CHAPTER – III
LATTICE OF LEVEL $\ell$ – IDEALS OF A FUZZY $\ell$ – IDEAL

In classical theory, $\ell$ – ideals associated to any $\ell$ – ring, play a central role. Keeping this in mind, the study of fuzzy $\ell$ – ideal of $\ell$ – ring is made in this chapter. The concepts of fuzzy $\ell$ – ideal and level $\ell$ – ideal of an $\ell$ – ring were introduced and their properties are studied. Further every fuzzy $\ell$ – ideal of an $\ell$ – ring is either a decreasing function or a constant function, the characterization theorem of a fuzzy $\ell$ – ideal, a procedure to construct a fuzzy $\ell$ – ideal from any given ascending chain of $\ell$ – ideals of an $\ell$ – ring, the necessary and sufficient condition for equality of two level $\ell$ – ideals, completely distributive lattice of the family of level $\ell$ – ideal are established.

To start with,

**Definition 3.1:**

A fuzzy subset $\mu$ of an $\ell$ – ring $R$, is called a fuzzy $\ell$ – ring ideal or fuzzy $\ell$ – ideal of $R$ if, for all $x, y \in R$ the following conditions are satisfied:

1. $\mu(x \lor y) \geq \min \{\mu(x), \mu(y)\}$
2. $\mu(x \land y) \geq \max \{\mu(x), \mu(y)\}$
3. $\mu(x - y) \geq \min \{\mu(x), \mu(y)\}$
4. $\mu(xy) \geq \max \{\mu(x), \mu(y)\}$

**Example 3.2:**

Consider the fuzzy subset $\mu$ of the $\ell$ – ring $R$, defined in Example 1.8

\[ \mu(x) = \begin{cases} 
0.9 & \text{if } x = a \\
0.6 & \text{if } x = b \\
0.4 & \text{if } x = c, d
\end{cases} \]

Then $\mu$ is a fuzzy $\ell$ – ideal of $R$. 

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Example 3.3:
Consider the ℓ – ring given in Example: 1.9. Then the fuzzy subset σ of S defined by
\[\sigma(x) = \begin{cases} 0.4 & \text{if } x = m \\ 0.1 & \text{if } x = n \end{cases}\]
Then σ is a fuzzy ℓ – ideal of R.

Remark 3.4:
Every fuzzy ℓ – ideal of an ℓ – ring R, is a fuzzy sub ℓ – ring of R. But the converse need not be true.

Proof:
By the Example,
Consider the fuzzy subset µ₁ of the ℓ – ring \((Z, +, \cdot, \lor, \land)\).
\[\mu_1(x) = \begin{cases} 0.6 & \text{if } x \not\in \langle 2 \rangle \\ 0.3 & \text{otherwise} \end{cases}\]
Then \(\mu_1\) is a fuzzy sub ℓ – ring of Z. But \(\mu_1\) is not a fuzzy ℓ – ideal of Z.

Proposition 3.5:
If µ is any fuzzy ℓ – ideal of an ℓ – ring R, then \(\mu(1) \leq \mu(x) \leq \mu(0)\), for all \(x \in R\) where 0 is the least element and 1 is the greatest element in R.

Proof:
Given µ is any fuzzy ℓ – ideal of an ℓ – ring R with least element 0 and greatest element 1.
To prove \(\mu(1) \leq \mu(x) \leq \mu(0)\), for all \(x \in R\).

Let \(x \in R\) be arbitrary. Then,
\[\mu(x) = \mu(1 \land x) \geq \max \{\mu(1), \mu(x)\} \geq \mu(1) \quad \rightarrow (1)\]
And \(\mu(0) = \mu(x - x) \geq \min \{\mu(x), \mu(x)\} \geq \mu(x) \quad \rightarrow (2)\)
From (1) and (2), we have \(\mu(1) \leq \mu(x) \leq \mu(0)\), for all \(x \in R\).

Remark 3.6:
Let µ be any fuzzy ℓ – ideal of an ℓ – ring R. Then \(\mu(x) = \mu(-x)\) for all \(x \in R\).
Proof:

Let µ be any fuzzy ℓ – ideal of an ℓ – ring R.
Then by Remark: 3.4, µ is fuzzy sub ℓ – ring of R.
Let x ∈ R be arbitrary.
Then by Proposition: 2.5, we get µ(x) = µ(-x) for all x ∈ R.

Proposition 3.7:

Let µ be any fuzzy ℓ – ideal of an ℓ – ring R. µ(x) ≥ µ(y) whenever x ≤ y, where x, y ∈ R.

Proof:

Given µ is any fuzzy ℓ – ideal of an ℓ – ring R.
Let x, y ∈ R be arbitrary.
Assume that x ≤ y.
⇒ x ∧ y = x and x ∨ y = y ----------→ (1)
Now, µ(x) = µ(x ∧ y), by (1)
≥ max {µ(x), µ(y)} ≥ µ(y)

Proposition 3.8:

Let µ be a fuzzy subset of an ℓ – ring R. If µ is a constant function, that is µ(x) = c, for all x ∈ R, where c is a constant, then µ is a fuzzy ℓ – ideal of R.

Proof:

Given µ is a fuzzy subset of an ℓ – ring R and µ is a constant function.
⇒ µ(x) = c, for all x ∈ R.
Then it is easy to prove the four inequalities.

Proposition 3.9:

Every constant function of an ℓ – ring R is a fuzzy ℓ – ideal of R. But the converse need not be true. That is, every fuzzy ℓ – ideal of R need not be a constant function.
Proof:

First part is shown in Proposition: 3.8

We prove the second part by giving a counter example.

\[ \theta(x) = \begin{cases} 
0.9 & \text{if } x = a, b \\
0.4 & \text{if } x = c, d 
\end{cases} \]

Then \( \theta \) is a fuzzy \( \ell \)– ideal of the \( \ell \)– ring \( R \) defined in Example: 1.8, but \( \theta \) is not a constant function. Thus every fuzzy \( \ell \)– ideal of \( R \) need not be a constant function.

Remark 3.10:

Let \( \mu \) be a fuzzy \( \ell \)– ideal of an \( \ell \)– ring \( R \). As \( x \land y \leq x \leq x \lor y \), then by the Proposition 3.7, we have \( \mu(x \land y) \geq \mu(x) \geq \mu(x \lor y) \), for all \( x \in R \).

Thus \( \mu \) is either a decreasing function or a constant function.

Proposition 3.11:

Let \( \mu \) be any fuzzy \( \ell \)– ideal of an \( \ell \)– ring \( R \). If \( \mu(x \lor y) = \mu(0) \) for some \( x, y \in R \), then \( \mu(x) = \mu(y) = \mu(0) \).

Also \( \mu(x) = \mu(y) = \mu(x \land y) = \mu(x \land y) = \mu(x - y) = \mu(xy) = \mu(0) \).

Proof:

Let \( \mu \) be any fuzzy \( \ell \)– ideal of an \( \ell \)– ring \( R \).

Let \( x, y \in R \) be arbitrary.

Assume that \( \mu(x \lor y) = \mu(0) \) \( \rightarrow (1) \)

To prove \( \mu(x) = \mu(y) = \mu(0) \).

Now, \( \mu(x) = \mu((x \lor y) \land x) \geq \max \{\mu(x \lor y), \mu(x)\} = \max \{\mu(0), \mu(x)\} \), by \( (1) \)

\[ = \mu(0) \rightarrow (2) \], by Proposition: 3.5

And \( \mu(x) \leq \mu(0) \rightarrow (3) \), by Proposition: 3.5

From (2) and (3), we have, \( \mu(x) = \mu(0) \).

Similarly, \( \mu(y) = \mu(0) \).

Hence \( \mu(x) = \mu(y) = \mu(0) \)

Here \( \min \{\mu(x), \mu(y)\} = \mu(0) \) and \( \max \{\mu(x), \mu(y)\} = \mu(0) \)

Now \( \mu(x - y) \geq \min \{\mu(x), \mu(y)\} = \mu(0) \)
\[ \Rightarrow \mu(x - y) \geq \mu(0) \]
But \( \mu(x - y) \leq \mu(0) \)
Hence \( \mu(x - y) = \mu(0) \).
Again \( \mu(x \wedge y) \geq \max \{\mu(x), \mu(y)\} = \mu(0) \)
\[ \Rightarrow \mu(x \wedge y) \geq \mu(0) \]
But \( \mu(x \wedge y) \leq \mu(0) \)
Hence \( \mu(x \wedge y) = \mu(0) \)
Similarly we can prove \( \mu(xy) = \mu(0) \)
Thus \( \mu(x) = \mu(y) = \mu(x \vee y) = \mu(x \wedge y) = \mu(x - y) = \mu(xy) = \mu(0) \).

**Proposition 3.12:**
Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \). If \( \mu(x - y) = \mu(0) \) for some \( x, y \in R \), then \( \mu(x) = \mu(y) \).

**Proof:**
Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \).
Then by Remark: 3.4, \( \mu \) is fuzzy sub \( \ell \) – ring of \( R \).
Let \( x, y \in R \) be arbitrary.
Assume that \( \mu(x - y) = \mu(0) \).
Then by Proposition: 2.7, we get \( \mu(x) = \mu(y) \)

**Remark 3.13:**
Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \). If \( \mu(x - y) = 1 \) for some \( x, y \in R \), then \( \mu(x) = \mu(y) \).

**Proof:**
Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \).
Let \( x, y \in R \) be arbitrary.
Assume that \( \mu(x - y) = 1 \) \( \Rightarrow (1) \)
Since \( R \) is an \( \ell \) – ring, \( x - y \in R \)
Then by Proposition: 3.5, \( \mu(x - y) \leq \mu(0) \)
\[ \Rightarrow 1 \leq \mu(0), \text{ by (1)} \]
But \( \mu(0) \leq 1 \), since \( \text{Im } \mu \in [0, 1] \)
\[ \Rightarrow \mu(0) = 1 \]

Hence (1) \( \Rightarrow \mu(x - y) = \mu(0) \)

Then by Proposition: 3.12, we have \( \mu(x) = \mu(y) \).

**Proposition 3.14:**

Let \( \mu \) be a fuzzy \( \ell \)-ideal of \( R \). If \( \mu(x) < \mu(y) \) for some \( x, y \in R \), then \( \mu(x - y) = \mu(x) \).

**Proof:**

Let \( \mu \) be a fuzzy \( \ell \)-ideal of \( R \).

Let \( x, y \in R \) be arbitrary.

Assume that \( \mu(x) < \mu(y) \) \( \Rightarrow (1) \)

Now \( \mu(x - y) \geq \min \{\mu(x), \mu(y)\} = \mu(x) \), by (1)

\[ \Rightarrow \mu(x - y) \geq \mu(x) \] \( \Rightarrow (2) \)

And \( x = x + 0 = x + (-y + y) = (x - y) + y \)

Therefore, \( \mu(x) = \mu((x-y) + y) \geq \min \{\mu(x-y), \mu(y)\} = \mu(x-y) \) as \( \mu(x) < \mu(y) \)

\[ \Rightarrow \mu(x) \geq \mu(x-y) \] \( \Rightarrow (3) \)

From (2) and (3), we have \( \mu(x) = \mu(x-y) \).

**Proposition 3.15:**

Let \( \mu \) and \( \theta \) be any two fuzzy \( \ell \)-ideals of \( R \). If \( \mu(x) \leq \theta(x) \) and \( \mu(y) \leq \theta(y) \), then \( \mu(x - y) \leq \theta(x - y) \) for some \( x, y \in R \).

**Proof:**

Let \( \mu \) and \( \theta \) be any two fuzzy \( \ell \)-ideals of an \( \ell \)-ring \( R \).

Then for some \( x, y \in R \), we have,

\[ \mu(x - y) \leq \min \{\mu(x), \mu(y)\} \] \( \Rightarrow (1) \)

\[ \theta(x - y) \leq \min \{\theta(x), \theta(y)\} \] \( \Rightarrow (2) \)

Assume that \( \mu(x) \leq \theta(x) \) \( \Rightarrow (3) \)

And \( \mu(y) \leq \theta(y) \) \( \Rightarrow (4) \)

To prove \( \mu(x - y) \leq \theta(x - y) \)

**Case (i):** \( \min\{\mu(x), \mu(y)\} = \mu(x) \) and \( \min\{\theta(x), \theta(y)\} = \theta(x) \)

\[ \Rightarrow \mu(x) \leq \mu(y) \text{ and } \theta(x) \leq \theta(y) \]
\( \Rightarrow \mu(x - y) = \mu(x) \) and \( \theta(x - y) = \theta(x) \rightarrow (5) \), by previous Proposition.

(5) and (3) \( \Rightarrow \mu(x - y) \leq \theta(x - y) \)

**Case (ii):** \( \min\{\mu(x), \mu(y)\} = \mu(y) \) and \( \min\{\theta(x), \theta(y)\} = \theta(y) \)

\( \Rightarrow \mu(y) \leq \mu(x) \) and \( \theta(y) \leq \theta(x) \)

\( \Rightarrow \mu(x - y) = \mu(y) \) and \( \theta(x - y) = \theta(y) \rightarrow (6) \), by previous Proposition.

(6) and (4) \( \Rightarrow \mu(x - y) \leq \theta(x - y) \)

**Case (iii):** \( \min\{\mu(x), \mu(y)\} = \mu(x) \) and \( \min\{\theta(x), \theta(y)\} = \theta(y) \)

\( \Rightarrow \mu(x) \leq \mu(y) \) and \( \theta(y) \leq \theta(x) \rightarrow (7) \)

By previous Proposition and (7), we have,

\( \mu(x \vee y) = \mu(x) \) and \( \theta(x \vee y) = \theta(y) \rightarrow (9) \)

(8) and (9) \( \Rightarrow \mu(x - y) \leq \theta(x - y) \)

**Case (iv):** \( \min\{\mu(x), \mu(y)\} = \mu(y) \) and \( \min\{\theta(x), \theta(y)\} = \theta(x) \)

\( \Rightarrow \mu(y) \leq \mu(x) \) and \( \theta(x) \leq \theta(y) \)

\( \Rightarrow \mu(x - y) = \mu(y) \) and \( \theta(x - y) = \theta(x) \rightarrow (10) \), by previous Proposition.

Again \( \mu(y) \leq \mu(x) \), by (3)

\( \Rightarrow \mu(y) \leq \theta(x) \rightarrow (11) \)

(10) and (11) \( \Rightarrow \mu(x - y) \leq \theta(x - y) \)

Thus \( \mu(x - y) \leq \theta(x - y) \) in all the cases.

**Proposition 3.16:**

Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \). If \( \mu(x) < \mu(y) \) for some \( x, y \in R \), then \( \mu(x \vee y) = \mu(x) \).

**Proof:**

Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \).

Let \( x, y \in R \) be arbitrary.

Assume that \( \mu(x) < \mu(y) \rightarrow (1) \)
Now $\mu(x \lor y) \geq \min \{\mu(x), \mu(y)\} = \mu(x)$, by (1) $\rightarrow$ (2)

Again $\mu(x) = \mu((x \lor y) \land x) \geq \max \{\mu(x \lor y), \mu(x)\} \geq \mu(x \lor y)$ $\rightarrow$ (3)

From (2) and (3), we have, $\mu(x \lor y) = \mu(x)$.

**Proposition 3.17:**

Let $\mu$ and $\theta$ be any two fuzzy $\ell$ – ideals of $R$. If $\mu(x) \leq \theta(x)$ and $\mu(y) \leq \theta(y)$, then $\mu(x \lor y) \leq \theta(x \lor y)$ for some $x, y \in R$.

**Proof:**

Let $\mu$ and $\theta$ be any two fuzzy $\ell$ – ideals of an $\ell$ – ring $R$.

Let $x, y \in R$ be arbitrary. Then, we have, $\mu(x \lor y) \geq \min \{\mu(x), \mu(y)\}$ $\rightarrow$ (1) and $\theta(x \lor y) \geq \min \{\theta(x), \theta(y)\}$ $\rightarrow$ (2)

Assume that $\mu(x) \leq \theta(x)$ $\rightarrow$ (3)

And $\mu(y) \leq \theta(y)$ $\rightarrow$ (4)

To prove $\mu(x \lor y) \leq \theta(x \lor y)$

**Case (i):** $\min\{\mu(x), \mu(y)\} = \mu(x)$ and $\min\{\theta(x), \theta(y)\} = \theta(x)$

$\Rightarrow \mu(x) \leq \mu(y)$ and $\theta(x) \leq \theta(y)$

$\Rightarrow \mu(x \lor y) = \mu(x)$ and $\theta(x \lor y) = \theta(x)$ $\rightarrow$ (5), by previous Proposition.

(5) and (3) $\Rightarrow \mu(x \lor y) \leq \theta(x \lor y)$

**Case (ii):** $\min\{\mu(x), \mu(y)\} = \mu(y)$ and $\min\{\theta(x), \theta(y)\} = \theta(y)$

$\Rightarrow \mu(y) \leq \mu(x)$ and $\theta(y) \leq \theta(x)$

$\Rightarrow \mu(x \lor y) = \mu(y)$ and $\theta(x \lor y) = \theta(y)$ $\rightarrow$ (6), by previous Proposition.

(6) and (4) $\Rightarrow \mu(x \lor y) \leq \theta(x \lor y)$

**Case (iii):** $\min\{\mu(x), \mu(y)\} = \mu(x)$ and $\min\{\theta(x), \theta(y)\} = \theta(y)$

$\Rightarrow \mu(x) \leq \mu(y)$ and $\theta(y) \leq \theta(x)$ $\rightarrow$ (7)

$\Rightarrow \mu(x) \leq \mu(y) \leq \theta(y) \leq \theta(x)$, by (4) & (7)

$\Rightarrow \mu(x) \leq \theta(y)$ $\rightarrow$ (8)

$\Rightarrow \mu(x \lor y) = \mu(x)$ and $\theta(x \lor y) = \theta(y)$ $\rightarrow$ (9), by previous Proposition and (7)

(8) and (9) $\Rightarrow \mu(x \lor y) \leq \theta(x \lor y)
**Case (iv):** \( \min \{\mu(x), \mu(y)\} = \mu(y) \) and \( \min \{\theta(x), \theta(y)\} = \theta(x) \)

\( \Rightarrow \mu(y) \leq \mu(x) \) and \( \theta(x) \leq \theta(y) \)

\( \Rightarrow \mu(x \vee y) = \mu(y) \) and \( \theta(x \vee y) = \theta(x) \) \( \Rightarrow (10) \), by previous Proposition.

Again \( \mu(y) \leq \mu(x) \leq \theta(x) \), by (3)

\( \Rightarrow \mu(y) \leq \theta(x) \) \( \Rightarrow (11) \)

(10) and (11) \( \Rightarrow \mu(x \vee y) \leq \theta(x \vee y) \)

Thus \( \mu(x \vee y) \leq \theta(x \vee y) \) in all the cases.

*Now, the necessary and sufficient condition of a fuzzy subset to be a fuzzy \( \ell \)– ideal in terms of \( \ell \)– ideals of the \( \ell \)– ring is proved here.*

**Theorem 3.18:**

Let \( H \) be any non– empty subset of an \( \ell \)– ring \( R \), \( H \neq R \). If \( \mu \) is a fuzzy \( \ell \) – ideal of \( R \), defined by

\[
\mu(x) = \begin{cases} 
    s & \text{if } x \in H \\
    t & \text{if } x \in R \sim H 
\end{cases}
\]

where \( s, t \in [0, 1] \), \( s > t \), then \( H \) is an \( \ell \)– ideal of \( R \).

**Proof:**

Assume that \( \mu \) is a fuzzy \( \ell \)– ideal of \( R \).

Let \( x, y \in R \) be arbitrary. Then,

\( \begin{align*}
(v) & \quad \mu(x \vee y) \geq \min \{\mu(x), \mu(y)\} \\
(vi) & \quad \mu(x \wedge y) \geq \max \{\mu(x), \mu(y)\} \\
(vii) & \quad \mu(x - y) \geq \min \{\mu(x), \mu(y)\} \\
(viii) & \quad \mu(x y) \geq \max \{\mu(x), \mu(y)\}
\end{align*} \)

Given \( \mu(x) = \begin{cases} 
    s & \text{if } x \in H \\
    t & \text{if } x \in R \sim H 
\end{cases} \)

where \( s, t \in [0, 1] \), \( s > t \).

To prove \( H \) is an \( \ell \)– ideal of \( R \). That is to prove

\( \begin{align*}
(i) & \quad x, y \in H \Rightarrow x \vee y, x - y \in H. \\
(ii) & \quad x \in H, r_1 \in R \text{ such that } r_1 \leq x \Rightarrow r_1 \in H. \\
(iii) & \quad x \in H, r \in R \Rightarrow rx, xr \in H.
\end{align*} \)
For (i):

Let \( x, y \in H \) be arbitrary.

Then \( \mu(x) = s, \mu(y) = s \)

Here \( \min \{\mu(x), \mu(y)\} = \min\{s, s\} = s \)

Hence all the values of \( \mu(x \lor y) \), and \( \mu(x - y) \) are greater than or equal to \( s \).

But \( \mu \) has only two values \( s \) and \( t \) with \( s > t \).

Therefore, all the values of \( \mu(x \lor y) \) and \( \mu(x - y) \) are equal to \( s \).

This implies \( x \lor y, x - y \in H \).

Thus \( x, y \in H \Rightarrow x \lor y, x - y \in H \).

For (ii):

Let \( x \in H \) and \( r_1 \in R \) be arbitrary, such that \( r_1 \leq x \)

Then \( \mu(x) = s \)

Now, \( \mu(r_1) = \mu(r_1 \land x) \geq \max \{\mu(r_1), \mu(x)\} = s \)

Then \( \mu(r_1) \geq s \)

\( \Rightarrow r_1 \in H \)

Thus \( x \in H \) and \( r_1 \in R \) such that \( r_1 \leq x \Rightarrow r_1 \in H \).

For (iii):

Let \( x \in H, r \in R \)

Then \( \mu(x) = s \)

Suppose \( r \in H \), then \( \mu(r) = s \)

Now, \( \mu(xr) \geq \max \{\mu(x), \mu(r)\} = \max \{s, s\} = s \)

\( \Rightarrow \mu(xr) = s \)

\( \Rightarrow xr \in H. \)

And \( \mu(rx) \geq \max \{\mu(r), \mu(x)\} = \max \{s, s\} = s \)

\( \Rightarrow \mu(rx) = s \)

\( \Rightarrow rx \in H. \)

Suppose \( r \not\in H \), then \( \mu(r) = t \)
Now, \( \mu(xr) \geq \max \{\mu(x), \mu(r)\} = \max \{s, t\} = s \)
\( \Rightarrow \mu(xr) = s \)
\( \Rightarrow xr \in H. \)
And \( \mu(rx) \geq \max \{\mu(r), \mu(x)\} = \max \{t, s\} = s \)
\( \Rightarrow \mu(rx) = s \)
\( \Rightarrow rx \in H. \)
Hence, \( x \in H \) and \( r \in R \Rightarrow xr, rx \in H. \)
Thus, \( H \) is an \( \ell \)– ideal of \( R. \)

**Theorem 3.19:**
If \( H \) is any \( \ell \)– ideal of an \( \ell \)– ring \( R, H \neq R \), then the fuzzy subset \( \mu \)
of \( R \) defined by
\[
\mu(x) = \begin{cases} 
s & \text{if } x \in H \\
t & \text{if } x \in R \sim H
\end{cases}
\]
where \( s, t \in [0, 1], s > t \) is a fuzzy \( \ell \)– ideal of \( R. \)

**Proof:**
Given \( H \) is any \( \ell \)– ideal of an \( \ell \)– ring \( R, H \neq R. \)
Consider, the fuzzy subset \( \mu \) of \( R \) defined by,
\[
\mu(x) = \begin{cases} 
s & \text{if } x \in H \\
t & \text{if } x \in R \sim H
\end{cases}
\]
where \( s, t \in [0, 1], s > t. \)

Let \( x, y \in R \) be arbitrary.
To prove \( \mu \) is a fuzzy \( \ell \)– ideal of \( R. \)
It is enough to prove that whenever \( x, y \in R, \)
(i) \( \mu(x \lor y) \geq \min \{\mu(x), \mu(y)\} \)
(ii) \( \mu(x \land y) \geq \max \{\mu(x), \mu(y)\} \)
(iii) \( \mu(x - y) \geq \min \{\mu(x), \mu(y)\} \)
(iv) \( \mu(xy) \geq \max \{\mu(x), \mu(y)\} \)
We prove this in three cases:
Case (i): \(x, y \in H\)
\(\mu(x) = s, \mu(y) = s\)
As \(x, y \in H\), we have \(x \vee y, x \wedge y, x - y, -x, xy \in H\)
\(\Rightarrow \mu(x \vee y) = s, \mu(x \wedge y) = s, \mu(x - y) = s, \mu(xy) = s\).
Then all the inequalities are satisfied in this case.

Case (ii): \(x, y \in R \sim H\)
\(\mu(x) = t, \mu(y) = t\)
\(\Rightarrow \min \{\mu(x), \mu(y)\} = \min \{t, t\} = t \quad \text{and} \quad \max \{\mu(x), \mu(y)\} = \max \{t, t\} = t\).
Here \(x \vee y, x \wedge y, x - y, xy \in R\)
Then all the values of \(\mu(x \vee y), \mu(x \wedge y), \mu(x - y)\) and \(\mu(xy)\) are either \(t\) or \(s\).
If \(\mu(x \vee y) = s\), then \(\mu(x \vee y) = s > t = \min \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}\).
If \(\mu(x \vee y) = t\), then \(\mu(x \vee y) = t = \min \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(x \vee y) \geq \min \{\mu(x), \mu(y)\}\).
If \(\mu(x \wedge y) = s\), then \(\mu(x \wedge y) = s > t = \max \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(x \wedge y) \geq \max \{\mu(x), \mu(y)\}\).
If \(\mu(x \wedge y) = t\), then \(\mu(x \wedge y) = t = \max \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(x \wedge y) \geq \max \{\mu(x), \mu(y)\}\).
If \(\mu(x - y) = s\), then \(\mu(x - y) = s > t = \min \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(x - y) \geq \min \{\mu(x), \mu(y)\}\).
If \(\mu(x - y) = t\), then \(\mu(x - y) = t = \min \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(x - y) \geq \min \{\mu(x), \mu(y)\}\).
If \(\mu(xy) = s\), then \(\mu(xy) = s > t = \max \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(xy) \geq \max \{\mu(x), \mu(y)\}\).
If \(\mu(xy) = t\), then \(\mu(xy) = t = \max \{\mu(x), \mu(y)\}\)
\(\Rightarrow \mu(xy) \geq \max \{\mu(x), \mu(y)\}\).
Then all the inequalities are satisfied in this case also.
**Case (iii):** \( x \in H, y \in R \sim H \)

\( \mu(x) = s, \mu(y) = t \)

Again, \( \min \{ \mu(x), \mu(y) \} = \min \{s, t\} = t \) and \( \max \{ \mu(x), \mu(y) \} = \max \{s, t\} = s \).

Clearly \( x \lor y, x \land y, x - y, xy \in R \)

Suppose \( x \lor y \in H \), then \( y \in H \) as \( y \leq x \lor y \) and \( H \) is an \( \ell \)–ideal of \( R \)

This is a contradiction to our assumption.

Thus \( x \lor y \notin H \) and hence \( x \lor y \in R \sim H \)

Therefore, \( \mu(x \lor y) = t \).

Again, \( x \land y \leq x \) and \( H \) is an \( \ell \)–ideal of \( R \), then \( x \land y \in H \)

Therefore, \( \mu(x \land y) = s \).

Suppose \( x - y \in H \)

Since \( x \in H \) and \( x - y \in H \) as \( H \) is an \( \ell \)–ideal, then \( x - (x - y) \in H \).

\( \Rightarrow y \in H \)

This is a contradiction to our assumption.

Thus \( x - y \notin H \) and hence \( x - y \in R \sim H \)

Therefore, \( \mu(x - y) = t \).

Here \( x \in H \) and \( y \in R \), and as \( H \) is an \( \ell \)–ideal, then \( xy \in H \).

Then \( \mu(xy) = s \).

Then all the inequalities are satisfied in this case also.

Thus \( \mu \) is a fuzzy \( \ell \)–ideal of \( R \).

**Corollary 3.20:**

If a non–empty subset \( H \) of an \( \ell \)–ring \( R \), is an \( \ell \)–ideal of \( R \), then \( \Sigma H \) is a fuzzy \( \ell \)–ideal of \( R \).

**Proof:**

Take \( s = 1, t = 0 \) in the above theorem.

Then \( \Sigma H \) is a fuzzy \( \ell \)–ideal of \( R \).
From Theorem: 3.18 and Theorem: 3.19, we have,

**Theorem 3.21:**

Let \( H \) be any non-empty subset of an \( \ell \) – ring \( R, H \neq R \). Let \( \mu \) be any fuzzy subset of \( R \) defined by

\[
\mu(x) = \begin{cases} 
s & \text{if } x \in H \\
t & \text{if } x \in R - H \end{cases}
\]

where \( s, t \in [0, 1], s > t \). Then \( \mu \) is a fuzzy \( \ell \) – ideal of \( R \) iff \( H \) is an \( \ell \) – ideal of \( R \).

Now the Characterization Theorem for a fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \) with respect to the level subsets of \( \mu \) is proved.

**Proposition 3.22:**

Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \). The level subsets \( \mu_t, t \in \text{Im} \mu \) are \( \ell \) – ideals of \( R \).

**Proof:**

Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \).

Consider the level subset \( \mu_t \) of \( R, \mu_t = \{x \in R/\mu(x) \geq t\}, t \in \text{Im} \mu \).

For all \( x \in R, \mu(x) \leq \mu(0) \).

\[ \Rightarrow \mu(0) \geq t \text{ for all } t \in \text{Im} \mu. \]

\[ \Rightarrow 0 \in \mu_t, \text{ for all } t. \]

Hence \( \mu_t \neq \phi \).

To prove \( \mu_t \) is an \( \ell \) – ideal of \( R \).

That is to prove

\[
\begin{align*}
(\text{i}) & \quad x, y \in \mu_t \Rightarrow x \lor y, x - y \in \mu_t. \\
(\text{ii}) & \quad x \in \mu_t, r_1 \in R \text{ such that } r_1 \leq x \Rightarrow r_1 \in \mu_t. \\
(\text{iii}) & \quad x \in \mu_t, r \in R \Rightarrow rx, xr \in \mu_t.
\end{align*}
\]

**For (i):**

Let \( x, y \in \mu_t, t \in \text{Im} \mu \) be arbitrary.

Then \( x, y \in R \) such that \( \mu(x) \geq t \) and \( \mu(y) \geq t \) \( \Rightarrow \)

\[ x \lor y \in R \text{ such that, } \mu(x \lor y) \geq \min \{\mu(x), \mu(y)\} \geq t, \text{ by (1)} \]

\[ \Rightarrow x \lor y \in R \text{ such that, } \mu(x \lor y) \geq t \]
$\Rightarrow x \vee y \in \mu_t.$
And $\mu(x - y) \geq \min \{\mu(x), \mu(y)\} \geq t$, by (1)
$\Rightarrow x \vee y \in \mu_t.$
Thus, $x, y \in \mu_t \Rightarrow x \vee y, x - y \in \mu_t.$

**For (ii):**

Let $x \in \mu_t, t \in \text{Im } \mu$ and $r_1 \in R$ be arbitrary, such that $r_1 \leq x$
Then $x \in R$ such that $\mu(x) \geq t$
And as $r_1 \leq x$, we have, $\mu(r_1) \geq \mu(x)$, by Proposition: 3.7
$\Rightarrow \mu(r_1) \geq t$ as $\mu(x) \geq t$
$\Rightarrow r_1 \in \mu_t.$

**For (iii):**

Let $x \in \mu_t, r \in R$ be arbitrary.
Then $\mu(x) \geq t$. ------- $\Rightarrow$ (3)
Suppose $r \in \mu_t$. Then $\mu(r) \geq t$ ------- $\Rightarrow$ (4)
Now, $\mu(rx) \geq \max \{\mu(r), \mu(x)\} \geq t$, by (3) and (4)
$\Rightarrow rx \in \mu_t.$
Now, $\mu(xr) \geq \max \{\mu(x), \mu(r)\} \geq t$, by (3) and (4)
$\Rightarrow xr \in \mu_t.$
Suppose $r \notin \mu_t$. Then $\mu(r) < t$ ------- $\Rightarrow$ (5)
And, $\max \{\mu(x), \mu(r)\} \geq \mu(x)$, by (3) and (5)
$\geq t$, by (3)
$\Rightarrow \max \{\mu(x), \mu(r)\} \geq t$ ------- $\Rightarrow$ (6)
Now, $\mu(rx) \geq \max \{\mu(r), \mu(x)\} \geq t$, by (6)
$\Rightarrow rx \in \mu_t.$
Now, $\mu(xr) \geq \max \{\mu(x), \mu(r)\} \geq t$, by (6)
$\Rightarrow xr \in \mu_t.$
Thus $x \in \mu_t, r \in R \Rightarrow rx, xr \in \mu_t.$
Hence the level subsets $\mu_t, t \in \text{Im } \mu$ are $\ell$ – ideals of $R.$
Theorem 3.23: [Characterization Theorem]

A fuzzy subset $\mu$ of an $\ell$– ring $R$, is a fuzzy $\ell$– ideal of $R$ iff, the level subsets $\mu_t$, $t \in \text{Im } \mu$ are $\ell$– ideals of $R$.

Proof:

Proposition: 3.22 complete the proof of first part.

Conversely, assume that the level subsets $\mu_t$, $t \in \text{Im } \mu$ are $\ell$– ideals of $R$.

To prove $\mu$ is a fuzzy $\ell$– ideal of $R$.

It is enough to prove that whenever $x, y \in R$,

\begin{align*}
(v) & \quad \mu(x \lor y) \geq \min \{\mu(x), \mu(y)\} \\
(vi) & \quad \mu(x \land y) \geq \max \{\mu(x), \mu(y)\} \\
(vii) & \quad \mu(x - y) \geq \min \{\mu(x), \mu(y)\} \\
(viii) & \quad \mu(xy) \geq \max \{\mu(x), \mu(y)\}
\end{align*}

For (i) and (iii):

Let $\min \{\mu(x), \mu(y)\} = r$

$\Rightarrow$ Either $\mu(x) = r$ and $\mu(y) \geq \mu(x) = r$ Or $\mu(y) = r$ and $\mu(x) \geq \mu(y) = r$

$\Rightarrow \mu(x) \geq r$ and $\mu(y) \geq r$

$\Rightarrow x, y \in \mu_r$.

$\Rightarrow x \lor y, x - y \in \mu_r$, since $\mu_r$ is an $\ell$– ideal of $R$.

$\Rightarrow \mu(x \lor y) \geq r = \min \{\mu(x), \mu(y)\}$

And $\mu(x - y) \geq r = \min \{\mu(x), \mu(y)\}$.

Thus $\mu(x \lor y) \geq \min \{\mu(x), \mu(y)\}$ and $\mu(x - y) \geq \min \{\mu(x), \mu(y)\}$.

For (ii) and (iv):

Let $\max \{\mu(x), \mu(y)\} = s$ ---------> (6)

Case (i): Suppose $\max \{\mu(x), \mu(y)\} = \mu(x)$

$\Rightarrow \mu(x) = s$

$\Rightarrow x \in \mu_s$.

Here $x \in \mu_s$ and $y \in R$

$\Rightarrow xy \in \mu_s$, since $\mu_s$ is an $\ell$– ideal of $R$. 

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\[ \Rightarrow \mu(xy) \geq s = \max \{\mu(x), \mu(y)\}, \text{by (6)} \]
\[ \Rightarrow \mu(xy) \geq \max \{\mu(x), \mu(y)\} \]
Again \( x \in \mu_s \) and \( x \wedge y \leq x \)
\[ \Rightarrow x \wedge y \in \mu_s, \text{since } \mu_s \text{ is an } \ell - \text{ideal of } R. \]
\[ \Rightarrow \mu(x \wedge y) \geq s = \max \{\mu(x), \mu(y)\}, \text{by (6)} \]
\[ \Rightarrow \mu(x \wedge y) \geq \max \{\mu(x), \mu(y)\} \]

**Case (ii):** Suppose \( \max \{\mu(x), \mu(y)\} = \mu(y) \)
\[ \Rightarrow \mu(y) = s \]
\[ \Rightarrow y \in \mu_s. \]
Here \( y \in \mu_s \) and \( x \in R \)
\[ \Rightarrow xy \in \mu_s, \text{since } \mu_s \text{ is an } \ell - \text{ideal of } R. \]
\[ \Rightarrow \mu(xy) \geq s = \max \{\mu(x), \mu(y)\}, \text{by (6)} \]
\[ \Rightarrow \mu(xy) \geq \max \{\mu(x), \mu(y)\} \]
Again \( y \in \mu_s \) and \( x \wedge y \leq y \)
\[ \Rightarrow x \wedge y \in \mu_s, \text{since } \mu_s \text{ is an } \ell - \text{ideal of } R. \]
\[ \Rightarrow \mu(x \wedge y) \geq s = \max \{\mu(x), \mu(y)\}, \text{by (6)} \]
\[ \Rightarrow \mu(x \wedge y) \geq \max \{\mu(x), \mu(y)\} \]
Thus \( \mu \) is a fuzzy \( \ell - \text{ideal of } R. \)

Next, the level \( \ell - \text{ideal and the family of level } \ell - \text{ideals are defined here.} \)

**Definition 3.24:**

Let \( \mu \) be any fuzzy \( \ell - \text{ideal of } R; \ t \in [0, 1]; \) and \( t \leq \mu(0). \) Then the \( \ell - \text{ideal } \mu_t \text{ of } R \) is called a level \( \ell - \text{ideal of } \mu. \)

**Remark 3.25:**

Let \( \mu \) be any fuzzy \( \ell - \text{ideal of an } \ell - \text{ring } R. \) From the above Theorem, \( \mu_t, \ t \in \text{Im } \mu \) are level \( \ell - \text{ideals of } R. \) Then \( F_\mu = \{\mu_t / t \in \text{Im } \mu\} \) is called as the family of level \( \ell - \text{ideals of } \mu. \)
Definition 3.26:
Let $\mu$ be any fuzzy $\ell$–ideal of an $\ell$–ring $R$. Let $\mu_t$ and $\mu_s$ be any two level $\ell$–ideals of $\mu$. Now, we define $\cup$, $\cap$ and $\sim$ on $F^l_\mu$ by,
\[
\mu_t \cup \mu_s = \{ x \in R / \mu(x) \geq \min \{t, s\} \}
\]
\[
\mu_t \cap \mu_s = \{ x \in R / \mu(x) \geq \max \{t, s\} \}
\]
\[
\sim (\mu_t) = \{ x \in R / x \notin \mu_t \}
\]

Theorem 3.27:
Let $\mu$ be any fuzzy $\ell$–ideal of an $\ell$–ring $R$. Let $\mu_t$ and $\mu_s$ be any two level $\ell$–ideals of $\mu$. Then $\mu_t \cup \mu_s$ and $\mu_t \cap \mu_s$ are also a level $\ell$–ideal of $\mu$.

Proof:
Let $\mu$ be any fuzzy $\ell$–ideals of an $\ell$–ring $R$.
Let $\mu_t$ and $\mu_s$ be any two level $\ell$–ideals of $\mu$.

$\Rightarrow \mu_t$ and $\mu_s$ are $\ell$–ideals of $R$.

To prove $\mu_t \cup \mu_s$ and $\mu_t \cap \mu_s$ are level $\ell$–ideals of $\mu$.

That is to prove (a) $\mu_t \cup \mu_s$ and (b) $\mu_t \cap \mu_s$ are $\ell$–ideal of $R$.

For (a):
That is to prove

(i) $x, y \in \mu_t \cup \mu_s \Rightarrow x \lor y, x - y \in \mu_t \cup \mu_s$.

(ii) $x \in \mu_t \cup \mu_s, r_1 \in R$ such that $r_1 \leq x \Rightarrow r_1 \in \mu_t \cup \mu_s$.

(iii) $x \in \mu_t \cup \mu_s, r \in R \Rightarrow rx, xr \in \mu_t \cup \mu_s$.

For (i):
Let $x, y \in \mu_t \cup \mu_s$ be arbitrary.

$\Rightarrow x, y \in R$ such that $\mu(x) \geq \min \{t, s\}$ and $\mu(y) \geq \min \{t, s\} \quad (1)$

Case (1): $\min \{t, s\} = t$

$\Rightarrow x \in R$ such that $\mu(x) \geq t$ and $y \in R$ such that $\mu(y) \geq t$, by (1)

$\Rightarrow x, y \in \mu_t$.

$\Rightarrow x \lor y, x - y \in \mu_t$, since $\mu_t$ is an $\ell$–ideal of $R$.  

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\[ x \lor y \in R \text{ such that } \mu(x \lor y) \geq t \text{ and } x - y \in R \text{ such that } \mu(x - y) \geq t. \]
\[ x \lor y, x - y \in R \text{ such that } \mu(x \lor y) \geq \min\{t, s\} \text{ and } \mu(x - y) \geq \min\{t, s\}, \]
by the assumption in this case.

\[ x \lor y, x - y \in \mu_t \cup \mu_s. \]

**Case (2):** \( \min\{t, s\} = s \)
\[ x \in R \text{ such that } \mu(x) \geq s \text{ and } y \in R \text{ such that } \mu(y) \geq s, \text{ by (1)} \]
\[ x, y \in \mu_s. \]
\[ x \lor y, x - y \in \mu_s, \text{ since } \mu_s \text{ is an } \ell\text{-}ideal \text{ of } R. \]
\[ x \lor y, x - y \in R \text{ such that } \mu(x \lor y) \geq s \text{ and } \mu(x - y) \geq s. \]
\[ x \lor y, x - y \in R \text{ such that } \mu(x \lor y) \geq \min\{t, s\} \text{ and } \mu(x - y) \geq \min\{t, s\}, \]
by the assumption in this case.

\[ x \lor y, x - y \in \mu_t \cup \mu_s. \]
Thus, \( x, y \in \mu_t \cup \mu_s \Rightarrow x \lor y, x - y \in \mu_t \cup \mu_s. \)

**For (ii):**
Let \( x \in \mu_t \cup \mu_s, r_1 \in R \text{ such that } r_1 \leq x. \)
\[ x \in R \text{ such that } \mu(x) \geq \min\{t, s\} \]
Now \( \mu(r_1) \geq \mu(x), \text{ since } \mu \text{ is a fuzzy } \ell\text{-}ideal \text{ of } R. \)
\[ \geq \min\{t, s\} \]
Thus \( r_1 \in R \text{ such that } \mu(r_1) \geq \min\{t, s\} \)
\[ \Rightarrow r_1 \in \mu_t \cup \mu_s. \]

**For (iii):**
Let \( x \in \mu_t \cup \mu_s \text{ be arbitrary and } r \in R. \)
\[ \Rightarrow x \in R \text{ such that } \mu(x) \geq \min\{t, s\}. \]
And \( x \in R \text{ and } r \in R \Rightarrow rx \in R \text{ and } xr \in R. \)
Now \( \mu(rx) \geq \max\{\mu(r), \mu(x)\} \geq \mu(x) \geq \min\{t, s\} \)
Thus \( rx \in R \text{ such that } \mu(rx) \geq \min\{t, s\}. \)
Again \( \mu(xr) \geq \max\{\mu(x), \mu(r)\} \geq \mu(x) \geq \min\{t, s\} \)
Thus \( xr \in R \text{ such that } \mu(xr) \geq \min\{t, s\}. \)
Hence $\mu_t \cup \mu_s$ is an $\ell$– ideal of $R$.
Thus, $\mu_t \cup \mu_s$ is a level $\ell$– ideal of $\mu$.

For (b):
That is to prove

(i) $x, y \in \mu_t \cap \mu_s \Rightarrow x \lor y, x - y \in \mu_t \cap \mu_s.$
(ii) $x \in \mu_t \cap \mu_s, r_1 \in R$ such that $r_1 \leq x \Rightarrow r_1 \in \mu_t \cap \mu_s.$
(iii) $x \in \mu_t \cap \mu_s, r \in R \Rightarrow rx, xr \in \mu_t \cap \mu_s.$

For (i):
Let $x, y \in \mu_t \cap \mu_s$ be arbitrary.
$
\Rightarrow x \in R$ such that $\mu(x) \geq \max \{t, s\}$
and $y \in R$ such that $\mu(y) \geq \max \{t, s\}$  ------$\rightarrow$ (1)

Case (1): $\max \{t, s\} = t$
$
\Rightarrow x, y \in R$ such that $\mu(x) \geq t$ and $\mu(y) \geq t$, by (1)
$
\Rightarrow x, y \in \mu_t.$
$
\Rightarrow x \lor y, x - y \in \mu_t$, since $\mu_t$ is an $\ell$– ideal of $R$.
$
\Rightarrow x \lor y, x - y \in R$ such that $\mu(x \lor y) \geq t$ and $\mu(x - y) \geq t$.
$
\Rightarrow x \lor y, x - y \in R$ such that $\mu(x \lor y) \geq \max \{t, s\}$ and $\mu(x - y) \geq \max \{t, s\},$
by the assumption in this case.
$
\Rightarrow x \lor y, x - y \in \mu_t \cap \mu_s.$

Case (2): $\max \{t, s\} = s$
$
\Rightarrow x \in R$ such that $\mu(x) \geq s$ and $y \in R$ such that $\mu(y) \geq s$, by (1)
$
\Rightarrow x, y \in \mu_s.$
$
\Rightarrow x \lor y, x - y \in \mu_s$, since $\mu_s$ is an $\ell$– ideal of $R$.
$
\Rightarrow x \lor y, x - y \in R$ such that $\mu(x \lor y) \geq s$ and $\mu(x - y) \geq s$.
$
\Rightarrow x \lor y, x - y \in R$ such that $\mu(x \lor y) \geq \max \{t, s\}$ and $\mu(x - y) \geq \max \{t, s\},$
by the assumption in this case.
$
\Rightarrow x \lor y \in \mu_t \cap \mu_s, x - y \in \mu_t \cap \mu_s.$
Thus, $x, y \in \mu_t \cap \mu_s \Rightarrow x \lor y, x - y \in \mu_t \cap \mu_s.$
For (ii):
Let $x \in \mu_t \cap \mu_s$, $r_1 \in R$ such that $r_1 \leq x$.
$
\Rightarrow x \in R$ such that $\mu(x) \geq \max \{t, s\}$
Now $\mu(r_1) \geq \mu(x)$, since $\mu$ is a fuzzy $\ell$–ideal of $R$.
$\geq \max \{t, s\}$
Thus $r_1 \in R$ such that $\mu(r_1) \geq \max \{t, s\}$
$\Rightarrow r_1 \in \mu_t \cap \mu_s$.

For (iii):
Let $x \in \mu_t \cap \mu_s$ be arbitrary and $r \in R$.
$\Rightarrow x \in R$ such that $\mu(x) \geq \max \{t, s\}$.
And $x \in R$ and $r \in R \Rightarrow rx \in R$ and $xr \in R$.
Now $\mu(rx) \geq \max \{\mu(r), \mu(x)\} \geq \mu(x) \geq \max \{t, s\}$
Thus $rx \in R$ such that $\mu(rx) \geq \max \{t, s\}$.
Again $\mu(xr) \geq \max \{\mu(x), \mu(r)\} \geq \mu(x) \geq \max \{t, s\}$
Thus $xr \in R$ such that $\mu(xr) \geq \max \{t, s\}$.
Hence $\mu_t \cap \mu_s$ is an $\ell$–ideal of $R$.
Thus, $\mu_t \cap \mu_s$ is a level $\ell$–ideal of $\mu$.

**Theorem 3.28:**
Let $\mu$ be any fuzzy $\ell$–ideal of an $\ell$–ring $R$. Then $(F^i_\mu, \cup)$ and $(F^i_\mu, \cap)$ are commutative monoids.

**Proof:**
Let $\mu$ be any fuzzy $\ell$–ideal of an $\ell$–ring $R$.
Let $\text{Im } \mu = \{a, b, c, ..., z\}$ with $a > b > ... > r > s > t > ... > y > z$.
Here the least value of $\text{Im } \mu$ is ‘$z$’ and greatest value of $\text{Im } \mu$ is ‘$a$’.
Then, by previous Theorem, we get, $\mu_a, \mu_b, ..., \mu_r, \mu_s, \mu_t, ..., \mu_y, \mu_z \in F^i_\mu$
such that $\mu_a \subseteq \mu_b \subseteq ... \subseteq \mu_r \subseteq \mu_s \subseteq \mu_t \subseteq ... \subseteq \mu_y \subseteq \mu_z$.

**1) Closure Property:**
Consider any two level $\ell$–ideals: $\mu_r, \mu_s \in F^i_\mu$.
(i) $\mu_r \cup \mu_s \in F^i_\mu$ by previous Theorem.
(ii) $\mu_r \cap \mu_s \in F^i_\mu$ by previous Theorem.
(2) **Associative Property:**
Consider any three level \( \ell \)– ideals: \( \mu_r, \mu_s, \mu_t \in F^i_{\mu} \).

(i) \[(\mu_r \cup \mu_s) \cup \mu_t = \mu_r \cup \mu_r = \mu_r \cup \mu_s = \mu_r \cup (\mu_s \cup \mu_t)\]

(ii) \[(\mu_r \cap \mu_s) \cap \mu_t = \mu_s \cap \mu_t = \mu_t \cap \mu_t = \mu_t \cap (\mu_s \cap \mu_t)\]

(3) **Identity Property:**
Consider any level \( \ell \)– ideal: \( \mu_r \in F^i_{\mu} \).

(i) \[\mu_r \cup \mu_z = \mu_r.\]

Hence \( \mu_z \) is the identity with respect to ‘\( \cup \)’.

(ii) \[\mu_r \cap \mu_a = \mu_r.\]

Hence \( \mu_a \) is the identity with respect to ‘\( \cap \)’.

(4) **Commutative Property:**

(i) \[(\mu_r \cup \mu_s) = \mu_r = \mu_s \cup \mu_r.\]

(ii) \[(\mu_r \cap \mu_s) = \mu_s = \mu_s \cap \mu_r.\]

Thus, \((F^i_{\mu}, \cup)\) and \((F^i_{\mu}, \cap)\) are commutative monoids.

**Theorem 3.29:**

Two level \( \ell \)– ideals \( \mu_s \) and \( \mu_t \) (with \( s < t \)) of a fuzzy \( \ell \)– ideal \( \mu \) of \( R \) are equal iff, there is no \( x \) in \( R \) such that \( s \leq \mu(x) < t \).

**Proof:**

Let \( \mu_s \) and \( \mu_t \) (with \( s < t \)) be two level \( \ell \)– ideals of a fuzzy \( \ell \)– ideals \( \mu \) of \( R \).

Assume that \( \mu_s \) and \( \mu_t \) are equal.

To prove there is no \( x \) in \( R \) such that \( s \leq \mu(x) < t \).

On the contrary, assume that \( s \leq \mu(x) < t \) for some \( x \) in \( R \).

\( \Rightarrow \mu(x) \geq s \) and \( \mu(x) < t \)

\( \Rightarrow x \in \mu_s \) and \( x \notin \mu_t \) and hence \( \mu_s \neq \mu_t \)

This is a contradiction to our assumption.

Hence there is no \( x \) in \( R \) such that \( s \leq \mu(x) < t \).
Conversely, assume that no \( x \) in \( R \) such that \( s \leq \mu(x) < t \). ------\( (1) \)

\[ \mu_s = \{ x \in R / \mu(x) \geq s \} \text{ and } \mu_t = \{ x \in R / \mu(x) \geq t \} \text{ and } s < t. \]

Then clearly \( \mu_t \subseteq \mu_s \). ------\( (2) \)

It is enough to show that \( \mu_s \subseteq \mu_t \).

Let \( x \in \mu_t \) be arbitrary.
Then \( \mu(x) \geq t \).
\[ \Rightarrow \mu(x) > t, \text{ by } (1) \]
\[ \Rightarrow x \in \mu_t \]
\[ \Rightarrow \mu_s \subseteq \mu_t \] ------\( (3) \)

From \( (2) \) and \( (3) \), we have \( \mu_s = \mu_t \).

Hence two level \( \ell \) – ideals are equal.

**Theorem 3.30:**

Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \). If \( \text{Im } \mu = \{ t_0, t_1, t_2, \ldots, t_n \} \) with \( t_0 > t_1 > t_2 > \ldots > t_n \), then we have the following chain of level \( \ell \) – ideal of \( \mu \): \( \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} \subseteq \ldots \subseteq \mu_{t_n} = R. \)

**Proof:**

Let \( \mu \) be any fuzzy \( \ell \) – ideal of an \( \ell \) – ring \( R \).

Let \( \text{Im } \mu = \{ t_0, t_1, t_2, \ldots, t_n \} \) with \( t_0 > t_1 > t_2 > \ldots > t_n \). \( \cdots \) ------\( (1) \)

Let \( x_1 \in \mu_{t_0} \) be arbitrary.
\[ \Rightarrow x_1 \in R \text{ such that } \mu(x_1) \geq t_0. \]
\[ \Rightarrow x_1 \in R \text{ such that } \mu(x_1) \geq t_1, \text{ by } (1) \]
\[ \Rightarrow x_1 \in \mu_{t_1}. \]

Thus, we have \( \mu_{t_0} \subseteq \mu_{t_1} \).

Again, let \( x_2 \in \mu_{t_1} \) be arbitrary.
\[ \Rightarrow x_2 \in R \text{ such that } \mu(x_2) \geq t_1. \]
\[ \Rightarrow x_2 \in R \text{ such that } \mu(x_2) \geq t_2, \text{ by } (1) \]
\[ \Rightarrow x_2 \in \mu_{t_2}. \]

Thus, we have \( \mu_{t_1} \subseteq \mu_{t_2}. \)
Hence we get $\mu_t \subseteq \mu_s \subseteq \mu_t$.

Proceeding like this, we get $\mu_t \subseteq \mu_t \subseteq \mu_t \subseteq ... \subseteq \mu_t = R$.

**Remark 3.31:**

Let $\mu$ be any fuzzy $l$ – ideal of an $l$ – ring $R$. Let $t, s \in \text{Im } \mu$ with $s \geq t$. Then, by previous Theorem, we have, $\mu_t$ and $\mu_s$ are two level $l$ – ideals of $\mu$ such that $\mu_s \subseteq \mu_t$. Here, $\min \{s, t\} = t$ and $\max \{s, t\} = s$.

Now, $\mu_s \cup \mu_t = \{x \in R / \mu(x) \geq \min \{s, t\}\} = \{x \in R / \mu(x) \geq t\} = \mu_t$.

And $\mu_s \cap \mu_t = \{x \in R / \mu(x) \geq \max \{s, t\}\} = \{x \in R / \mu(x) \geq s\} = \mu_s$.

**Theorem 3.32:**

Let $A_0 \subset A_1 \subset ... \subset A_n = R$ be any finite chain of $l$ – ideals of an $l$ – ring $R$. Then there exists a fuzzy $l$ – ideal $\mu$ of $R$ whose chain of level $l$ – ideals is precisely $A_0 \subset A_1 \subset ... \subset A_n = R$.

**Proof:**

Choose numbers $t_i \in [0, 1]$, $0 \leq i \leq n$, such that $t_0 > t_1 > ... > t_n$.

Let $\hat{A}_0 = A_0$ and $\hat{A}_i = A_i - A_{i-1}$, $1 \leq i \leq n$.

Consider the fuzzy $l$ – subset $\mu$ of $R$ defined by $\mu(x) = t_i$, whenever $x \in \hat{A}_i$, $0 \leq i \leq n$.

Clearly, $\text{Im } \mu = \{t_0, t_1, ..., t_n\}$.

Now to prove: $\mu_{t_i} = A_i$.

Let $x \in R$

**Case (i):** $i = 0$

Let $x \in \mu_{t_i}$

Then $x \in \mu_{t_0}$

$\iff \mu(x) = t_0$

$\iff x \in \hat{A}_0 = A_0$.

That is $x \in A_i$. 
**Case (ii):** $i \in \{1, 2, \ldots, n\}$

Then $x \in \mu_i$,

$\iff \mu(x) \geq t_i$

If $\mu(x) = t_i$, then $x \in \hat{A}_i$

$\Rightarrow x \in A_i$

If $\mu(x) > t_i$, then $\mu(x) = t_j$ for some $j < i$.

Hence, $x \in \hat{A}_j \subseteq A_i$

Thus $x \in A_i$

Therefore, $\mu_i \subseteq A_i$ ------ $\rightarrow$ (1)

Again, let $x \in A_i$

Then $x \in \hat{A}_k$ for some $k \leq i$

$\Rightarrow \mu(x) = t_k \geq t_i$

$\Rightarrow x \in \mu_i$

Therefore, $A_i \subseteq \mu_i$ ------ $\rightarrow$ (2)

From (1) and (2), we have, $\mu_i = A_i$.

As $A_i$ is an $\ell$ – ideal of $R$, $\mu_i$ is an $\ell$ – ideal of $R$.

The chain of level $\ell$ – ideals $A_0 \subset A_1 \subset \ldots \subset A_n = R$ implies the chain of level $\ell$ – ideals of $\mu$ as $\mu_{i_0} \subset \mu_{i_1} \subset \ldots \subset \mu_{i_n} = R$

Clearly $t_i \in \text{Im} \mu$, for $i = 1$ to $n$.

Then by Theorem 3.23, $\mu$ is a fuzzy $\ell$ – ideal of $R$.

Thus there exists a fuzzy $\ell$ – ideal $\mu$ of $R$ whose chain of level $\ell$ – ideals is precisely $A_0 \subset A_1 \subset \ldots \subset A_n = R$.

**Theorem 3.33:**

Let $\mu$ be any fuzzy $\ell$ – ideal of an $\ell$ – ring $R$. Then $(F_{\mu}^i, \lor, \land)$ is a completely distributive lattice and $(F_{\mu}^i, \lor, \land)$ is a sublattice of $(F_{\mu}, \lor, \land)$.

**Proof:**

Define $\lor$ and $\land$ on $F_{\mu}^i$ by $\mu_t \lor \mu_s = \mu_t \cup \mu_s$ and $\mu_t \land \mu_s = \mu_t \cap \mu_s$, where $\mu_t$ and $\mu_s \in F_{\mu}$. Then it is easy to verify that $(F_{\mu}^i, \lor, \land)$ is a completely distributive lattice.

Theorem 2.27 and Remark 3.4 complete the rest.