Chapter IV

ON THE ABSOLUTE NÖRLUND SUMMABILITY OF A SERIES

ASSOCIATED WITH A FOURIER SERIES II

4.1. Definitions and Notations: In the present chapter, we follow the definitions and notations, as given in the Chapter III. There are few exceptions, which are incorporated below.

We denote the $n$-th partial sum of Fourier series by $a_n$, and we write throughout

$$\phi^*(t) = \frac{1}{2} \left( f(x+t) + f(x-t) - 2s \right),$$

$$a_n = \frac{2}{\pi} \int_0^\infty \frac{\phi^*(t)}{t} \sin nt \, dt,$$

$$b_n = \frac{2}{\pi} \int_0^\infty \frac{\phi^*(t)}{t} \left( \frac{1}{2 \tan \frac{t}{2}} - \frac{1}{t} \right) \sin nt \, dt,$$

$$v_n = \frac{1}{\pi} \int_0^\infty \phi^*(t) \cos nt \, dt,$$

$$c_n(t) = \int_0^t \frac{\log(t)}{t} \sin nt \, dt,$$
\[ s(n, t) = \frac{1}{p_{n+1}} \sum_{k=0}^{n-1} \frac{p_{n-k}}{\log(n+2)} \eta(t). \]

\[ \mathcal{C} = \left[ \frac{k}{n} \right], \quad \mathcal{C}' = \left[ \frac{k}{3} \right], \quad t \leq \frac{n}{3}. \]

\[ n = \left\lfloor \frac{n}{2} \right\rfloor. \]

\[ R_n = \left\{ \frac{(n+1)p_n}{p_n} \right\} \]

\[ v_n = P_n \sum_{\nu=0}^{n} \frac{1}{(n+1)p_n}. \]

\[ s_n = \frac{1}{p_n} \sum_{\nu=0}^{n} \frac{p_n}{n+1}. \]

By "\( q_n \in \mathbb{B} \)" we mean that \( \{ q_n \} \) is a bounded sequence.

4.2. In the year 1971, Nayak [38] has proved the following theorem.

**Theorem A.** If

(1) \( \phi^*(t) \in SV(0, a), \)

(2) \( \frac{\phi^*(t)}{t \log \left( \frac{t}{b} \right)} \in L(0, a), \)
then the series

\[(4.2.1) \quad \sum_{n=2}^{\infty} \frac{a_n - c}{n \log n}\]

is summable \(|C, a|\) (\(a > -1\)).

The object of the present chapter is to obtain a corresponding result for \(|N, p_n|\) summability, so as to include the Theorem A, as a special case, when \(p_n = A_{n-1}^{-1}\), \((a > -1)\).

4.3. We establish the following theorem.

**Theorem.** Let \(\{p_n\}\) be a non-negative, non-increasing sequence of numbers such that

1. \(\{p_n\} \in BV\),
2. \(\{v_n\} \in B\).

If \(\phi(t) \in BV(0, \infty)\) and \(\frac{\phi(t)}{t \log(\frac{t}{\epsilon})} \in L(0, \infty)\), then the series (4.2.1) is summable \(|N, p_n|\).

4.4. We shall require the following lemmas for the proof of our theorem.
Lemma 1. If \( \{ R_n \} \subset B \), then, for any positive sequence \( \{ v_n \} \subset B \) is equivalent to \( \{ w_n \} \subset B \), \( (n=0,1,2,\ldots) \).

The proof of this lemma has been discussed by Mikhit (18), pp.167-168), see also Singh [49] and Pati [41].

Lemma 2. ([52], p.440). Uniformly in \( 0 < t \leq n \),

\[
\frac{n \sin vt}{v^m} \leq C,
\]

where \( n \) and \( m \) are any positive integers, such that \( n \geq m \).

C is an absolute positive constant, not necessarily same at each occurrence.

Lemma 3. ([29], Lemma 5.11, p.182). If \( q_n \) is non-negative and non-increasing, then, for \( 0 \leq n \leq b \leq \alpha \leq t \leq x \), and any \( n \),

\[
\left| \sum_{k=a}^{b} \frac{1(n-k)t}{k} q_k \right| \leq C q_{\alpha},
\]

where \( C = [\frac{1}{n}] \) and \( q_n = q_0 + q_1 + \ldots + q_n \).

Lemma 4. If
(1) \( \phi^*(t) \in BV(a, b) \).

(ii) \[ \frac{\phi^*(t)}{t \log \left( \frac{K}{t} \right)} \in L(a, b) \],

then we have

\[ \int_a^b \log \left( \frac{K}{t} \right) \, d \left\{ \frac{\phi^*(t)}{\log \left( \frac{K}{t} \right)} \right\} < \infty. \]

and

\[ \lim_{t \to 0} \frac{\phi^*(t)}{\log \left( \frac{K}{t} \right)} = 0. \]

**Proof.** By using the formula (see [44]),

\[ \int_a^b |f(x)| \, d|g(x)| \leq \int_a^b |f(x)g(x)| + \int_a^b |g(x)||df(x)|, \]

we have

\[ \int_a^b \log \left( \frac{K}{t} \right) \, d \left\{ \frac{\phi^*(t)}{\log \left( \frac{K}{t} \right)} \right\} \leq \int_a^b |\phi^*(t)| \, d|\log \left( \frac{K}{t} \right)| \]

\[ = o(1) + \int_a^b \left| \frac{\phi^*(t)}{\log \left( \frac{K}{t} \right)} \right| \, dt \]

\[ = o(1) + \int_a^b \frac{\phi^*(t)}{t \log \left( \frac{K}{t} \right)} \, dt \]

\[ = o(1). \]
by hypothesis of the lemma.

Since \( \phi'(t) \in BV \), \( \lim_{t \to 0} \phi'(t) \) is finite, hence

\[
\frac{\phi'(t)}{\log\left(\frac{x}{t}\right)} = o\left(\frac{1}{\log\left(\frac{x}{t}\right)}\right)
\]

and we have

\[
\lim_{t \to 0} \frac{\phi'(t)}{\log\left(\frac{x}{t}\right)} = 0.
\]

**Lemma 5.** For \( 0 \leq t \leq x \), \( t \leq \xi \leq x \),

\[
\int_t^x \frac{1}{\log\left(\frac{x}{u}\right)} \sin(n-k)u \, du = \left\{ \log\left(\frac{x}{t}\right) \right\}^2 \sin(n-k)t - \frac{1}{2} \left\{ \log\left(\frac{x}{t}\right) \right\}^2 \sin(n-k)\xi.
\]

We obtain the result by applying integration by parts and the second mean-value theorem for integral calculus.

**Lemma 6.** Let \( \{p_n\} \) satisfies the same conditions as given in the theorem. If \( \phi'(t) \in BV(0,x) \), then the series \( \sum_{n=1}^\infty p_n \) is summable \( |N(p_n)| \).
The proof is contained in [34].

**Lemma 7.** Under the same hypothesis as in Lemma 6, the series \( \sum p_n \) is summable \( |N, p_n| \).

It can be easily proved by following the analysis of Pati [40] together with Lemma 1.

**Lemma 8.** Let all the conditions of the theorem be satisfied. Then a necessary and sufficient condition for the series (4.2.1) to be summable \( |N, p_n| \) is that

\[
(4.4.1) \quad \sum_{n=1}^{\infty} \frac{1}{n+1} \left| \dot{S}_n(x) \right| < \infty,
\]

where

\[
\dot{S}_n(x) = \frac{1}{p_n} \sum_{k=1}^{n} \frac{p_{n-k}}{\log(n-k+2)} c_k.
\]

**Proof.** Let

\[
S_n = \frac{1}{p_n} \sum_{k=0}^{n} \frac{p_{n-k}}{(k+1)\log(k+2)} c_k,
\]

so that
\[
\begin{align*}
\frac{a_n}{a_{n-1}} &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{P_{n-k}a_k}{P_n} - \frac{1}{n} \sum_{k=0}^{n-1} \frac{P_{n-1-k}a_k}{P_n} \\
\frac{a_n}{a_{n-1}} &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{P_{n-k}P_{n-1-k}}{P_n P_{n-1}} a_k \\
\frac{a_n}{a_{n-1}} &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{(n+1)P_{n-k}a_k}{(n+1)P_n P_{n-1}} \\
\frac{a_n}{a_{n-1}} &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{P_{n-k}}{\log(k+2)} + \frac{1}{n} \sum_{k=0}^{n-1} \frac{(R_{n-k} - R_n)}{\log(k+2)} a_k \\
\frac{a_n}{a_{n-1}} &= \frac{1}{n+1} \sum_{k=0}^{n} a_k + \frac{1}{n+1} \sum_{k=0}^{n} \frac{(R_{n-k} - R_n)}{\log(k+2)} P_{n-k} a_k
\end{align*}
\]

Therefore, in order to prove the lemma, it is sufficient to show that

\[4.4.2 \quad \varepsilon = \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \frac{1}{k+1} \sum_{k=0}^{n} \frac{(R_{n-k} - R_n)}{P_k} \frac{a_{n-k}}{a_n} \quad \quad \varepsilon \rightarrow \frac{1}{(n-k) \log(n-k+1)}\]
Now, since

\[ a_n = \frac{2}{\pi} \int_0^\pi \frac{\phi^n(t)}{t} \sin nt \, dt \]

\[ = \frac{2}{\pi} \int_0^\pi \frac{\phi^n(t)}{\log \left( \frac{\pi}{2} \right)} \cdot \frac{\log \left( \frac{\pi}{2} \right)}{t} \sin nt \, dt \]

\[ = \frac{2}{\pi} \left[ \int_0^\pi \frac{\phi^n(t)}{\log \left( \frac{\pi}{2} \right)} \cdot \frac{\log \left( \frac{\pi}{2} \right)}{t} \sin nt \, dt \right]_0^\pi + \]

\[ + \frac{2}{\pi} \int_0^\pi d\left( \frac{\phi^n(t)}{\log \left( \frac{\pi}{2} \right)} \right) \int_0^\pi \frac{\log \left( \frac{\pi}{2} \right)}{t} \sin nt \, dt, \]

\[ = \int_0^\pi d\left( \frac{\phi^n(t)}{\log \left( \frac{\pi}{2} \right)} \right) g_n(t) \, dt, \]

and

\[ \frac{2}{\pi} \int_0^\pi \left| d\left( \frac{\phi^n(t)}{\log \left( \frac{\pi}{2} \right)} \right) \right| = \frac{|h(n,t)|}{n+1} \frac{1}{(n+1)p_{n+1}} \, dt, \]

where

\[ h(n,t) = \frac{1}{n+1} \sum_{k=1}^{n} \frac{(R_{k,n})p_k}{(n-k)p_{n+1}} g_{n-k}(t). \]

By virtue of Lemma 4, in order to prove (4.4.2), it is sufficient to show that, for \( 0 \leq t \leq \pi \),
\[ x' = \frac{1}{n+1} \sum_{m=1}^{n-1} h(m) = O(\log \frac{n}{t}) \]

We see that

\[ x' \leq \frac{1}{n+1} \sum_{m=1}^{n-1} \sum_{k=0}^{n-k-1} (R_k - R_n) P_k \]

\[ \sum_{m=1}^{n+1} (n+1)p_n \log(n-k+1) \]

\[ \leq \frac{1}{n+1} \sum_{m=1}^{n-1} \sum_{k=0}^{n-k-1} (P_{k-1/n} - P_k) \]

\[ \sum_{m=1}^{n+1} (n+1)p_n \log(n-k+1) \]

\[ = \epsilon_1 + \epsilon_2 + \epsilon_3 \text{ say.} \]

Now, by hypotheses and Lemma 5, we have

\[ \epsilon_1 \leq \frac{(\log \frac{K}{t})^2}{2} \frac{1}{n+1} \sum_{m=1}^{n-1} \sum_{k=0}^{n-k-1} \frac{P_k (R_k - R_n) \sin(n-k)t}{(n-k)\log(n-k+1)} \]

\[ + \frac{1}{2} (\log \frac{K}{t})^2 \frac{1}{n+1} \sum_{m=1}^{n-1} \sum_{k=0}^{n-k-1} \frac{P_k (R_k - R_n) \sin(n-k)t}{(n-k)\log(n-k+1)} \]

\[ = \epsilon_{11} + \epsilon_{12} \text{ say,} \]

Where
\[ z_{11}^l \leq c(\log(\frac{k}{t}))^2 + \frac{\tau}{t} \sum_{n=1}^{\infty} \frac{1}{(n+1)p_{n-1}} \frac{m-1}{b_\tau \log(n-k+1)} \]
\[ \leq c(\log(\frac{k}{t}))^2 + \tau \sum_{n=1}^{\infty} (1) \]
\[ = c(\log(\frac{k}{t}))^2 \cdot \tau \]
\[ = o(\log(\frac{k}{t})). \]

Similarly, we can have \[ z_{12} = o(\log(\frac{k}{t})). \]

Thus

\[ z_{11} = r(\log(\frac{k}{t})). \]

Applying Lemma 5, we have

\[ z_{21} = (\log(\frac{k}{t}))^2 \sum_{n=1}^{\infty} \frac{1}{(n+1)p_{n-1}} \frac{(R_k-R_n)p_k \sin(n-k)\tau}{b_\tau \log(n-k+1)} + \]
\[ + \frac{(\log(\frac{k}{t}))^2}{2} \sum_{n=1}^{\infty} \frac{1}{(n+1)p_{n-1}} \frac{m-1}{b_\tau \log(n-k+1)} \]
\[ = \sum_{21} + \sum_{22}, \]
where

\[
\begin{align*}
    \mathcal{E}_1' = & \left( \log \left( \frac{E'}{E} \right) \right)^2 \sum_{n=1}^{m} \frac{1}{n^2+1} \left( \frac{R_k-R_n}{L_{n-1}} \right)^2 P_k \sin(n-k) t \\
    \mathcal{E}_2' = & \left( \log \left( \frac{E'}{E} \right) \right)^2 \sum_{n=1}^{m} \frac{1}{n^2+1} \left( \frac{R_k-R_n}{L_{n-1}} \right)^2 \frac{P_k \sin(n-v) t}{\log(n-k+1)}
\end{align*}
\]

Thus, in order to show that

\[ \mathcal{E}_{21} = o \left( \log \left( \frac{E'}{E} \right) \right) \]
It is enough to establish that

$$\sum_{21}^{'} = o(\log \left( \frac{k}{t} \right)), \ r = 1, 2, 3. $$

we see that

$$\sum_{21}^{'} = (\log(\frac{k}{t})) \sum_{t = 1}^{1} \frac{1}{n} \frac{1}{n(n+1)} \sum_{k=0}^{m-2} \frac{|R_k - R_{n-k+1}|}{(n-k)\log(n-k+1)}$$

$$\leq (\log \left( \frac{k}{t} \right)) \sum_{t = 1}^{1} \frac{1}{n} \frac{1}{n(n+1)} \sum_{k=0}^{m-2} \frac{|R_{n-k+1} - R_n|}{(n-k-1)^2\log(n-k+1)^2}$$

$$\leq C (\log \left( \frac{k}{t} \right)) \sum_{t = 1}^{1} \frac{1}{n} \frac{1}{n(n+1)}$$

$$= o (\log \left( \frac{k}{t} \right));$$

$$\sum_{21}^{'} = (\log(\frac{k}{t})) \sum_{t = 1}^{1} \frac{1}{n} \frac{1}{n(n+1)} \sum_{k=0}^{m-2} \frac{|R_k - R_{n-k+1}|}{(n-k)\log(n-k+1)}$$

$$\leq C (\log \left( \frac{k}{t} \right)) \sum_{t = 1}^{1} \frac{1}{n} \frac{1}{n(n+1)}$$

$$= o (\log \left( \frac{k}{t} \right));$$

and
\[
\varepsilon_2' = \left( \log \frac{\xi}{\eta} \right)^2 \sum_{\eta = \xi' + 1}^{\xi} \frac{1}{(\eta - 1)P_{\eta-1}} \frac{|R_{\eta-1} - R_{\eta}|}{(n-\eta)\log(n-\eta)}
\]

\[
\leq \left( \log \frac{\xi}{\eta} \right) \sum_{\eta = \xi' + 1}^{\xi} \frac{1}{\eta(n+1)} \frac{1}{n(n+1)}
\]

\[
= o(\log \frac{\xi}{\eta}).
\]

Similarly, we can have

\[
\varepsilon_{22}' = o(\log \frac{\xi}{\eta}).
\]

Thus,

\[
\varepsilon_2' = o(\log \frac{\xi}{\eta}).
\]

\[
\varepsilon_2'(\log \frac{\xi}{\eta})^2 \sum_{\eta = \xi' + 1}^{\xi} \frac{1}{(\eta - 1)P_{\eta-1}} \frac{|A R_k|}{(n+1)P_{n-1}} \frac{k}{(n-\eta)\log(n-\eta)} R_v \sin(n-v)t
\]

\[
+ (\log \frac{\xi}{\eta})^2 \sum_{\eta = \xi' + 1}^{\xi} \frac{1}{(\eta - 1)P_{\eta-1}} \frac{|A R_k|}{(n+1)P_{n-1}} \frac{k}{(n-\eta)\log(n-\eta)} R_v \sin(n-v)t
\]

\[
= \varepsilon_{21}' + \varepsilon_{32}' \text{ say.}
\]

Since, by Abel's transformation
\[ \sum_{k} \frac{P_{k}}{\log(n-k)} \log(n-k) \]

\[ = \frac{\sum_{k=1}^{n} P_{k}}{(n+1) \log(n+1)} \frac{\sum_{k=1}^{n-1} \Delta R_k}{\log(n-k)} \]

\[ \log(n-k) \]

we have

\[ \sum_{k=1}^{n-1} \Delta R_k \]

\[ = 0 \left\{ \frac{1}{(n+1) \log(n+1)} \right\} \]

\[ = 0 \left\{ \frac{1}{(n+1) \log(n+1)} \sum_{k=1}^{n-1} \Delta R_k \right\} \]
\[ L_{32} = o(\log \frac{K}{\varepsilon}). \]

Similarly, we can have

\[ L_{32} = o(\log \frac{K}{\varepsilon}). \]

This completes the proof of the lemma.

4.5. Proof of the theorem. We have

\[ n = 2 \int_{x}^{x} \sin(n \cdot \frac{1}{2}) \frac{2 \sin \frac{1}{2}}{\int_{0}^{2 \tan \frac{1}{2}} \sin nt dt + \frac{1}{2} \int_{0}^{2 \tan \frac{1}{2}} \cos nt dt} \]

\[ = 2 \int_{0}^{x} \phi'(t) \sin nt dt + \frac{1}{2} \int_{0}^{2 \tan \frac{1}{2}} \phi'(t) \cos nt dt \]

\[ = 2 \int_{0}^{x} \phi''(t) \sin nt dt + \frac{2}{x} \int_{0}^{x} \phi'(t) \left\{ \frac{1}{x} \frac{1}{2 \tan \frac{1}{2}} - \frac{1}{x} \right\} \sin nt dt + \]

\[ + \int_{0}^{x} \phi''(t) \cos nt dt \]

\[ = a_n + \beta_n + \gamma_n. \]
If \( \phi^\circ(t) \) satisfies the conditions of the theorem, then so does \( \phi^\circ(t) \left\{ \frac{1}{2 \tan \frac{t}{2}} - \frac{1}{t} \right\} \).

Since, by Lemma 6 and 7 respectively, the series \( \sum \frac{\phi_n}{n \log n} \) and \( \sum \frac{\phi_n}{n \log n} \) are summable \( |\pi, p_n| \), whenever \( \phi^\circ(t) \in BV(0, \pi) \), we proceed to prove that, under the hypotheses of the theorem: \( \frac{\pi}{n \log n} \) is summable \( |\pi, p_n| \).

By virtue of Lemma 8, it is enough to show that

\[
\left| \sigma_n(x) \right| < + \infty
\]

(4.5.1) \( n=1 \)

Proof of (4.5.1).

\[
\sigma_n(x) = \int_0^x \frac{\phi^\circ(t)}{t} g(n, t) \, dt
\]

\[
= \left[ - \frac{\phi^\circ(t)}{\log(t)} \int_t^x \frac{\log(t)}{t} g(n, t) \, dt \right]_0^x
\]

\[
+ \int_0^x d \left\{ \frac{\phi^\circ(t)}{\log(t)} \right\} \int_t^x \frac{1}{\log(\frac{t}{u})} g(n, u) \, du
\]

\[
= \int_0^x \left\{ \frac{\phi^\circ(t)}{\log(t)} \right\} \frac{c}{p_{n-1}} \sum_{k=0}^{2^n - 1} \frac{p_{2n-k}}{\log(\log(k+2))} \, g_n(t)
\]
Thus, in view of Lemma 4, it is sufficient to prove that

\[(4.5.2) \quad \varepsilon'' = \varepsilon - \sum_{n=1}^{\infty} \left( \frac{1}{(n+1)p_{n-1}} \right) \varepsilon_{n-1}^{(n)} \leq c \left( \log \frac{L}{r} \right). \]

We have

\[\varepsilon'' = (\log \frac{L}{r})^2 \sum_{n=1}^{\infty} \left( \frac{1}{(n+1)p_{n-1}} \right) \left[ \sum_{k=1}^{n} \frac{p_k}{(n-k+1)p_{n-k+1}} \sin(n-k)t \right] \]

\[+ \sum_{k=1}^{n} \frac{p_k}{(n-k+1)p_{n-k+1}} \sin(n-k)t \]

\[= \varepsilon_1'' + \varepsilon_2'', \text{ say.} \]

Now

\[L_1'' = (\log \frac{L}{r})^2 \sum_{n=1}^{\infty} \left( \frac{1}{(n+1)p_{n-1}} \right) \left[ \sum_{k=1}^{n} \frac{p_k}{(n-k+1)p_{n-k+1}} \sin(n-k)t \right] \]

\[= \varepsilon_1'' \cdot \varepsilon_2'', \text{ say,} \]