CHAPTER III

SEPERABILITY OF AN S-OPERATOR

3.1 Introduction:

In this chapter we deal with the separability of the range of an $S$-operator. It is well known that the range, $R(T)$, of a completely continuous linear $T$ on a normed space $X$ to a normed space $Y$ is separable (Theorem 0.23). In 1963, Goldberg and Thorp in (Goldberg and Thorp [63] pp. 335-336) showed that the range of a strictly singular operator need not be separable by an example which is given below.

If $Q$ is an arbitrary set, $p(Q), 1 < p < \infty$ is the space of scalar valued functions $x$ with domain $Q$, having almost countably non-zero coordinates such that

$$||x|| = \left( \sum_{q \in Q} |x(q)|^p \right)^{\frac{1}{p}}$$

is finite.

It is a Banach space with this norm. It is asserted that all the continuous operators $T : \ell_2(Q) \rightarrow \ell_p(Q)$ ($2 < p < \infty$) where $Q$ is an uncountable set, are strictly singular and that the inclusion map is such an operator and has a non-separable range. From this it is obvious that the range of an $S$-operator on a normable $F$-space $X$ to a normable $F$-space $Y$, is not separable because an $S$-operator becomes strictly singular operator on Banach spaces (normable $F$-spaces). We prove in the next section that the range of an $S$-operator on a non-normable $F$-space $X$ to a non-normable $F$-space $Y$ is separable.
3.2 Separability of an S-operator:

We study the separability of an S-operator on a non-normable F-space X with the topology induced by the total paranorm p on X as in [Def. 0.76] to a non-normable F-space Y with the topology induced by the total paranorm q on Y as in [Def. 0.76].

Now we state and prove the following theorem:

Theorem 3.1: The range of an S-operator on a non-normable F-space X with the topology induced by the total paranorm p on X to a non-normable F-space Y with the topology induced by the total paranorm q on Y, is separable.

Proof: Let T be an S-operator on X to Y implying the restriction of T to a closed subspace U of X is a homeomorphism, U is a Montel space. Now T(U) is the image of U under T is the homeomorphic image of the Montel space U implying T(U) is a Montel space. Since U is a closed subspace of X, T(U) being the homeomorphic image of a closed subspace U of X, is a closed subspace of Y. Let V be a closed and bounded set (sub-space) of T(U). From Theorem 0.11 V is bounded in T(U) or in q restriction to T(U). Since T(U) is Montel space and by lemma 1.3, a bounded and closed set in T(U) or in q restriction to T(U) is compact if and only if it is compact in each q(m=1,2,3,..) restriction to T(U), V is compact in T(U). We know that
every compact metric space is complete and separable (Kelley [63] p. 32) therefore $V$ is separable. From Remark 2.1, $T(U)$ being a closed subspace of $Y$, is of finite codimension. It is easy to see as in the proof of Theorem 2.2, $V$ is a closed convex body in $T(U)$. By theorem 0.14, $V$ is homeomorphic with $Y$, therefore $Y$ is separable. $T(X)$ is a closed subspace of $Y$ and so $T(X)$ is separable because every closed subspace of a separable metric space is separable.

Hence the range of $T$ is separable.

REFERENCES

Bachman [66], Goldberg and Thorp [63], Wilansky [64], Kelley [63], Bessaga and Klee [66].