CHAPTER IV

HOMOMORPHISM ON FUZZY JOIN SEMI L-IDEAL

4.1 INTRODUCTION

In this chapter the concept of fuzzy join semi L-ideal homomorphism in fuzzy join semi L-ideal is introduced and the invariant property of fuzzy join semi L-ideal is derived. Fuzzy join semi L-cosest in join semi L-ideal, fuzzy join semi L-quotient ideal of join semi L-ideal and the properties of fuzzy join semi L-ideal are discussed. Some related theorems are established.

Definition: 4.1.1

Let \((A, \lor)\) and \((A', \lor)\) be two fuzzy join semilattices. Let \(f\) be a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of \(A\) onto a fuzzy join semi L-ideal of \(A'\). If \(S(\mu)\) and \(S(\sigma)\) are fuzzy join semi L-ideals of \(A\) then the following is true:

\[ f [ S(\mu) \lor S(\sigma) ] = f [ S(\mu) ] \lor f [ S(\sigma) ], \forall S(\mu), S(\sigma) \in A. \]

Definition: 4.1.2

A one-one and onto fuzzy join semi L-ideal homomorphism is called a fuzzy join semi L-ideal isomorphism.

Thorem: 4.1.3

Let \(f\) be a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of \(A\) onto a fuzzy join semi L-ideal of \(A\). If \(S(\mu)\) and \(S(\sigma)\) are fuzzy join semi L-ideals of \(A\), then the following are true:
(i) \( f [ S(\mu) \vee S(\sigma) ] = f [ S(\mu) ] \vee f [ S(\sigma) ] \)

(ii) \( f [ S(\mu) \cap S(\sigma) ] \subseteq f [ S(\mu) ] \cap f [ S(\sigma) ] \), with equality if atleast one of \( S(\mu) \) or \( S(\sigma) \) is \( f \)-invariant.

**Proof:**

Let \( y \in A' \) and let \( \varepsilon > 0 \) be given.

(i) Let \( S(\alpha) = \{ f [ S(\mu) \vee S(\sigma) ] (y) \} \) and \( S(\beta) = \{ f [ S(\mu) ] \vee f [ S(\sigma) ] \} (y) \)

Then \( S(\alpha) - \varepsilon < \max_{x \in f^{-1}(y)} [ S(\mu) \vee S(\sigma) ] (x) \)

\( \Rightarrow S(\alpha) - \varepsilon < [ S(\mu) \vee S(\sigma) ] (x_0) \) for some \( x_0 \in A \) such that \( f(x_0) = y \)

\( = \max \{ \min \{ S[\mu(a)], S[\sigma(b)] \} \}, \) where \( a, b \in A \)

\( \Rightarrow S(\alpha) - \varepsilon < \min \{ S[\mu(a_0)], S[\sigma(b_0)] \} \) \( \cdots (1) \)

for some \( a_0, b_0 \in A \) such that \( x_0 = a_0 \vee b_0 \).

Now,

\( S(\beta) = \max_{y = y_1 \vee y_2} \{ \min \{ f [ S(\mu(y_1)] \}, f [ S(\sigma(y_2)] \} \}, \) where \( y_1, y_2 \in A' \)

\( \Rightarrow S(\beta) \geq \min \{ f [ S(\mu) ] f(a_0), f [ S(\sigma) ] f(b_0) \}, \) since \( y = f(x_0) = f(a_0) \vee f(b_0) \)

\( = \min \{ f^{-1}[ S(\mu(a_0))], f^{-1}[ S(\sigma(b_0))] \} \)

\( \geq \min \{ S[\mu(a_0)], S[\sigma(b_0)] \} \)

\( > S(\alpha) - \varepsilon, \) by (1).

\( \Rightarrow S(\beta) \leq S(\alpha). \) Since \( \varepsilon \) is arbitrary. \( \cdots \)

Next, to show that \( S(\beta) \leq S(\alpha) \)
\[ S(\beta) - \varepsilon < \max \left\{ \min \left\{ f( S(\mu( y_1 ))), f( S(\sigma( y_2 ))) \right\} \right\}, \text{where } y_1, y_2 \in A'. \]

\[ y = y_1 \lor y_2 \]

\[ \Rightarrow S(\beta) - \varepsilon < f \{ S(\mu( y_1 )) \} \text{ and } f \{ S(\sigma( y_2 )) \}, \text{for some } y_1, y_2 \in A \text{ such that } y = y_1 \lor y_2. \]

\[ \Rightarrow S(\beta) - \varepsilon < S(\mu( x_1 )) \text{ and } S(\sigma( x_2 )), \text{for some } x_1 \in f^{-1}( y_1 ) \text{ and } x_2 \in f^{-1}( y_2 ), \text{by definition.} \]

\[ \Rightarrow S(\beta) - \varepsilon < \min \{ S(\mu( x_1 )), S(\sigma( x_2 )) \} \]

\[ \leq [ S(\mu) \lor S(\sigma) ]( x_1 \lor x_2 ), \text{by definition.} \]

\[ \leq \max \{ [ S(\mu) \lor S(\sigma) ]( x ) \}, \text{Since } x_1 \lor x_2 \in f^{-1}( y) \]

\[ x \in f^{-1}( y) \]

\[ = f [ S(\mu) \lor S(\sigma) ]( y) \]

\[ = S(\alpha) \]

\[ \Rightarrow S(\beta) - \varepsilon \leq S(\alpha). \]

\[ \Rightarrow S(\beta) \leq S(\alpha), \text{Since } \varepsilon \text{ is arbitrary } \]

Therefore (1) and (2) imply \( S(\alpha) = S(\beta). \)

\[ \Rightarrow f [ S(\mu) \lor S(\sigma) ] = f [ S(\mu) ] \lor f [ S(\sigma) ]. \]

(i) It is true that, \( S(\mu) \cap S(\sigma) \subseteq S(\mu) \) and \( S(\mu) \cap S(\sigma) \subseteq S(\sigma) \)

\[ \Rightarrow f [ S(\mu) \cap S(\sigma) ] \subseteq f [ S(\mu) ] \text{ and } f [ S(\mu) \cap S(\sigma) ] \subseteq f [ S(\sigma) ] \]

\[ \Rightarrow f [ S(\mu) \cap S(\sigma) ] \subseteq f [ S(\mu) ] \cap f [ S(\sigma) ] \]----------------------(4).
Next assume that, $S(\sigma)$ is $f$-invariant.

Then $f^{-1}f \left[ S(\sigma) \right] = S(\sigma)$

Now put $S(\alpha) = \{ f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \} (y)$ and $S(\beta) = \{ f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \} (y)$

Then $S(\alpha) - \varepsilon < \max \{ \{ f ( S[ \mu(y)] ) \}, \{ f ( S[ \sigma(y)] ) \} \}$

$$= \max \{ \max \{ f \left[ S(\mu) \right], f \left[ S(\sigma) \right] \} \}$$

$\Rightarrow S(\alpha) - \varepsilon < S[ \mu(z)]$, for some $z \in f^{-1}(y)$ and $S(\alpha) - \varepsilon < \{ f ( S[ \sigma(y)] ) \}$

$\Rightarrow S(\alpha) - \varepsilon < S[ \mu(z)]$ and $S(\alpha) - \varepsilon < f ( S[ \sigma(z)] ) = f^{-1}f ( S[ \sigma(z)] ) = S[ \sigma(z)]$

$\Rightarrow S(\alpha) - \varepsilon < \max \{ S[ \mu(z)], S[ \sigma(z)] \}$

$$= \left[ S(\mu) \cap S(\sigma) \right] (z)$$

$\Rightarrow S(\alpha) - \varepsilon < \max \left[ S(\mu) \cap S(\sigma) \right] (z)$, Since $z \in f^{-1}(y)$.

$$= f \left[ S(\mu) \cap S(\sigma) \right] (y)$$

$$= S(\beta)$$

Hence $f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \subseteq f \left[ S(\mu) \cap S(\sigma) \right]$ -----------------------------(5).

Therefore (6) and (7) imply $f \left[ S(\mu) \cap S(\sigma) \right] = f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right]$.

**Theorem: 4.1.4**

If $f$ is a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of $A$ onto a fuzzy join semi L-ideal of $A'$ then for each fuzzy join semi L-ideal $S(\mu)$ of $A$, $f \left[ S(\mu) \right]$ is a fuzzy join semi L-ideal of $A'$ and for each fuzzy join semi L-ideal $S(\mu')$ of $A'$, $f^{-1}[S(\mu')]$ is a fuzzy join semi L-ideal of $A$. 

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Proof:

Let f be a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of A onto fuzzy join semi L-ideal of A'. Assume that S(µ) is a fuzzy join semi L-ideal of A and define S[ µ'f(x) ] = S[ µ(x) ] and for y ∈ A', S[ µ'(y) ] = S[ µ(y) ].

To prove: S( µ' ) is a fuzzy join semi L-ideal of A' corresponding to a fuzzy join semi L-ideal S(µ) of A.

(i.e) to prove

S( µ' ) [ f(x) ∨ f(y) ] ≥ max { S[ µ'f(x) ] , { S[ µ'f(y) ] } }

(i.e) to prove

S( µ' ) [ f(x) ∨ f(y) ] = S[ µ'f( x ∨ y ) ], Since f is a fuzzy join semi L-ideal homomorphism.

= S[ µ( x ∨ y ) ], Since S[ µ'f(x) ] = S[ µ(x) ]

≥ max { S[ µ(x) ], S[ µ(y) ] }

= max { S[ µ'f(x) ] , S[ µ'f(y) ] }.

Therefore S( µ' ) is a fuzzy join semi L-ideal of A'.

Let S( µ' ) is a fuzzy join semi L-ideal of A'.

To Prove: S( µ'^{-1} ) is a fuzzy join semi L-ideal of A.

(i) S(µ) [ f^{-1}(x) ∨ f^{-1}(y) ] = S[ µ'^{-1}( x ∨ y ) ]

= S[ µ'( x ∨ y ) ]. Since S[ µ'^{-1}(x) ] = S[ µ'(x) ].
\[ \geq \max \{ S[\mu'(x)], S[\mu'(y)] \} \]

Hence, \( S(\mu f^{-1}) \) is a fuzzy join semi L-ideal of \( A \) corresponding to \( S(\mu') \) of \( A' \).

**Definition: 4.1.5**

Let \( f \) be any function from a fuzzy join semi L-ideal of \( A \) onto a fuzzy join semi L-ideal of \( A' \). Then \( S(\mu) \) is called \( f \)-invariant if \( S[f(\mu(x))] = S[f(\mu(y))] \) then \( S[\mu(x)] = S[\mu(y)] \) where \( x, y \in A \).

**Theorem 4.1.6**

Let \( f \) be a fuzzy join semi L-ideal isomorphism from a fuzzy join semi L-ideal of \( A \) onto a fuzzy join semi L-ideal of \( A' \). Let \( S(\mu) \) and \( S(\mu') \) be fuzzy join semi L-ideals of \( A \) and \( A' \) respectively and let \( S(\mu) \) be \( f \)-invariant. Let \( t = S[\mu(x)] = S[\mu'(f(x))] \). Then the following statements are true:

(i) \( F_{f[S(\mu)]} = \{ f[S(\mu)] / t \in \text{Im } S(\mu) \} \) and

(ii) \( F_{f^{-1}[S(\mu)]} = \{ f^{-1}[S(\mu')]/ s \in \text{Im } S(\mu') \} \)

**Proof:**

(i) \( t \in \text{Im } S(\mu) \Leftrightarrow S[\mu(x)] = t \), for some \( x \in A \).

\[ \Leftrightarrow f^{-1}(S[\mu(x)]) = t \]

\[ \Leftrightarrow S[\mu f(x)] = t \]

\[ \Leftrightarrow S[\mu(y)] = t \]

\[ \Leftrightarrow t \in \text{Im } S(\mu) = S(\mu') \]

Therefore, \( \text{Im } S(\mu) = \text{Im } S(\mu') \)
Claim: $f[S(\mu_t)] = [f\{S(\mu)\}]_t$

Let $y \in f[S(\mu_t)] \Rightarrow y = f(x)$, for some $x \in S(\mu_t)$, $x \in A$.

$\Rightarrow y = f(x)$ and $S[\mu(x)] \geq t$

$\Rightarrow \text{Sup} \{S[\mu(z)] / y = f(z)\} \geq t$

$\Rightarrow f(S[\mu(y)]) \geq t$

$\Rightarrow y \in f[S(\mu)]_t$

Therefore, $f[S(\mu)]_t \subseteq f[S(\mu_t)]$

Also, $[f(S(\mu))]_t \subseteq f[S(\mu_t)] \geq t$

$\Rightarrow \{f[S(\mu)]\} f(x) \geq t$, Since $y = f(x)$, for some $x \in A$.

$\Rightarrow f^{-1}f\{S[\mu(x)]\} \geq t$

$\Rightarrow S[\mu(x)] \geq t$

$\Rightarrow x \in S(\mu_t)$

$\Rightarrow y = f(x) \in f[S(\mu_t)]$

Hence $F_{f[S(\mu)]} = \{[f\{S(\mu)\}]_t / t \in \text{Im} f[S(\mu)]\}$

$= \{f[S(\mu_t)] / t \in \text{Im} f[S(\mu)]\}$

(ii) $s \in f^{-1}[S(\mu')] \Leftrightarrow$ there exists $x \in A$ such that $f^{-1}\{S[\mu'(x)]\} = s$

$\Leftrightarrow S[\mu'f(x)] = s$, for some $x \in A$. 101
\[ \Leftrightarrow S[\mu'(ff^1(x))] = s \]

\[ \Leftrightarrow S[\mu'(y)] = s \]

\[ \Leftrightarrow s \in \text{Im} S(\mu') \]

Next,

\[ x \in f^{-1}[S(\mu')] \iff f^{-1}\{S[\mu'(x)]\} \geq s \]

\[ \Leftrightarrow S[\mu'f(x)] \geq s \]

\[ \Leftrightarrow f(x) \in S(\mu'_s) \]

\[ \Leftrightarrow x \in f^{-1}[S(\mu'_s)] \]

Hence \( F_{f'[S(\mu')] = \{f^{-1}[S(\mu')]_s \in \text{Im} f^{-1}[S(\mu')] \} = \{f^{-1}[S(\mu'_s)]_s \in \text{Im} S(\mu'_s) \} \]

**Theorem: 4.1.7**

Let \( f \) be a join semi L-ideal homomorphism from a fuzzy join semi L-ideal of \( A \) onto a fuzzy join semi L-ideal of \( A' \). If \( S(\mu') \) and \( S(\theta') \) are any two fuzzy join semi L-ideal of \( A' \), then \( S(\mu'f^{-1}) \lor S(\theta'f^{-1}) \subseteq [S(\mu') \lor S(\theta')] (f^{-1}) \)

**Proof:**

Let \( x \in A \) and let \( \epsilon > 0 \) be given.

Let \( S(\alpha) = \{S[\mu'f^{-1}] \lor S[\theta'f^{-1}] \} (y) \) and \( S(\beta) = \{S(\mu') \lor S(\theta') \} (f^{-1}) \} (y) \)

Then \( S(\alpha) - \epsilon < \min \{ \max \{S[\mu'f^{-1}(y_1)], S[\theta'f^{-1}(y_2)]\}, \ y_1, y_2 \in A' \} = \min \{ \max \{S[\mu'f^{-1}(y_1)], S[\theta'f^{-1}(y_2)]\}, \ y_1, y_2 \in A' \} \)
\[ \leq [ S(\mu') \lor S(\theta') ] [ f^{-1}(y_1 \lor y_2) ] \]

\[ = [ S(\mu') \lor S(\theta') ] [ f^{-1}(y) ] \]

\[ = S(\beta) \]

\[ \Rightarrow S(\alpha) \leq S(\beta), \text{Since } \varepsilon \text{ is arbitrary.} \]

Hence, \( S[\mu'f^{-1}] \lor S[\theta'f^{-1}] \subseteq [ S(\mu') \lor S(\theta') ] f^{-1} \)

### 4.2 FUZZY JOIN SEMI L-QUOTIENT IDEAL

**Definition: 4.2.1**

Let \( S(\mu) \) be any fuzzy join semi L-ideal of a fuzzy join semi L-ideal of \( A \). Then the fuzzy join semi L-ideal \( S(\mu_x^*) \) of \( A \), where \( x \in A \) defined by \( S[\mu_x^*(y)] = S[\mu(y \lor x)] \), for all \( y \in A \) is termed as the fuzzy join semi L-quotient ideal determined by \( x \) and \( S(\mu) \).

**Remark: 4.2.2**

If \( S(\mu) \) is constant, then \( A_{S(\mu)} = S[\mu^*(0)] \).

**Theorem: 4.2.3**

Let \( S(\mu) \) be any fuzzy join semi L-ideal of a fuzzy join semilattice \( A \). Then \( S(\mu_x^*) \), for all \( x \in A \), the fuzzy join semi L-quotient ideal \( S(\mu) \) of \( A \) is also a fuzzy join semi L-ideal of \( A \).
Proof:

Given $S(\mu)$ be any fuzzy join semi L-ideal of $A$ and $S(\mu_x^\ast)$ is a fuzzy join semi L-quotient of $x$ in $A / S(\mu)$.

To prove: $S(\mu_x^\ast)$ is a fuzzy join semi L-ideal.

That is to prove,

(i) For all $y, z \in A$.

$$S[\mu_x^\ast(y \lor z)] = S(\mu)[(y \lor z) \lor x],$$

by definition.

$$= S(\mu)[(y \lor x) \lor (z \lor x)]$$

$$\geq \max \{S[\mu(y \lor x)], S[\mu(z \lor x)]\}$$

$$\geq \max \{S[\mu_x^\ast(y)], S[\mu_x^\ast(z)]\}$$

Hence $S(\mu_x^\ast)$ is a fuzzy join semi L-ideal of $A$.

Lemma: 4.2.4

If $S(\mu)$ be any fuzzy join semi L-ideal of a fuzzy join semi L-ideal of $A$ then the following holds:

$$S[\mu(x)] = S[\mu(0)] \Leftrightarrow S(\mu_x^\ast) = S[\mu(0)], \forall x \in A.$$

Proof:

Let $S(\mu) = S[\mu(0)]$ -----------(1).

$$S[\mu(y)] \leq S[\mu(0)]$""""""---------(2).

From (1) and (2), We have $S[\mu(y)] \leq S[\mu(x)]$.

Case (i):

If $S[\mu(y)] < S[\mu(x)]$, then

$$S[\mu(y \lor x)] \geq \max \{S[\mu(y)], S[\mu(x)]\}$""""""
= S[ \mu(x) ].

**Case (ii):**

If \( S[ \mu(y) ] = S[ \mu(x) ] \), then \( x, y \in S[ \mu(x) ] \), where \( t = S[ \mu(0) ] \).

Hence \( S[ \mu( y \lor x ) ] \geq \max \{ S[ \mu(y) ], S[ \mu(x) ] \} \)

= \( S[ \mu(x) ] \)

= \( S[ \mu(0) ] \)

Therefore \( S[ \mu( y \lor x ) ] = S[ \mu(0) ] = S[ \mu(y) ] = S[ \mu(x) ] \)

Thus in either case,

\( S[ \mu( y \lor x ) ] = S[ \mu(x) ] \), \( \forall \ y \in A. \)

\( S[ \mu^*(y) ] = S[ \mu(x) ] = S[ \mu^*(0) ] \)

Therefore, \( S( \mu^*_x ) = S( \mu_0^* ) \)

The converse is straightforward.

**Lemma: 4.2.5**

If \( S(\mu) \) is a fuzzy join semi L-ideal of a fuzzy join semi L-ideal of \( A \), then \( A / S( \mu_t ) \cong A_{S(\mu)} \), where \( t = S[ \mu(0) ] \).

**Proof:**

To prove \( f : A \rightarrow A_{S(\mu)} \) is a map defined by \( f(x) = S( \mu_x^* ) \), for all \( x \in A \) is an onto fuzzy join semi L-ideal homomorphism.

(i.e) to prove

\( f( x \lor y ) = S( \mu_{x \lor y}^* ) 

= S[ \mu_{x \lor y}^*(z) ] \)
\[ = S[ \mu( (x \lor y) \lor z) ] \]
\[ = S[ \mu( x \lor z ) ] \lor S[ \mu( y \lor z ) ] \]
\[ = S[ \mu( x \lor z ) ] \lor S[ \mu( y \lor z ) ] \]
\[ = S( \mu^*_x ) \lor S( \mu^*_y ) \]

(ii) \[ f( x \lor y ) = S( \mu^*_{x \lor y} ) \]
\[ = S[ \mu^*_{x \lor y} (z) ] \]
\[ = S[ \mu( (x \lor y) \lor z) ] \]
\[ = S[ \mu( x \lor y ) ] \lor S[ \mu( y \lor z ) ] \]
\[ = S[ \mu( x \lor z ) ] \lor S[ \mu( y \lor z ) ] \]
\[ = S( \mu^*_x ) \lor S( \mu^*_y ) \]

Therefore, \( f \) is a fuzzy join semi L-ideal homomorphism.

Now \( f(x) = S( \mu^*_x ) \iff S( \mu^*_x ) = S( \mu^*_0 ) \)

\[ \iff S[ \mu(x) ] = S[ \mu(0) ] \]

This shows that kernel of \( f \) equal \( S( \mu_t ) \).

Therefore \( A / S( \mu_t ) \cong A_{S(\mu)} \).

**Theorem: 4.2.6**

Let \( f \) be a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal \( A \) onto a fuzzy join semi L-ideal of \( A' \) and let \( S(\mu) \) be any \( f \)-invariant fuzzy join semi L-ideal of \( A \), then \( A_{S(\mu)} \cong A'[ S(\mu) ] \).
Proof:

Since $S(\mu)$ is $f$-invariant, $K_f \subseteq S( \mu_t )$, where $t = S[ \mu(0) ]$.

Now, $f( S[ \mu(0') ] ) = t$, because

$$f( S[ \mu(0') ] ) = \sup S[ \mu(x) ], x \in f^{-1}(0').$$

$$= S[ \mu(0) ]$$

Next, $f[ S(\mu) ] = f[ S(\mu_t) ]$, Since $f(x) \in f[ S(\mu) ] \Leftrightarrow f\{ S(\mu[ f(x) ] ) \} \geq t$

$$\Leftrightarrow f^{-1}( S[ \mu(x) ] ) \geq t$$

$$\Leftrightarrow S[ \mu(x) ] \geq t, as f^{-1}f[ S(\mu) ] = S(\mu)$$

$$\Leftrightarrow x \in S(\mu_t)$$

$$\Leftrightarrow f(x) \in f[ S(\mu_t) ], because K_f \subseteq S(\mu_t).$$

Therefore, by theorem 4.2.5

$$A_{S(\mu)} \cong \frac{A}{S(\mu)} and A'_{f(S(\mu))} \cong A'/[f(S(\mu))]:$$

Also, note that $A/S(\mu) \cong A'_{f(S(\mu))}$

From this, it can be shown that $A_{S(\mu)} \cong \frac{A'}{S(\mu)} \cong A'_{f(S(\mu))} \cong A'/[f(S(\mu))]: \cong A'_{f(S(\mu))}$

$A_{S(\mu)} \cong A'_{f(S(\mu))}$
Definition: 4.2.7

Let $S(\mu)$ be any fuzzy join semi L-ideal of $A$. The fuzzy join semi L-quotient ideal $S(\mu)$ of $A_{S(\mu)}$ ($= A / S(\mu_t)$) is defined by $S(\mu^*[x \lor S(\mu_t)]) = S[\mu(x)]$, $\forall x \in A$,

$S(\mu_t) = \{ x / S[\mu(x)] = S[\mu(0)] = t \}$.

Theorem: 4.2.8

If $S(\mu)$ is any fuzzy join semi L-ideal of a join semilattice $A$ then the fuzzy join subset $S(\mu^*)$ of $A_{S(\mu)}$ defined by $S*[x \lor S(\mu_t)] = S[\mu(x)]$, where $x \in A$.

Proof:

Given that $S(\mu)$ is a fuzzy join semi L-ideal of a join semilattice $A$.

To show that the fuzzy join semi L-ideal $S(\mu^*)$ of $A_{S(\mu)}$ defined by $S*[x \lor S(\mu_t)]$, where $x \in A$ is a fuzzy join semi L-ideal of $A$.

For this, let $x, y \in A$.

Then,

\begin{align*}
(i) \quad S*[x \lor S(\mu_t)] \lor (y \lor S(\mu_t)) &= S*[x \lor S(\mu_t)] \\
&= S[\mu(x)] \\
&\geq \max \{ S[\mu(x)], S[\mu(y)] \}
\end{align*}

Therefore $S(\mu^*)$ is a fuzzy join semi L-ideal of $A_{S(\mu)}$.  

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Theorem : 4.2.9

(i) Let $S(\mu)$ be any fuzzy join semi L-ideal of a fuzzy join semilattice $A$ and let $t = S[\mu(0)]$. Then the fuzzy join semi L-ideal $S(\mu^*)$ of $A / S(\mu_t)$ defined by $S[\mu^*(x \vee S(\mu_t))]$, for all $x \in A$ is a fuzzy join semi L-ideal of $A / S(\mu_t)$.

(ii) If $B$ is a fuzzy join semilattice of $A$ and $S(\theta)$ is a fuzzy join semi L-ideal of $A / B$ such that $S[\theta(x \vee A)] = A$ only when $x \in A$, then there exists a fuzzy join semi L-ideal $S(\mu)$ of $A$ such that $S(\mu_t) = B$, $t = S[\mu(0)]$ and $S(\theta) = S(\mu^*)$.

Proof:

(i) Since $S(\mu)$ is a fuzzy join semi L-ideal of $A$, $S(\mu_t)$ is an fuzzy level join semi L-ideal of $A$.

Now,

$x \vee S(\mu_t) = y \vee S(\mu_t)$

$\Rightarrow x \vee y \in S(\mu_t)$

$\Rightarrow S[\mu(x \vee y)] = t = S[\mu(0)]$

$\Rightarrow S[\mu(x)] = S[\mu(y)]$

$\Rightarrow S[\mu^*(x \vee S(\mu_t))] = S[\mu^*(y \vee S(\mu_t))]$

Therefore, $S(\mu^*)$ is well defined.

Next, for all $x, y \in A$.

$S[\mu^*\{x \vee S(\mu_t) \vee (y \vee S(\mu_t))\}] = S[\mu(\mu(x) \vee S(\mu_t))]$

$= S[\mu(x \vee y)]$

$\geq \max \{ S[\mu(x)], S[\mu(y)] \}$

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\[
= \max \{ S[\mu^* (x \lor S(\mu_t))], S[\mu^* (y \lor S(\mu_t))] \}
\]

\[
S[\mu^* (x \lor S(\mu_t)) \lor (y \lor S(\mu_t))] = S[\mu^* ((x \lor y) \lor S(\mu_t))]
\]

\[
= S[\mu (x \lor y)]
\]

\[
\geq \max \{ S[\mu(x)], S[\mu(y)] \}
\]

\[
= \max \{ S[\mu^* (x \lor S(\mu_t))], S[\mu^* (y \lor S(\mu_t))] \}
\]

(ii) Define \(S(\mu) : A \to [0, 1]\) be \(S[\mu(x)] = S[\theta(x \lor B)]\)

\[
\geq \max \{ S[\mu(x)], S[\mu(y)] \}
\]

\[
S[\mu (x \lor y)] = S[\theta(x \lor y \lor B)]
\]

\[
\geq \max \{ S[\theta(x \lor B)], S[\theta(y \lor B)] \}
\]

\[
= \max \{ S[\mu(x)], S[\mu(y)] \}
\]

Therefore, \(S(\mu)\) is a fuzzy join semi L-ideal.

Also, \(S(\mu_t) = B\)

\(x \in S(\mu_t) \iff S[\mu(x)] = S[\mu(0)]\)

\(\iff S[\theta(x \lor B)] = S[\theta(B)]\)

\(\iff x \in B\)

Now,

\[
S[\mu^* (x \lor S(\mu_t))] = S[\mu(x)]
\]

\[
= S[\theta(x \lor B)].
\]

\[
= S[\theta(x \lor S(\mu_t))]
\]
Hence, \( S(\mu^*) = S(\theta) \).

**Theorem: 4.2.10**

Let \( A \) be any fuzzy join semilattice. Let \( S(\mu^*) \) be any fuzzy join semi L-ideal of the quotient fuzzy join semilattice \( A / K \), where \( K \) is any fuzzy subset of \( A \). Then corresponding to \( S(\mu^*) \) in \( A / K \), there exists a fuzzy join semi L-ideal in \( A \).

Let \( S(\mu^*) \) be any fuzzy join semi L-ideal of \( A / K \).

Define the fuzzy join semi L-ideal \( S(\theta) \) of \( A \) by

\[
S(\theta)(x) = S(\mu^* (x \lor k)), \quad \forall x \in A.
\]

To Prove: \( S(\theta) \) is a fuzzy join semi L-ideal of \( A \).

\[
S(\theta( x \lor y )) = S(\mu^*((x \lor y) \lor k))
\]

\[
= S(\mu^* ((x \lor k) \lor (y \lor k)))
\]

\[
\geq \max \{ S(\mu^*(x \lor k)), S(\mu^*(y \lor k)) \}
\]

\[
= \max \{ S(\theta(x)), S(\theta(y)) \}
\]

Therefore, \( S(\theta( x \lor y )) \geq \max \{ S(\theta(x)), S(\theta(y)) \} \)

Hence \( S(\theta) \) is a fuzzy join semi L-ideal of \( A \).

**Theorem: 4.2.11**

Let \( f \) be a fuzzy join semi L-ideal homomorphism from a semi L-ideal of \( A \) onto a fuzzy join semi L-ideal of \( A' \) and let \( S(\mu) \) be any fuzzy join semi L-ideal of \( A \) such that \( S(\mu_t) \subseteq K_\theta \), where \( t = S(\mu(0)) \). Then there exists a unique fuzzy join semi L-ideal homomorphism \( f' \) from \( A_{S(\mu)} \) onto \( A' \) with the property that \( f = f \circ g \) where \( g(x) = S(\mu^*_x), \quad \forall x \in A \).
Proof:

Define a function $f' : A_{S(\mu)} \rightarrow A'$ by $f'[ S( \mu_x^* ) ] = f(x), \ \forall \ x \in A.$

Now, $S( \mu_x^* ) = S( \mu_y^* )$

$\Rightarrow S( \mu_{x\lor y}^* ) = S( \mu_0^* )$

$\Rightarrow S[ \mu( x\lor y ) ] = S[ \mu(0) ] = t$

$\Rightarrow x\lor y \in S( \mu_{t} ) \subseteq K_f$

$\Rightarrow f'[ S( \mu_x^* ) ] = f'[ S( \mu_y^* ) ]$

Therefore $f'$ is well defined.

Since $f$ is onto, $f'$ is also onto.

Therefore, $f'$ is fuzzy join semi L-ideal homomorphism.

Now,

$f(x) = f'[ S( \mu_x^* ) ]$

$= f'[ g(x) ]$

$= [ f' \circ g ] (x), \ \forall \ x \in A.$

$\xymatrix{ A_{S(\mu)} \ar[rd]^f \ar[rd]_{f'} & \\
\quad & g \ar[ru]^f}$
Finally, to show that this factorization of $f$ is unique.

Suppose that $f = h \circ g$ for some function $h : A_{S(\mu)} \rightarrow A'$

Then $f' \left[ S(\mu_x^*) \right] = f(x)$

$$= \left[ h \circ g \right] (x)$$

$$= h \left[ g(x) \right]$$

$$= h \left[ S(\mu_x^*) \right], \forall x \in A.$$

Hence, there is a unique fuzzy join semi L-ideal $f'$ from $A_{S(\mu)}$ onto $A'$ with the property that $f = f' \circ g$, where $g(x) = S(\mu_x^*), \forall x \in A$.

**Corollary : 4.2.12**

The induced $f'$ is a fuzzy join semi L-ideal isomorphism iff $S(\mu)$ is $f$-invariant.

**Proof:**

Let $f'$ be one-one.

**Claim:**

Let $x, y \in A$.

$f(x) = f(y)$

$\Rightarrow f' \left[ S(\mu_x^*) \right] = f' \left[ S(\mu_y^*) \right]$

$\Rightarrow S(\mu_x^*) = S(\mu_y^*)$

$\Rightarrow S(\mu_{xy}^*) = S(\mu_0^*)$
\[ \Rightarrow S[\mu(x\lor y)] = S[\mu(0)] \]

\[ \Rightarrow S[\mu(x)] = S[\mu(y)] \]

On the otherhand, let \( S(\mu) \) be \( f \)-invariant.

**Claim:** \( f' \) is one-one.

\[ S[\mu(x)] = S[\mu(y)] \]

\[ \Rightarrow f'(S[\mu(x)]) = f'(S[\mu(y)]) \]

\[ \Rightarrow f'[S(\mu_x^*)] = f'[S(\mu_y^*)] \]

\[ \Rightarrow S[\mu(x)] = S[\mu(y)], \text{Since } f \text{ is invariant.} \]

\[ \Rightarrow S(\mu_x^*) = S(\mu_y^*) \]

\[ \Rightarrow f \text{ is one – one.} \]