similar items under test.

1.2. Recurrence Relations And Identities Of Order Statistics

Order statistics and their moments have concentrated
attention and interest from the beginning of this century since
Galton(1902) and Pearson(1902) studied the distribution of the
difference of two successive order statistics. Moments of order
statistics have considerable and fetching importance in the
statistical literature and have been numerically tabulated
extensively for several distributions. For example, one can refer
to David(1981), Arnold and Balakrishnan(1989), Arnold et
al.(1992) for a detailed list of these tables. Recurrence
relations of order statistics reduce the amount of direct
computation and hence reduce the time and labour. To reduce the
amount of direct computation, many authors including Jones(1948),
Godwin(1949), Khan et al.(1963a,b), Ali and Khan(1967) and
Sillitto(1951, 1964) carried out independent investigations
satisfied by the moments of order statistics. Many of these
relations and identities are quite useful as they express the higher order moments in terms of the lower order moments thus making the evaluation of higher order moments easy and in addition, provide some simple checks to test the accuracy of computation of moments of order statistics. Govindarajulu(1963), Arnold and Balakrishnan(1989) nicely summarized all these results and established some more recurrence relations and identities satisfied by single and product moments of order statistics. They then systematically applied these results in order to determine the maximum number of single and double integrals to be evaluated for the calculation of means, variances and covariances of order statistics in a sample of size n, assuming the quantities for all sample sizes less than n are known. By a simple generalization of one of the results of Govindarajulu(1963), Joshi(1971) determined that for distributions symmetric about zero the number of double integrals to be evaluated for even values of n is in fact zero. Joshi and Balakrishnan(1982) established similar results for any arbitrary continuous distribution and applied them to improve over the bounds of Govindarajulu(1963). Yet another interesting application of these recurrence relations and identities among order statistics is in establishing some combinatorial identities and this has been demonstrated by Joshi(1973) and Joshi and Balakrishnan(1981). Malik et al.(1988) have listed and analyzed all these results for moments of order statistics from an arbitrary continuous distribution in their expository review
article on this line. Also Balakrishnan et al. (1988) cited similar results on moments of order statistics from some specific continuous distributions.

Order statistics play an important role in statistics to characterize the distributions. The recurrence relations for single and product moments of order statistics obtained by Khan et al. (1983a,b) have nicely been applied to characterize the concern distributions (Khan and Khan, 1987; Khan and Ali, 1987) Reference may be made to Huang (1989), Khan and Abu-Salih (1989), Govindarajulu (1975), Lin (1988, 1989), Hwang and Lin (1984), Kamps (1991) and others.

1.3. ORDER STATISTICS AND THEIR DISTRIBUTIONS

In this section we will discuss the basic distribution theory of order statistics by assuming that population is absolutely continuous. Let us assume that \( X_1, X_2, \ldots, X_n \) be a random sample from an absolutely continuous population with probability density function (pdf) \( f(x) \) and cumulative distribution function (cdf) \( F(x) \) and let \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \) be the order statistics obtained by arranging the preceding random sample in increasing order of magnitude. Consider the event \( x < X_{r:n} \leq x + \delta x \), where \( \delta x \) is a small positive increment and we have

\[
P(x < X_{r:n} \leq x + \delta x) = \frac{n!}{(r-1)! (n-r)!} [F(x)]^{r-1} \left(1 - [F(x + \delta x)]^n - [F(x + \delta x) - F(x)] + O(\delta x^2)\right) \quad \ldots \ldots \quad (1.3.1)
\]
where $\alpha(\delta x)^2$, a term of order $(\delta x)^2$, is the probability corresponding to the event having more than one $X_i$ in the interval $(x,x+\delta x]$. From (1.3.1) we have the pdf of $X_{r:n}$ ($1 \leq x \leq n$) as

$$f_{r:n}(x) = \lim_{\delta x \to 0} \frac{P(x < X_{r:n} \leq x + \delta x)}{\delta x}$$

$$= \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x); -\infty < x < \infty \quad (1.3.2)$$

The pdf's of smallest and largest order statistics follows from (1.3.2) at $r = 1$ and $n$ respectively

$$f_{1:n}(x) = n[1-F(x)]^{n-1} f(x); -\infty < x < \infty \quad (1.3.3)$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x); -\infty < x < \infty \quad (1.3.4)$$

The cumulative distribution functions of the smallest and the largest order statistics are easily derived by integrating the pdf's in (1.3.3) and (1.3.4) are as

$$F_{1:n}(x) = 1-[1-F(x)]^n; -\infty < x < \infty \quad (1.3.5)$$

and

$$F_{n:n}(x) = [F(x)]^n; -\infty < x < \infty \quad (1.3.6)$$

In general, the cdf of $X_{r:n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \leq x)$$

$$= P(\text{at least}\ r\ \text{of}\ X_1, X_2, \ldots, X_n\ \text{are\ at\ most}\ x)$$

$$= \sum_{i=r}^n P(\text{exactly}\ i\ \text{of}\ X_1, X_2, \ldots, X_n\ \text{are\ at\ most}\ x)$$
\[ \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i}, \quad -\infty < x < \infty \]  \quad (1.3.7)

The cdf of \( X_{r:n} \) may also be obtained by integrating the pdf of \( X_{r:n} \) in (1.3.2) as

\[ F_{r:n}(x) = \int_{-\infty}^{x} f_{r:n}(t)dt \]
\[ = \frac{n!}{(r-1)! (n-r)!} \int_{-\infty}^{x} [F(t)]^{r-1} [1-F(t)]^{n-r} f(t)dt \]
\[ = \frac{n!}{(r-1)! (n-r)!} \int_{0}^{F(x)} u^{r-1} (1-u)^{n-r} du \]
\[ = I_{F(x)}(r, n-r+1) \]

which is just Pearson's (1934) incomplete beta function.

From the above density function, we obtain the k-th moment of \( X_{r:n} \) to be

\[ \mu_{r:n}^{(k)} = E(X_{r:n}^k) = \int_{-\infty}^{\infty} x^k f_{r:n}(x)dx \]

(1.3.9)

To derive the joint density function of two order statistics \( X_{r:n} \) and \( X_{s:n} \) (\( 1 \leq x < s \leq n \)), let us consider the event

\( (x < X_{r:n} \leq x + \delta x, \ y < X_{s:n} \leq y + \delta y); \quad -\infty < x < y < \infty. \)

For small positive \( \delta x \) and \( \delta y \) we may write

\[ P(x < X_{r:n} \leq x + \delta x, \ y < X_{s:n} \leq y + \delta y) \]
\[ = \frac{n!}{(r-1)! (s-r-1)! (n-s)!} [F(x)]^{r-1} [F(y)-F(x+\delta x)]^{s-r-1} \]
\[
[1-F(y+\delta y)]^{n-s}[F(x+\delta x)-F(x)][F(y+\delta y)-F(y)] + O(\delta x^2 \delta y) + O(\delta x(\delta y)^2)
\]

where \(O(\delta x^2 \delta y)\) and \(O(\delta x(\delta y)^2)\) are higher-order terms which correspond to the probabilities of the event having more than one \(X_i\) in the interval \((x,x+\delta x)\) and at least one \(X_i\) in the interval \((y,y+\delta y)\), and of the event of having one \(X_i\) in \((x,x+\delta x)\) and more than one \(X_i\) in \((y,y+\delta y)\) respectively.

From (1.3.10) the joint pdf of \(X_{r:n}\) and \(X_{s:n}\) is given by

\[
f_{r,s:n}(x,y) = \lim_{\delta x \to 0, \delta y \to 0} \frac{P(x < X_{r:n} \leq x + \delta x, y < X_{s:n} \leq y + \delta y)}{\delta x \delta y}
\]

\[
= \frac{n!}{(r-1)! (s-r-1)! (n-s)!} [F(x)^r-1] [F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s} f(x) f(y)
\]

\[-\infty < x < y < \infty \quad \cdots (1.3.11)\]

The joint pdf of the smallest and the largest order statistics is obtained on putting \(r = 1, s = n\) in (1.3.11) and is given by

\[
f_{1,n:n}(x,y) = n(n-1) [F(y)-F(x)]^{n-2} f(x) f(y) \quad -\infty < x < y < \infty \quad \cdots (1.3.12)
\]

Similarly, by setting \(s = r+1\) in (1.3.11), we obtain the joint pdf of two consecutive order statistics, \(X_{r:n}\) and \(X_{r+1:n}\) \((1 \leq r \leq n-1)\), to be

\[
f_{r,r+1:n}(x,y) = \frac{n!}{(r-1)! (n-r-1)!} [F(x)^r-1] [1-F(y)]^{n-r-1} f(x) f(y)
\]

\[-\infty < x < y < \infty \quad \cdots (1.3.13)\]

The joint cumulative distribution function of \(X_{r:n}\) and \(X_{s:n}\)
\begin{align*}
(1 \leq r < s \leq n) \text{ can be obtain as} \\
F_{r,s:n}(x,y) &= P(X_{r:n} \leq x, X_{s:n} \leq y) \\
&= P(\text{at least } r \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x \text{ and} \\
&\quad \text{at least } s \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } y) \\
&= \sum_{j=s}^{n} \sum_{i=r}^{j} P(\text{exactly } i \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } x \\
&\quad \text{and exactly } j \text{ of } X_1, X_2, \ldots, X_n \text{ are at most } y) \\
&= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{(j-i)!(s-r-1)!(n-j)!} [F(x)]^{i-1}[F(y)-F(x)]^{j-i-1}[1-F(y)]^{n-j} \\
&\quad \ldots \ldots \ldots \quad (1.3.14)
\end{align*}

We can write the joint cdf of $X_{r:n}$ and $X_{s:n}$ in (1.3.14) equivalently as

\begin{align*}
F_{r,s:n}(x,y) &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{0}^{F(x)} \int_{0}^{F(y)} u^{r-1}(v-u)^{s-r-1} \\
&\quad (1-v)^{n-s}dudv \quad \ldots \ldots \ldots \quad (1.3.15)
\end{align*}

which is incomplete bivariate beta function.

It may be noted that for $x \geq y$

\begin{align*}
F_{r,s:n}(x,y) &= F_{s:n}(y) \\
&\quad \ldots \ldots \ldots \quad (1.3.17)
\end{align*}

The product moments of $j$-th and $k$-th order of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is given by

\begin{align*}
\mu_{r,s:n}^{(j,k)} &= E\left[X_{r:n}^j X_{s:n}^k\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{y} x^j y^k f_{r,s:n}(x,y)dx dy \quad (1.4.10)
\end{align*}
In general, the joint pdf of $X_{i_1:n}$, $X_{i_2:n}$, ..., $X_{i_k:n}$ for $1 \leq i_1 < i_2 < ... < i_k \leq n$ is given by

$$f_{i_1,i_2,...,i_k:n}(x_{i_1:n}, x_{i_2:n}, ..., x_{i_k:n}) = \frac{n!}{\prod_{j=1}^{k}} \left\{ \frac{\left( F(x_{i_j+1}) - F(x_{i_j}) \right)^{j+1 - i_j - 1}}{(j+1 - i_j - 1)!} \right\} f(x_{i_j})$$

$$\text{for } -\infty < x_{i_1} < x_{i_2} < ... < x_{i_k} < \infty$$

................................. (1.3.19)

where $x_0 = -\infty$, $x_{k+1} = +\infty$, $i_0 = 0$, $i_{k+1} = n+1$.

The distribution function of $X_{r:n}$ can be expressed in terms of negative binomial probabilities instead of the binomial given in (1.3.7) as (Khan, 1981)

$$F_{r:n}(x) = \sum_{r=0}^{n-r} \left( \begin{array}{c} n-1 \end{array} \right) r F(x)^r (1-F(x))^{n-r-1}; \ -\infty < x < \infty . \quad (1.3.20)$$

**Some Comments**

(1) The ranking of random variables $X_1, X_2, ..., X_n$ is preserved under any monotonic increasing transformation of the random variables.

(2) Regarding the probability-integral transformation, if $X_{r:n}$, $1 \leq r \leq n$, are the order statistics from a continuous distribution $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the $r$-th order statistics from a uniform distribution on $(0,1)$.  

10
(3) Even if $X_1, X_2, \ldots, X_n$ are independent random variables, order statistics are not independent random variables.

(4) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a continuous distribution. Then the set of order statistics $\{X_{1:n}, X_{2:n}, \ldots, X_{n:n}\}$ is both sufficient and complete (Lehmann, 1959).

(5) Let $X$ be a continuous random variable. Let $E\{X_{r:n}\} = \mu_{r:n}$.
   
   (a) $\mu_{r:n}$ exists provided $\mu$ exists, but converse is not necessarily true. Specifically, if $\mu$ does not exist, $\mu_{r:n}$ may exist for certain (but not all) values of $r$.

   (b) $\mu_{r:n}$ for all $n$ determine the distribution completely.

1.4. TRUNCATED AND CONDITIONAL DISTRIBUTION OF ORDER STATISTICS

Let $X$ be a continuous random variable having pdf $f_1(x)$ and the cdf $F_1(x)$ in the interval $[-\infty, \infty]$.

Let $\int_{-\infty}^{Q_1} f_1(x)dx = Q$ and $\int_{-\infty}^{P_1} f_1(x)dx = P$ \ldots \ (1.4.1)\n
where $Q_1$ and $P_1$ are known constants. Then the doubly truncated pdf $f(x)$ of $f_1(x)$ is given by

$$f(x) = \frac{1}{P-Q} f_1(x); \ x \in (Q_1, P_1) \quad \ldots \ldots \ldots \ (1.4.2)$$

and the corresponding cdf $F(x)$ is given by

$$F(x) = \frac{1}{P-Q} \left[F_1(x) - Q\right]; \ x \in (Q_1, P_1) \quad \ldots \ldots \ldots \ (1.4.3)$$

The lower and upper truncation points are $Q_1$, $P_1$ respectively; the degrees of truncation are $Q$ (from below) and $1-P$ (from above). If we put $Q=0$, the distribution will be truncated to the
right and for $P=\frac{1}{2}$, the distribution will be truncated to the left. Whereas for $Q=0, P=\frac{1}{2}$, we get the non-truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

In the following we will relate the conditional distribution of order statistics (conditioned on another order statistic) to the distribution of order statistics from a population whose distribution is a truncated form of the original population distribution function $F(x)$.

**STATEMENT 1.4.1.** (David, 1981) Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with cdf $F(x)$ and density function $f(x)$, and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$, given that $X_{r:n} = x$ for $r < s$, is the same as the distribution of the $(s-r)$-th order statistic obtained from a sample of size $n-r$ from a population whose distribution is truncated on the left at $x$.

**STATEMENT 1.4.2.** (David, 1981) Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with cdf $F(x)$ and density function $f(x)$, and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$, given that $X_{s:n} = y$ for $s > r$, is the same as the distribution of the $r$-th order statistic.
obtained from a sample of size s-1 from a population whose
distribution truncated on the right at y.

**STATEMENT 1.4.3.** Let \( X_1, X_2, \ldots, X_n \) be a random sample from an
absolutely continuous population with cdf \( F_X(x) \) and density
function \( f_X(x) \), and let \( X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \) denote the order
statistics obtained from this sample. Then the conditional
distribution of \( X_{s:n} \), given that \( X_{r:n} = x \) and \( X_{k:n} = z \) for \( r < s \)< \( k \), is the same as the distribution of the \((s-r)\)th order
statistic obtained from a sample of size \( k-r-1 \) from a population
whose distribution is truncated on the left at \( x \) and on the right
at \( z \).

**PROOF:** From (1.3.19), we can show that the joint density
function of \( X_{r:n}, X_{s:n}, \) and \( X_{k:n} \) \((1 \leq r < s < k \leq n)\) is given by

\[
f_{r,s,k:n}(x,y,z) = \frac{n!}{(r-1)! (s-r-1)! (k-s-1)! (n-k)!} [F_X(x)]^{r-1}
\]

\[
[F_X(x)-F_X(y)]^{s-r-1} [F_X(z)-F_X(x)]^{k-s-1} [1-F_X(z)]^{n-k}
\]

\[
f(x)f(y)f(z); \quad -\infty < x < y < z < \infty \quad \ldots \quad (1.4.6)
\]

From equations (1.4.6) and (1.3.11), we obtain the conditional
density function of \( X_{s:n} \) given that \( X_{r:n} = x \) and \( X_{k:n} = z \), to be

\[
f_{s:n}(y|X_{r:n} = x, X_{k:n} = z) = \frac{f_{r,s,k:n}(x,y,z)}{f_{r,k:n}(x,z)}
\]
The result follows immediately from (1.4.4) upon noting that $f(x) = \frac{f(y) f(z) - f(y) f(x)}{f(z) f(x)}$ and $F(x) = \frac{F(y) - F(x)}{F(z) - F(x)}$ are the cdf and density function of the population whose distribution is obtained by truncating the distribution $F(x)$ on the left at $x$ and on the right at $z$.

1.5. SOME CONTINUOUS DISTRIBUTIONS.

I. **PARETO DISTRIBUTION:** A random variable $X$ is said to have the Pareto distribution if its probability density function (pdf) and distribution function (df) are of the form

$$f(x) = p\lambda^p x^{-(p+1)}; \quad 0 < x < \omega; \quad \lambda, p > 0$$

$$F(x) = 1 - \lambda^p x^{-p}; \quad 0 < x < \omega; \quad \lambda, p > 0$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc..

II. **POWER FUNCTION DISTRIBUTION:** A random variable $X$ is said to have a Power function distribution if its pdf $f(x)$ and $F(x)$ are of the form

$$f(x) = p\lambda^{-p} x^{p-1}; \quad 0 \leq x \leq \lambda; \quad \lambda, p > 0$$

and the cumulative distribution function is given by

$$F(x) = \lambda^p x^p; \quad 0 \leq x \leq \lambda; \quad \lambda, p > 0$$

The Power function distribution is used to approximate
representation of the lower tail of the distribution of random variable having a fixed lower bound. It may be noted that if X has a Power function distribution, then Y = 1/X has a Pareto distribution.

III. Beta Distribution: A random variable X is said to have the Beta distribution if its probability density function is of the form

\[ f(x) = \frac{1}{B(p, q)} x^{p-1}(1-x)^{q-1}; 0 \leq x \leq 1, \quad p, q > 0 \]

Beta distribution arises as the distribution of an ordered variable from a rectangular distribution. Suppose \( X_{r:n} \) is an ordered sample from \( U(0,1) \), then \( X_{r:n} \sim B(r, n-r+1) \). The standard rectangular distribution \( R(0,1) \) is the special case of Beta distribution obtained by putting the exponents p and q equal to 1. If q = 1, the distribution is sometimes called Power function distribution. The Beta distribution is one of the most frequently employed distribution to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

IV. Weibull Distribution: A random variable X is said to have a Weibull distribution if its probability density function is given by

\[ f(x) = \theta px^{p-1} e^{-\theta x^p}; 0 \leq x < \infty; \theta > 0, \quad p > -1 \]

and the cumulative distribution function is given by
If we put \( p=1 \) in Weibull distribution, we get the pdf of Exponential distribution whereas for \( p=2 \), it gives Rayleigh distribution. The use of Weibull distribution in reliability and quality control is well known. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

V. **Burr Type XII Distribution:** A random variable \( X \) is said to have a Burr type XII distribution if its probability density function is given by

\[
f(x) = \lambda \theta x^{p-1}(1+\theta x^p)^{-(\lambda+1)}; \quad 0 \leq x < \infty; \quad \lambda, \ p, \ \theta > 0
\]

and the cumulative distribution function is given by

\[
F(x) = 1-(1+\theta x^p)^{-\lambda}; \quad 0 \leq x < \infty; \quad \lambda, \ p, \ \theta > 0
\]

This distribution is frequently used for purpose of graduation and in reliability theory. At \( p=1 \), it is called Lomax distribution whereas for \( \lambda=1 \), it is known as Log-logistic distribution.

VI. **Rectangular Distribution:** A variable \( X \) is said to have a Rectangular distribution if its probability density function is given by

\[
f(x) = \frac{1}{\lambda-\beta}; \quad \beta \leq x \leq \lambda
\]

and the cdf is given by
The standard Rectangular distribution \( R(0,1) \) is obtained by putting \( \beta=0 \) and \( \lambda=1 \). It is noted that every distribution function follows rectangular distribution \( R(0,1) \). This distribution is used in "rounding off" errors, probability integral transformation, random number generation, traffic flow, generation of Normal, Exponential distribution etc.

VII. Exponential Distribution: A random variable \( X \) is said to have a Exponential distribution if its probability density function is given by

\[
f_X(x) = \theta e^{-\theta x}; \quad 0 \leq x < \infty; \quad \theta > 0
\]

and the cdf is given by

\[
F_X(x) = 1 - e^{-\theta x}; \quad 0 \leq x < \infty; \quad \theta > 0
\]

The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable \( X \) assuming non-negative values satisfies the assumption,

\[
P(X > s + t | X > s) = P(X > t)
\]

for all \( s \) and \( t \), then \( X \) will have an exponential distribution. This is particularly very much appropriate failure law when present does not depend on past, for example in studying the life of a bulb etc.
VIII. **Generalized Linear-Exponential Distribution:** A random variable $X$ have a Generalized Linear-Exponential distribution if its pdf is given by

$$f(x) = (\lambda + \theta x)^{p-1} e^{-(\lambda x + \theta x^p)}; \quad 0 \leq x < \infty; \quad \lambda, \theta, p > 0$$

and the cdf is given by

$$F(x) = 1 - e^{-(\lambda x + \theta x^p)}; \quad 0 \leq x < \infty; \quad \lambda, \theta, p > 0$$

If we put $\lambda = 0$, it becomes Weibull distribution whereas for $\theta = 0$; and for $\lambda = 0$, $p = 2$, it is Exponential and Rayleigh distribution respectively.

IX. **Cauchy Distribution:** The pdf of Cauchy distribution is given by

$$f(x) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + (x-\theta)^2}; \quad -\infty < x < \infty; \quad \lambda > 0, \quad -\infty < \theta < \infty$$

and the cdf is given by

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{x-\theta}{\lambda} \right); \quad -\infty < x < \infty; \quad \lambda > 0, \quad -\infty < \theta < \infty$$

For $\lambda = 1, \theta = 0$ we get the standard form of the distribution and it is also the t-distribution with 1 degree of freedom. It is thus the distribution of the ratio of two independent unit normal variates. The distribution does not admit mean and variance.