CHAPTER- II

FUZZY JOIN
SUBSEMILATTICE
2.1 Introduction

In this chapter the concept of fuzzy join subsemilattice is introduced. Fuzzy join subsemilattice is defined and examples are discussed. Fuzzy level join subsemilattice is discussed. Fuzzy level join subsemilattices are defined and properties are established.

Definition: 2.1.1

Let A be a fuzzy join semilattice. A fuzzy join subset $S(\mu) : A \rightarrow [0, 1]$ of a fuzzy join semilattice A is called a fuzzy join subsemilattice of A, if $\forall x, y \in A$

$$S[\mu(x \lor y)] \geq \min \{ S[\mu(x)], S[\mu(y)] \}.$$ 

Example: 2.1.2

Let $A = \{0, a, b, c, 1\}$. Let $S(\mu) : A \rightarrow [0, 1]$ is a fuzzy join set in A defined by

$S[\mu(\ )] = 0$, $S[\mu(a)] = 0.5$, $S[\mu(b)] = 0.6$, $S[\mu(a, b, c)] = 0.7$.

Then $S(\mu)$ is a fuzzy join subsemilattice of A.
Definition: 2.1.3

Let $S(\mu)$ be any fuzzy join subsemilattice of a fuzzy join semilattice $A$ and let $t \in [0, 1]$. Then $S(\mu_t) = \{ x \in A / S[\mu(x)] \geq t \}$ is called a fuzzy level join subsemilattice of $S(\mu)$.

Example: 2.1.4

From example 2.1.2, let $t = 0.6$. Then $S(\mu_t) = \{ 0, c, 1 \}$. Then $S(\mu_t)$ is a fuzzy level join subsemilattice of $S(\mu)$.

Note: $S(\mu_t) \subseteq S(\mu_s)$ whenever $t > s$.

Definition: 2.1.5

Let $S(\mu_1)$ and $S(\mu_2)$ be any two fuzzy join subsemilattices of $A$. Then $S(\mu_1)$ is said to be contained in $S(\mu_2)$ if $S[\mu_1(x)] \leq S[\mu_2(x)], \forall x \in A$ and is denoted by $S(\mu_1) \subseteq S(\mu_2)$.

Definition: 2.1.6

Let $S(\mu_1)$ and $S(\mu_2)$ be any two fuzzy join subsemilattices of $A$. If $S[\mu_1(x)] = S[\mu_2(x)], \forall x \in A$, then $S(\mu_1)$ and $S(\mu_2)$ are said to be equal and it is written as $S(\mu_1) = S(\mu_2)$.

Definition: 2.1.7

The complement of a fuzzy join subsemilattice $S(\mu)$ of $A$ symbolized $\sim S(\mu)$ is defined by $\sim S[\mu(x)] = 1 - S[\mu(x)], \forall x \in A$. 
Definition: 2.1.8

The union of two fuzzy join subsemilattices $S(\mu_1)$ and $S(\mu_2)$ of $A$ is defined as 
$[S(\mu_1) \cup S(\mu_2)](x) = \max \{ S[\mu_1(x)], S[\mu_2(x)] \}, \forall x \in A.$

Definition: 2.1.9

The intersection of two fuzzy join subsemilattices $S(\mu_1)$ and $S(\mu_2)$ of $A$ is defined as 
$[S(\mu_1) \cap S(\mu_2)] = \min \{ S[\mu_1(x)], S[\mu_2(x)] \}, \forall x \in A.$

Lemma: 2.1.10

Let $S(\mu)$ be a fuzzy join subsemilattice of $A$ and $t, s \in \text{Im } S(\mu)$. Then $S(\mu_t) = S(\mu_s)$ iff $t = s$.

Proof:

If $t = s$ then clearly $S(\mu_t) = S(\mu_s)$.

Conversely,

Let $S(\mu_t) = S(\mu_s)$.

Since $t \in \text{Im } S(\mu) \exists x \in A$ such that $S[\mu(x)] = t$.

$\Rightarrow x \in S(\mu_s)$

Hence $t = S[\mu(x)] \geq s$----------------(1).

Similarly, it can be proved $s \geq t$------(2).

Therefore, (1) and (2) imply $t = s$. 
**Theorem 2.1.11**

Two fuzzy join subsemilattices \( S(\mu) \) and \( S(\theta) \) of \( A \) such that \( \text{card } \text{Im} \ S(\mu) < \infty \) are equal iff \( \text{Im} \ S(\mu) = \text{Im} \ S(\theta) \) and \( F_{S(\mu)} = F_{S(\theta)} \), where \( F_{S(\mu)} = \{ S(\mu_t) / S(\mu_t) \text{ is a fuzzy level join subsemilattice of } A \text{ for all } t \in \text{Im} \ S(\mu) \} \) and \( F_{S(\theta)} = \{ S(\theta_t) / S(\theta_t) \text{ is a fuzzy level join subsemilattice of } A \text{ for all } t \in \text{Im} \ S(\theta) \} \).

**Proof:**

Let \( S(\mu) \) and \( S(\theta) \) be two fuzzy join subsemilattices of a fuzzy join semilattice \( A \) such that \( \text{card } \text{Im} \ S(\mu) < \infty \).

Assume that \( S(\mu) \) and \( S(\theta) \) are equal.

(i.e.) \( S[ \mu(x) ] = S[ \theta(x) ] \), \( \forall x \in A \)-----------------(1).

\( S[ \mu(x) ] \in \text{Im} \ S(\mu) \)

\( \Rightarrow S[ \theta(x) ] \in \text{Im} \ S(\mu) \), by (1).

But \( S[ \theta(x) ] \in \text{Im} \ S(\theta) \).

\( \Rightarrow \text{Im} \ S(\theta) \subseteq \text{Im} \ S(\mu) \)-----------------(2).

Similarly, it can be proved \( \text{Im} \ S(\mu) \subseteq \text{Im} \ S(\theta) \)-----------------(3).

Therefore, (2) and (3) imply \( \text{Im} \ S(\mu) = \text{Im} \ S(\theta) \)-----------------(4).

Let \( S(\mu_t) \in F_{S(\mu)} \) and \( x \in S(\mu_t) \), \( t \leq S[ \mu(0) ] \).

\( \Rightarrow S[ \mu(x) ] \geq t, t \in \text{Im} \ S(\mu) \)

\( \Rightarrow S[ \theta(x) ] \geq t, t \in \text{Im} \ S(\theta) \) by (1) and (4).
\( \Rightarrow x \in S(\theta_t), t \leq S[\mu(0)] = S[\theta(0)], \text{ by (1)}. \)

\( \Rightarrow S(\mu_t) \subseteq S(\theta_t) \)

Similarly, it can be proved \( S(\theta_t) \subseteq S(\mu_t) \).

Hence \( S(\mu_t) = S(\theta_t) \).

\( \Rightarrow S(\theta_t) \in F_{S(\mu)} \)

But, \( S(\theta_t) \in F_{S(\theta)} \)

\( \Rightarrow F_{S(\theta)} \subseteq F_{S(\mu)} \)..........................(5).

Similarly it can be proved \( F_{S(\mu)} \subseteq F_{S(\theta)} \)..........................(6).

Therefore (5) and (6) imply \( F_{S(\mu)} = F_{S(\theta)} \).................. (7).

Equations (4) and (7) complete the proof of this part.

Conversely, assume that \( \text{Im } S(\mu) = \text{Im } S(\theta) \) and \( F_{S(\mu)} = F_{S(\theta)} \).

To prove: \( S(\mu) \) and \( S(\theta) \) are equal.

Suppose, \( S[\mu(x)] \neq S[\theta(x)] \) for some \( x \in A \).

Then either \( \text{Im } S(\mu) \neq \text{Im } S(\theta) \) or \( F_{S(\mu)} \neq F_{S(\theta)} \).

This is a contradiction.

Hence \( S[\mu(x)] = S[\theta(x)] \), for all \( x \in A \).

Therefore \( S(\mu) \) and \( S(\theta) \) are equal.
Theorem 2.1.12

If B is any fuzzy join semilattice of A, B ≠ A, then the fuzzy join subsemilattice S(µ) of A is defined by

\[
S[ \mu(x) ] = \begin{cases} 
  s, & \text{if } x \in B \\
  t, & \text{if } x \in A \sim B
\end{cases}
\]

where s, t ∈ [0, 1], s > t is a fuzzy join subsemilattice of A.

Proof:

Let x, y ∈ A.

To prove: S(µ) is a fuzzy join subsemilattice of A.

(i.e) to prove S[ µ( x ∨ y ) ] ≥ min { S[ µ(x) ], S[ µ(y) ] }

It is proved by considering exhaustive three cases.

Case (i):

Let x, y ∈ B, S[ µ(x) ] = s, S[ µ(y) ] = s

As x, y ∈ B, x ∨ y ∈ B, Since B is a fuzzy join semilattice.

Now, S[ µ( x ∨ y ) ] ≥ min { S[ µ(x) ], S[ µ(y) ] }

≥ min { s, s }

= s

Hence S(µ) is a fuzzy join subsemilattice.
Case (ii):

Let \( x \in B, \ y \in A \sim B, \ S[\ \mu(x) \ ] = s \) and \( S[\ \mu(y) \ ] = t. \)

As \( x \in B, \ y \in A \sim B, \ x \lor y \in A \)

Now, \( S[\ \mu(x) \ ] = s > t = S[\ \mu(y) \ ] \)

(i.e.) \( S[\ \mu(x) \ ] > S[\ \mu(y) \ ] \)

\( x \lor y \in B \Rightarrow x \lor y = w, \) for some \( w \in B. \)

\( S[\ \mu(w) \ ] = S[\ \mu(x \lor y) \ ] \geq \min \{ S[\ \mu(x) \ ], S[\ \mu(y) \ ] \} \)

\[ \geq \min \{ s, s \} \]

\[ = s \]

\( x \lor y \in A \sim B \Rightarrow x \lor y = w, \) for some \( w \in A \sim B. \)

\( S[\ \mu(w) \ ] = S[\ \mu(x \lor y) \ ] \geq \min \{ S[\ \mu(x) \ ], S[\ \mu(y) \ ] \} \)

\[ \geq \min \{ t, t \} \]

\[ = t \]

Hence \( S(\mu) \) is a fuzzy join subsemilattice.

Case (iii):

Let \( x, y \in A \sim B, \ S[\ \mu(x) \ ] = t \) and \( S[\ \mu(y) \ ] = t. \)

As \( x, y \in A \sim B, \ x \lor y \in A \) or \( B. \)

If \( x \lor y \in A \sim B \) then
\[ S[\mu(x \lor y)] \geq \min\{S[\mu(x)], S[\mu(y)]\} \]

\[ \geq \min\{t, t\} \]

\[ = t \]

Therefore \( S(\mu) \) is fuzzy join subsemilattice.

Hence \( S(\mu) \) is a fuzzy join subsemilattice of \( A \) in all the three cases.

**Proposition: 2.1.13**

A nonempty fuzzy join subset \( C \) of \( A \) is a fuzzy join semilattice of \( A \) iff \( \chi_c \) is a fuzzy join subsemilattice of \( A \).

**Proof:**

\( \chi_c \) is nothing but the characteristic function of the fuzzy join subsemilattice of \( C \).

(i.e.) \( \chi_c(x) = \begin{cases} 1, & \text{if } x \in C, \\ 0, & \text{if } x \in A \sim C, \end{cases} \)

where \( s, t \in [0, 1], s > t. \)

Then by the theorem 2.2.12, the proof is complete.

**Theorem: 2.1.14**

The intersection of two fuzzy join subsemilattices of \( A \) is also a fuzzy join subsemilattice of \( A \).
Proof:

Let $A$ be a fuzzy join semilattice.

Let $S(\mu_1)$ and $S(\mu_2)$ be any two fuzzy join subsemilattices of $A$.

To prove: $S(\mu_1) \cap S(\mu_2)$ is a fuzzy join subsemilattice of $A$.

Let $a, b \in S(\mu_1) \cap S(\mu_2)$.

Then $a, b \in S(\mu_1)$ and $a, b \in S(\mu_2)$.

$\Rightarrow a \lor b \in S(\mu_1)$ and $a \lor b \in S(\mu_2)$

Therefore $S(\mu_1) \cap S(\mu_2)$ is a fuzzy join subsemilattice of $A$.

Remark: 2.1.15

The intersection of any family of fuzzy join subsemilattices of $A$ is also a fuzzy join subsemilattice of $A$.

Theorem: 2.1.16

The union of fuzzy join subsemilattices of $A$ is also a fuzzy join subsemilattice of $A$ iff one is contained in the other.

Proof:

Let $A$ be a fuzzy join semilattice.

Let $S(\mu_1)$ and $S(\mu_2)$ be any two fuzzy join subsemilattices of $A$ such that one is contained in the other.

Hence, $S(\mu_1) \subseteq S(\mu_2)$ or $S(\mu_2) \subseteq S(\mu_1)$.
⇒ \( S(\mu_1) \cup S(\mu_2) = S(\mu_1) \) or \( S(\mu_1) \cup S(\mu_2) = S(\mu_2) \)

Therefore, \( S(\mu_1) \cup S(\mu_2) \) is a fuzzy join subsemilattice of \( A \).

Conversely, suppose \( S(\mu_1) \cup S(\mu_2) \) is a fuzzy join subsemilattice of \( A \).

**To Prove:** \( S(\mu_1) \subseteq S(\mu_2) \) or \( S(\mu_2) \subseteq S(\mu_1) \)

Suppose that \( S(\mu_1) \) is not contained in \( S(\mu_2) \) and \( S(\mu_2) \) is not contained in \( S(\mu_1) \).

Then there exist elements \( a, b \) such that

\[ a \in S(\mu_1) \) and \( a \not\in S(\mu_2) \] \( \quad \)------------------(1).

\[ b \in S(\mu_2) \) and \( b \not\in S(\mu_1) \] \( \quad \)------------------(2).

Clearly \( a, b \in S(\mu_1) \cup S(\mu_2) \).

Since \( S(\mu_1) \cup S(\mu_2) \) is a fuzzy join subsemilattice of \( A \).

**Case (i) :**

Let \( a \lor b \in S(\mu_1) \).

Since \( a \in S(\mu_1) \), \( a' \in S(\mu_1) \).

Hence \( a' \lor (a \lor b) = (a' \lor a) \lor b = 1 \lor b = b \in S(\mu_1) \).

which is a contradiction to the assumption \( b \not\in S(\mu_1) \) by (2).

**Case (ii) :**

Let \( a \lor b \in S(\mu_2) \).

Since \( b \in S(\mu_2) \), \( b' \in S(\mu_2) \).
Hence \((a \lor b) \lor b' = a \lor (b \lor b') = a \lor 1 = a \in S(\mu_2)\).

Which is a contradiction to the assumption \(a \not\in S(\mu_2)\) by (1).

Hence the assumption that \(S(\mu_1)\) is not contained in \(S(\mu_2)\) and \(S(\mu_2)\) is not contained in \(S(\mu_1)\) is false.

Therefore either \(S(\mu_1) \subseteq S(\mu_2)\) or \(S(\mu_2) \subseteq S(\mu_1)\).

**Theorem: 2.1.17**

Let \(S(\theta)\) be any fuzzy join subsemilattice of \(A\) such that \(\text{Im } S(\theta) = \{ t \}\),

where \(t \in [0, 1]\). If \(S(\theta) = S(\mu) \cup S(\sigma)\), where \(S(\mu)\) and \(S(\sigma)\) are fuzzy join subsemilattices of \(A\), then either \(S(\mu) \subseteq S(\sigma)\) or \(S(\sigma) \subseteq S(\mu)\).

**Proof:**

Suppose \(S(\mu)\) does not contained or equal to \(S(\sigma)\) or \(S(\sigma)\) does not contained or equal to \(S(\mu)\) then there exists some \(x, y \in A\) such that \(S[\mu(x)] > S[\sigma(x)]\) and \(S[\sigma(y)] > S[\mu(y)]\).

Then \(t = S[\theta(x)] = [S(\mu) \cup S(\sigma)](x)\)

\[= \max \{S[\mu(x)], S[\sigma(x)]\}\]

\[= S[\mu(x)], \text{Since } S[\mu(x)] > S[\sigma(x)]\]

and

\(t = S[\theta(y)] = [S(\mu) \cup S(\sigma)](y)\)
\[ = \max \{ S[\mu(y)], S[\sigma(y)] \} \]
\[= S[\sigma(y)], \text{ since } S[\sigma(y)] > S[\mu(y)] \]

Therefore, \[ S[\theta(x)] = t = S[\theta(y)] \]

\[ \Rightarrow S[\mu(x)] = t = S[\sigma(y)] \]
\[ \Rightarrow S[\sigma(y)] > S[\sigma(x)] \text{ and } S[\mu(x)] > S[\mu(y)] \]

Then \[ S[\mu(x \lor y)] \geq \min \{ S[\mu(x)], S[\mu(y)] \} \]
\[ = S[\mu(y)] < t \text{ -------------------(1).} \]

and \[ S[\sigma(x \lor y)] \geq \min \{ S[\sigma(x)], S[\sigma(y)] \} \]
\[ = S[\sigma(x)] < t \text{ -------------------(2).} \]

Hence \[ t = S[\theta(x \lor y)] = [S[\mu] \cup S[\sigma]](x \lor y) \]
\[ = \max \{ S[\mu(x \lor y)], S[\sigma(x \lor y)] \} \]
\[ = \max \{ S[\mu(y)], S[\sigma(x)] \} \]
\[ < t, \text{ by (1) & (2).} \]

Which is a contradiction.

Therefore if \( S(\theta) = S(\mu) \cup S(\sigma) \), then either \( S(\mu) \subseteq S(\sigma) \) or \( S(\sigma) \subseteq S(\mu) \).

**Theorem: 2.1.18**

Let \( S(\theta) \) be any fuzzy join subsemilattice of \( A \) such that \( \text{Im } S(\theta) = \{ 0, t \} \),

where \( t \in [0, 1] \). If \( S(\theta) = S(\mu) \cup S(\sigma) \), where \( S(\mu) \) and \( S(\sigma) \) are fuzzy join
subsemilattices of \( A \), then either \( S(\mu) \subseteq S(\sigma) \) or \( S(\sigma) \subseteq S(\mu) \).

**Proof:**

Suppose \( S(\mu) \not\subseteq S(\sigma) \) or \( S(\sigma) \not\subseteq S(\mu) \) then there exists some \( x, y \in A \), \( S[ \mu(x) ] > S[ \sigma(x) ] \) and \( S[ \sigma(y) ] > S[ \mu(y) ] \).

Then \( t = S[ \theta(x) ] = [ S(\mu) \cup S(\sigma) ] (x) \)

\[ = \max \{ S[ \mu(x) ], S[ \sigma(x) ] \} \]

\[ = S[ \mu(x) ] \geq 0, \text{ Since } S[ \mu(x) ] > S[ \sigma(x) ]. \]

and \( t = S[ \theta(y) ] = [ S(\mu) \cup S(\sigma) ] (y) \)

\[ = \max \{ S[ \mu(y) ], S[ \sigma(y) ] \} \]

\[ = S[ \sigma(y) ] \geq 0, \text{ Since } S[ \sigma(y) ] > S[ \mu(y) ]. \]

Therefore \( S[ \theta(x) ] = t = S[ \theta(y) ]. \)

\[ \Rightarrow S[ \mu(x) ] = t = S[ \sigma(y) ] \]

\[ \Rightarrow S[ \sigma(y) ] > S[ \sigma(x) ] \text{ and } S[ \mu(x) ] > S[ \mu(y) ]. \]

Then \( S[ \mu( x \lor y ) ] \geq \min \{ S[ \mu(x) ], S[ \mu(y) ] \} \)

\[ = S[ \mu(y) ] < t \quad \text{----------(1).} \]

and \( S[ \sigma( x \lor y ) ] \geq \min \{ S[ \sigma(x) ], S[ \sigma(y) ] \} \)

\[ = S[ \sigma(x) ] < t \quad \text{----------(2).} \]

Hence
\[
t = S[\theta(x \vee y)] = [S(\mu) \cup S(\sigma)] (x \vee y)
\]

\[
= \max \{ S[\mu(x \vee y)], S[\sigma(x \vee y)] \}
\]

\[
= \max \{ S[\mu(y)], S[\sigma(x)] \}
\]

\[< t = \min \{ S[\theta(x)], S[\theta(y)] \}, \text{by (1)\& (2), which is a contradiction.}\]

Therefore if \(S(\theta) = S(\mu) \cup S(\sigma)\), then either \(S(\mu) \subseteq S(\sigma)\) or \(S(\sigma) \subseteq S(\mu)\).

2.2 FUZZY LEVEL JOIN SUBSEMILATTICES

**Definition: 2.2.1**

Let \(S(\mu)\) be a fuzzy join subsemilattice of \(A\). The fuzzy level join subsemilattices are defined by,

\[
S(\mu_t) = \{ x \in A / S[\mu(x)] \geq t \}
\]

\[
S(\mu_s) = \{ x \in A / S[\mu(x)] \geq s \}
\]

Clearly, \(S(\mu_t) \subseteq S(\mu_s)\) whenever \(t > s\).

**Theorem: 2.2.2**

Let \(A\) be a fuzzy join semilattice. If \(S(\mu) : A \rightarrow [0, 1]\) is a fuzzy join subsemilattice then the fuzzy level join subset \(S(\mu_t), t \in \text{Im } S(\mu)\) is fuzzy level join subsemilattice of \(A\).
**Proof:**

Let \( x, y \in S(\mu_t) \).

Then \( S[\mu(x)] \geq t; S[\mu(y)] \geq t \).

\( S[\mu(x \lor y)] \geq \min\{S[\mu(x)], S[\mu(y)]\} \). Since \( S(\mu) \) is a fuzzy join subsemilattice.

\( \geq t \).

Therefore, \( x \lor y \in S(\mu_t) \).

**Theorem: 2.2.3**

A fuzzy join subset \( S(\mu) \) of \( A \) is a fuzzy join subsemilattice iff the fuzzy level join subset \( S(\mu_t), t \in \text{Im } S(\mu) \) is a fuzzy level join subsemilattice of \( A \).

**Proof:**

Let \( A \) be a fuzzy join semilattice.

Assume that \( S(\mu) \) is a fuzzy join subset of \( A \). Then \( S(\mu_t), t \in \text{Im } S(\mu) \) a fuzzy level join subsemilattice of \( A \). by Theorem 2.2.2

Conversely, assume that the fuzzy level join subset \( S(\mu_t), t \in \text{Im } S(\mu) \) is fuzzy level join subsemilattice of \( A \).

**To prove:** \( S(\mu) \) is a fuzzy join subsemilattice of \( A \).

It is enough to prove that

\[ S[\mu(x \lor y)] \geq \min\{S[\mu(x)], S[\mu(y)]\} \]

Let \( x, y \in S(\mu_t) \).
$\Rightarrow S[\mu(x)] \geq t$ and $S[\mu(y)] \geq t$

$\Rightarrow \min\{S[\mu(x)], S[\mu(y)]\} > t.$

Now, $x\lor y \in S(\mu) \Rightarrow S[\mu(x\lor y)] \geq t$

$\Rightarrow S[\mu(x\lor y)] \geq \min\{S[\mu(x)], S[\mu(y)]\}$

Hence $S(\mu)$ is a fuzzy join subsemilattice.

**Theorem: 2.2.4**

Two fuzzy level join subsemilattices $S(\mu_s)$ and $S(\mu_t)$ (with $s < t$) of a fuzzy join subsemilattice $S(\mu)$ of $A$ are equal iff there is no $x \in A$ such that $s \leq S[\mu(x)] < t$.

**Proof:**

Let $S(\mu_s)$ and $S(\mu_t)$ be two fuzzy level join subsemilattices of a fuzzy join subsemilattice of $A$, where $s < t$.

Assume that $S(\mu_s)$ and $S(\mu_t)$ are equal.

**To prove:** There is no $x$ in $A$ such that $s \leq S[\mu(x)] < t$.

On the contrary, assume that $s \leq S[\mu(x)] < t$, for some $x \in A$.

$\Rightarrow S[\mu(x)] \geq s$ and $S[\mu(x)] < t$.

$\Rightarrow x \in S(\mu_s)$ and $x \notin S(\mu_t)$.

$\Rightarrow S(\mu_s) \neq S(\mu_t)$.

This is a contradiction to the assumption.

Hence there is no $x$ in $A$ such that $s \leq S[\mu(x)] < t$. 

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Conversely, assume that there is no x in A such that \( s \leq S[\mu(x)] < t \) \hspace{1cm} (1).

\[
S(\mu_s) = \{ x \in A / S[\mu(x)] \geq s \} \quad \text{and} \quad S(\mu_t) = \{ x \in A / S[\mu(x)] \geq t \} \quad \text{and} \quad s < t.
\]

Then \( S(\mu_t) \subseteq S(\mu_s) \) \hspace{1cm} (2).

It is enough that to show that \( S(\mu_s) \subseteq S(\mu_t) \).

Let \( x \in S(\mu_s) \). Then \( S[\mu(x)] \geq s \).

\[
\Rightarrow S[\mu(x)] \geq t, \text{ by (1)}.
\]

\[
\Rightarrow x \in S(\mu_t)
\]

\[
\Rightarrow S(\mu_s) \subseteq S(\mu_t) \hspace{1cm} (3)
\]

Therefore (2) and (3) imply \( S(\mu_s) = S(\mu_t) \).

Hence the two fuzzy level join subsemilattices are equal.

**Theorem : 2.2.5**

The intersection of two fuzzy level join subsemilattices of A is also a fuzzy level join subsemilattice of A.

**Proof:**

Let \( S(\mu) \) be a fuzzy join subsemilattice of A.

Let \( S(\mu_s) \) and \( S(\mu_t) \) be two fuzzy level join subsemilattices of a fuzzy join subsemilattice of \( S(\mu) \).

Let \( x, y \in S(\mu_s) \cap S(\mu_t) \).

Then \( x, y \in S(\mu_s) \) and \( x, y \in S(\mu_t) \).
\[ \Rightarrow x \lor y \in S(\mu_t) \]

Also, \( x \lor y \in S(\mu_s) \)

\[ \Rightarrow x \lor y \in S(\mu_t) \cap S(\mu_s). \]

Therefore, \( S(\mu_t) \cap S(\mu_s) \) is a fuzzy level join subsemilattice of \( A \).

**Remark : 2.2.6**

The intersection of any family of fuzzy level join subsemilattices of \( A \) is also a fuzzy level join subsemilattice of \( A \).

**Theorem: 2.2.7**

The union of two fuzzy level join subsemilattices of \( A \) is also a fuzzy level join subsemilattice of \( A \) iff one is contained in the other.

**Proof:**

Let \( A \) be a fuzzy join semilattice.

Let \( S(\mu_t) \) and \( S(\mu_s) \) be any two fuzzy level join subsemilattice of \( A \) such that one is contained in the other.

Hence \( S(\mu_t) \subseteq S(\mu_s) \) or \( S(\mu_s) \subseteq S(\mu_t) \).

Therefore, \( S(\mu_t) \cup S(\mu_s) = S(\mu_t) \) or \( S(\mu_s) \cup S(\mu_t) = S(\mu_s) \).

Therefore \( S(\mu_t) \cup S(\mu_s) \) is a fuzzy level join subsemilattice of \( A \).

Conversely, suppose \( S(\mu_t) \cup S(\mu_s) \) is a fuzzy level join subsemilattice of \( A \).
We claim that $S(\mu_{t}) \subseteq S(\mu_{s})$ or $S(\mu_{s}) \subseteq S(\mu_{t})$.

Suppose that $S(\mu_{t})$ is not contained in $S(\mu_{s})$ and $S(\mu_{s})$ is not contained in $S(\mu_{t})$.

Then there exists elements $a$, $b$ such that $a \in S(\mu_{t})$ and $a \notin S(\mu_{s})$.-----------------(1).

$b \in S(\mu_{s})$ and $b \notin S(\mu_{t})$.------------------(2).

Clearly $a$, $b \in S(\mu_{t}) \cup S(\mu_{s})$.

Since $S(\mu_{t}) \cup S(\mu_{s})$ is a fuzzy join subsemilattice of $A$.

$a \vee b \in S(\mu_{t})$ or $S(\mu_{s})$.

**Case (i):**

Let $a \vee b \in S(\mu_{t})$.

Since $a \in S(\mu_{t})$, $a' \in S(\mu_{t})$.

Hence $a' \vee (a \vee b) = (a' \vee a) \vee b = 1 \vee b = b \in S(\mu_{t})$.

Which is a contradiction to the assumption $b \notin S(\mu_{t})$, by (2).

**Case (ii):**

Let $a \vee b \in S(\mu_{s})$.

Since $b \in S(\mu_{s})$, $b' \in S(\mu_{s})$.

Hence $(a \vee b) \vee b' = a \vee (b \vee b') = a \vee 1 = a \in S(\mu_{s})$. 

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Which is a contradiction to the assumption $a \not\in S(\mu_s)$ by (1).

Hence the assumption that $S(\mu_t)$ is not contained in $S(\mu_s)$ and $S(\mu_s)$ is not contained in $S(\mu_t)$ is false.

Therefore $S(\mu_t) \subseteq S(\mu_s)$ or $S(\mu_s) \subseteq S(\mu_t)$. 
2.3 FUZZY JOIN SUBSEMILATTICE HOMOMORPHISM

Definition: 2.3.1

Let \((A, \lor)\) and \((A', \lor)\) be two fuzzy join semilattices. Let \(S(\mu)\) and \(S(\sigma)\) be any two fuzzy join subsemilattices of \(A\). A function \(f : S(\mu) \rightarrow S(\sigma)\) is a fuzzy join subsemilattice homomorphism if
\[
f[ S(\mu) \lor S(\sigma) ] (x \lor y) = f( S[ \mu(x) ] ) \lor f( S[ \sigma(y) ] ),
\]
where \(S[\mu(x)], S[\sigma(y)] \in A\).

Definition: 2.3.2

A one-one and onto fuzzy join subsemilattice homomorphism is called a fuzzy join subsemilattice isomorphism.

Definition: 2.3.3

Let \(S(\mu)\) and \(S(\sigma)\) be any two fuzzy join subsemilattices of \(A\). \(S(\mu) \lor S(\sigma)\) is defined by
\[
[ S(\mu) \lor S(\sigma) ] (x) = \max_{x = y \lor z} \{ \min \{ S[\mu(y)], S[\sigma(z)] \} \}
\]
where \(x, y, z \in A\).

Theorem: 2.3.4

Let \(f\) be a fuzzy join subsemilattice homomorphism from a fuzzy join subsemilattice of \(A\) onto a fuzzy join subsemilattice of \(A'\). If \(S(\mu)\) and \(S(\sigma)\) are fuzzy join subsemilattices of \(A\), then the following are true:
(i) \( f[ S(\mu) \lor S(\sigma) ] = f[ S(\mu) ] \lor f[ S(\sigma) ] \)

(ii) \( f[ S(\mu) \land S(\sigma) ] \subseteq f[ S(\mu) ] \land f[ S(\sigma) ] \), with equality if atleast one of \( S(\mu) \)
or \( S(\sigma) \) is \( f \)-invariant.

**Proof:**

Let \( y \in A' \) and let \( \varepsilon > 0 \) be given.

(i) Let \( S(\alpha) = f\{ [ S(\mu) \lor S(\sigma) ] \} (y) \) and \( S(\beta) = \{ f[ S(\mu) ] \lor f[ S(\sigma) ] \} (y) \)

Then \( S(\alpha) - \varepsilon < \max \{ S(\mu) \lor S(\sigma) \} (x) \)

\[ x \in f^{-1}(y) \]

\[ \Rightarrow S(\alpha) - \varepsilon < [ S(\mu) \lor S(\sigma) ] (x_0) \text{ for some } x_0 \in A \text{ such that } f(x_0) = y \]

\[ = \max \{ \min \{ S[\mu(a)], S[\sigma(b)] \} \}, \text{ where } a, b \in A. \]

\[ x = a \lor b \]

\[ \Rightarrow S(\alpha) - \varepsilon < \min \{ S[\mu(a_0)], S[\sigma(b_0)] \} \text{----------}(1). \]

for some \( a_0, b_0 \in A \) such that \( x_0 = a_0 \lor b_0 \).

Now,

\[ S(\beta) = \max \{ \min \{ f[ S(\mu(y_1)) ], f[ S(\sigma(y_2)) ] \} \}, \text{ where } y_1, y_2 \in A' \]

\[ y = y_1 \lor y_2 \]

\[ \Rightarrow S(\beta) \geq \min \{ f[ S(\mu) ] f(a_0), f[ S(\sigma) ] f(b_0) \}, \text{ since } y = f(x_0) = f(a_0) \lor f(b_0). \]

\[ = \min \{ f^{-1}f \{ S[\mu(a_0)] \}, f^{-1}f \{ S[\sigma(b_0)] \} \}. \]

\[ \geq \min \{ S[\mu(a_0)], S[\sigma(b_0)] \}. \]

\[ > S(\alpha) - \varepsilon, \text{ by (1)}. \]
$\Rightarrow S(\beta) \geq S(\alpha)$. Since $\varepsilon$ is arbitrary.-------------------------(2).

Next, to show that $S(\beta) \leq S(\alpha)$

$S(\beta) - \varepsilon < \max \{ \min \{ f(S[\mu(y_1)]), f(S[\sigma(y_2)]) \} \} \quad \text{where } y_1, y_2 \in A'.$

$\Rightarrow S(\beta) - \varepsilon < f(S[\mu(y_1)]) \text{ and } f(S[\sigma(y_2)]), \text{ for some } y_1, y_2 \in A \text{ such that } y = y_1 \lor y_2.$

$\Rightarrow S(\beta) - \varepsilon < S[\mu(x_1)] \text{ and } S[\sigma(x_2)], \text{ for some } x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2), \text{ by definition.}$

$\Rightarrow S(\beta) - \varepsilon < \min \{ S[\mu(x_1)], S[\sigma(x_2)] \}$

$\leq [S(\mu) \lor S(\sigma)(x_1 \lor x_2), \text{ by definition.}$

$\leq \max \{ [S(\mu) \lor S(\sigma)](x) \}, \text{ Since } x_1 \lor x_2 \in f^{-1}(y).$

$\quad x \in f^{-1}(y)$

$= f[S(\mu) \lor S(\sigma)](y) = S(\alpha).$

$\Rightarrow S(\beta) - \varepsilon \leq S(\alpha).$

$\Rightarrow S(\beta) \leq S(\alpha), \text{ Since } \varepsilon \text{ is arbitrary } \-------------------------(3).$

Therefore (1) and (2) imply $S(\alpha) = S(\beta) \ h$

$\Rightarrow f[S(\mu) \lor S(\sigma) = f[S(\mu)] \lor f[S(\sigma)].$

(ii) Now, $S(\mu) \cap S(\sigma) \subseteq S(\mu) \text{ and } S(\mu) \cap S(\sigma) \subseteq S(\sigma)$

$\Rightarrow f[S(\mu) \cap S(\sigma)] \subseteq f[S(\mu)] \text{ and } f[S(\mu) \cap S(\sigma)] \subseteq f[S(\sigma)].$
\[ \Rightarrow f \left[ S(\mu) \cap S(\sigma) \right] \subseteq f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \]----------------------(6).

Next assume that \( S(\sigma) \) is \( f \)-invariant.

Then \( f^{-1} f \left[ S(\sigma) \right] = S(\sigma) \)

Now put \( S(\alpha) = \left\{ f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \right\} (y) \) and \( S(\beta) = \left\{ f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \right\} (y) \)

Then \( S(\alpha) - \varepsilon < \max \left\{ f \left[ S(\mu) \right], f \left[ S(\sigma) \right] \right\} (y) \)

\[ = \max_{x \in f^{-1}(y)} \left\{ \max \left\{ f \left[ \mu(x) \right], f \left[ \sigma(y) \right] \right\} \right\} \]

\[ \Rightarrow S(\alpha) - \varepsilon < S[ \mu(z) ], \text{ for some } z \in f^{-1}(y) \text{ and } S(\alpha) - \varepsilon < f \left[ S(\sigma) \right] (y) \]

\[ \Rightarrow S(\alpha) - \varepsilon < S[ \mu(z) ] \text{ and } S(\alpha) - \varepsilon < f \left[ S(\sigma) \right] f(z) = f^{-1} f \left\{ S[ \sigma(z) ] \right\} = S[ \sigma(z) ] \]

\[ \Rightarrow S(\alpha) - \varepsilon < \max \left\{ S[ \mu(z) ], S[ \sigma(z) ] \right\} \]

\[ = \left[ S(\mu) \cap S(\sigma) \right] (z) \]

\[ \Rightarrow S(\alpha) - \varepsilon < \max_{z \in f^{-1}(y)} \left[ \left( S(\mu) \cap S(\sigma) \right)(z) \right], \text{ Since } z \in f^{-1}(y). \]

\[ = f \left[ S(\mu) \cap S(\sigma) \right] (y) = S(\beta) \]

Hence \( f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \subseteq f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] \)----------------------(7).

Therefore (6) and (7) imply \( f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right] = f \left[ S(\mu) \right] \cap f \left[ S(\sigma) \right]. \)

**Theorem: 2.3.5**

If \( f \) is a fuzzy join subsemilattice homomorphism from a fuzzy join subsemilattice of \( A \) onto a fuzzy join subsemilattice of \( A' \) then for each fuzzy join
Proof:

Let $f$ be a fuzzy join subsemilattice homomorphism from a fuzzy join subsemilattice of $A$ onto a fuzzy join subsemilattice of $A'$. Assume that $S(\mu)$ is a fuzzy join subsemilattice of $A$ and define $S[\mu f(x)] = S[\mu(x)]$.

To prove: $S(\mu')$ is a fuzzy join subsemilattice of $A'$ corresponding to a fuzzy join subsemilattice $S(\mu)$ of $A$.

(i.e.) to prove $S(\mu')[f(x) \lor f(y)] \geq \min \{ S[\mu f(x)], S[(\mu f(y))] \}$.

(i.e.) to prove $S(\mu')[f(x) \lor f(y)] = S(\mu')[f(x \lor y)]$. Since $f$ is a fuzzy join subsemilattice homomorphism.

$$= S[\mu(x \lor y)]$$. Since $S[\mu f(x)] = S[\mu(x)]$.

$$\geq \min \{ S[\mu(x)], S[\mu(y)] \}$$

$$= \min \{ S[\mu f(x)], S[\mu f(y)] \}.$$ 

Therefore $S(\mu')$ is a fuzzy join subsemilattice of $A'$.

Let $S(\mu')$ is a fuzzy join subsemilattice of $A'$.

To Prove: $S(\mu f^{-1})$ is a fuzzy join subsemilattice of $A$.

(i) $S(\mu)[f^{-1}(x) \lor f^{-1}(y)] = S[\mu f^{-1}(x \lor y)]$

$$= S[\mu'(x \lor y)]$$. Since $S[\mu f^{-1}(x)] = S[\mu'(x)]$. 

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\[ \geq \min \{ S[\mu'(x)], S[\mu'(y)] \} \]

\[ = \min \{ S[\mu^{-1}(x)], S[\mu^{-1}(y)] \}. \]

Hence \(S(\mu^{-1})\) is a fuzzy join subsemilattice of \(A\) corresponding to \(S(\mu')\) of \(A'\).

**Definition: 2.3.6**

Let \(f\) be any function from a fuzzy join subsemilattice of \(A\) onto a fuzzy join subsemilattice of \(A'\) and let \(S(\mu)\) be any fuzzy join subsemilattice of \(A\). Then \(S(\mu)\) is called \(f\)-invariant if \(f(\{ S[\mu(x)] \}) = f(\{ S[\mu(y)] \})\) implies \(S[\mu(x)] = S[\mu(y)]\) where \(x, y \in A\).

**Theorem: 2.3.7**

Let \(f\) be a fuzzy join subsemilattice isomorphism from a fuzzy join subsemilattice of \(A\) onto a fuzzy join subsemilattice of \(A'\). Let \(S(\mu)\) and \(S(\mu')\) be two fuzzy join subsemilattices of \(A\) and \(A'\) respectively and let \(S(\mu)\) be \(f\)-invariant. Then the following statements are true:

(iii) \(F_{S(\mu')} = \{ S(\mu') / t \in \text{Im } S(\mu') \} \) and

(iv) \(F_{S(\mu)} = \{ S(\mu) / t \in \text{Im } S(\mu) \} \).

**Proof:**

(i) \(t \in \text{Im } S(\mu) \iff S[\mu(x)] = t\), for some \(x \in A\).

\[ \iff S(\mu) \sqcap f^{-1}(x) = t \]

\[ \iff S[\mu f^{-1}(y)] = t \]

\[ \iff t \in \text{Im } S(\mu') \]
Therefore, $\text{Im } S(\mu) = \text{Im } S(\mu')$.

**Claim:** $S(\mu f) = [S(\mu f')]_t$

$y \in S(\mu f) \Rightarrow y = f(x)$, for some $x \in S(\mu), x \in A$.

$\Rightarrow y = f(x)$ and $S[\mu(x)] \geq t$

$\Rightarrow \text{Sup } \{ S[\mu f(z)] / y = f(z) \} \geq t$

$\Rightarrow [S(\mu')f](y) \geq t$

$\Rightarrow y \in [S(\mu'f)]_t$

Therefore, $S(\mu'f) \subseteq [S(\mu'f)]_t$

Also, $[S(\mu'f)]_t \Rightarrow S[\mu'f(y)] \geq t$

$\Rightarrow [S(\mu'f)] f(x) \geq t$, Since $y = f(x)$, for some $x \in A$.

$\Rightarrow [S(\mu')(f^1 f)](x) \geq t$

$\Rightarrow S[\mu'(x)] \geq t$

$\Rightarrow x \in S(\mu_t)$

$\Rightarrow y = f(x) \in S(\mu_t f)$

$\Rightarrow [S(\mu f)]_t \subseteq S(\mu_t f)$

Therefore, $[S(\mu f)]_t = S(\mu_t f)$.

Hence $F_{S(\mu')} = \{ S(\mu') / t \in \text{Im } S(\mu') \}$

(ii) To show that $\text{Im } S(\mu) = \text{Im } S(\mu')$. 
For, \( t \in \text{Im } S(\mu') \Leftrightarrow S[\mu' f^{-1}(x)] = t \), for some \( x \in A \).

\[
\Leftrightarrow S(\mu') [f f^{-1}(x)] = t.
\]

\[
\Leftrightarrow S[\mu' f^{-1}(y)] = t
\]

\[
\Leftrightarrow t \in \text{Im } S(\mu).
\]

Therefore, \( \text{Im } S(\mu') = \text{Im } S(\mu). \)

Next to prove \( S(\mu' f^{-1}) = [S(\mu f^{-1})] \),

\( y \in S(\mu' f^{-1}) \Rightarrow y = f^{-1}(x) \), for some \( x \in S(\mu') \), \( x \in A \).

\[
\Rightarrow y = f^{-1}(x) \text{ and } S[\mu'(x)] \geq t
\]

\[
\Rightarrow \text{Sup } \{ S[\mu'(z)] / y = f^{-1}(z) \} \geq t
\]

\[
\Rightarrow [S(\mu' f^{-1})](y) \geq t
\]

\[
\Rightarrow y \in [S(\mu' f^{-1})].
\]

Therefore, \( S(\mu' f^{-1}) \subseteq [S(\mu' f^{-1})] \)

Also \( [S(\mu' f^{-1})] \) = \( [S(\mu' f^{-1})] \) \( (y) \geq t \)

\[
\Rightarrow [S(\mu' f^{-1})] f^{-1}(x) \geq t, \text{ since } y = f^{-1}(x), \text{ for some } x \in A.
\]

\[
\Rightarrow [S(\mu') (f^{-1} f)](x) \geq t
\]

\[
\Rightarrow S[\mu'(x)] \geq t
\]

\[
\Rightarrow x \in S(\mu')
\]

\[
\Rightarrow y = f^{-1}(x) \in S(\mu' f^{-1})
\]
\[ \Rightarrow [ S(\mu f^{-1}) ]_t \subseteq S(\mu' f^{-1}) \]

Therefore, \( [ S(\mu f^{-1}) ]_t = S(\mu' f^{-1}) \).

Hence \( F_{S(\mu)} = \{ S(\mu_t) / t \in \text{Im } S(\mu) \} \).

**Theorem: 2.3.8**

Let \( f \) be a fuzzy join subsemilattice homomorphism from a fuzzy join subsemilattice of \( A \) onto a fuzzy join subsemilattice of \( A' \). If \( S(\mu') \) and \( S(\theta') \) are any two fuzzy join subsemilattice of \( A' \) then \( [ S(\mu' f^{-1}) \lor S(\theta' f^{-1}) ] \subseteq [ S(\mu') \lor S(\theta') ] f^{-1} \).

**Proof:**

Let \( x \in A \) and let \( \varepsilon > 0 \) be given.

Let \( S(\alpha) = [ S(\mu' f^{-1}) \lor S(\theta' f^{-1}) ] (x) \) and \( S(\beta) = \{ [ S(\mu') \lor S(\theta') ] (f^{-1}) \} (x) \).

Then \( S(\alpha) - \varepsilon > \min \{ \min \{ [ S(\mu' f^{-1}) ] (x_1), [ S(\theta' f^{-1}) ] (x_2) \} \} \), \( x_1, x_2 \in A \).

\[
S(\alpha) - \varepsilon > \min \{ \min \{ [ S(\mu' f^{-1}(x_1)) ], [ S(\theta' f^{-1}(x_2)) ] \} \}
\]

\[
S(\alpha) - \varepsilon > \min \{ [ S(\mu' f^{-1}(x_1)) ], [ S(\theta' f^{-1}(x_2)) ] \}, \text{ for some } x_1, x_2 \in A \exists : x = x_1 \lor x_2
\]

\[
\geq [ S(\mu') \lor S(\theta') ] f^{-1}(x_1 \lor x_2)
\]

\[
= [ S(\mu') \lor S(\theta') ] f^{-1}(x)
\]

\[
= S(\beta)
\]

\[ \Rightarrow S(\alpha) \geq S(\beta) \), Since \( \varepsilon \) is arbitrary.\]

Hence, \( f^{-1}[ S(\mu') ] \lor f^{-1}[ S(\theta') ] \subseteq f^{-1}[ S(\mu') \lor S(\theta') ] \).
Definition: 2.3.9

Let $S(\mu)$ be any fuzzy join subsemilattice of $A$. Then the fuzzy join subset $S(\mu_x^*)$ of $A$, where $x \in A$, defined by $S[\mu_x^*(y)] = S[\mu(y \lor x)]$, $\forall y \in A$ is termed as the fuzzy join subsemilattice determined by $x$ and $S(\mu)$.

Remark: 2.3.10

If $S(\mu)$ is constant then $A_{S(\mu)} = S(\mu_0^*)$.

Definition: 2.3.11

Let $S(\mu)$ be any fuzzy join subsemilattice of $A$. Then we define the fuzzy join subsemilattice $S(\mu')$ of $A_{S(\mu)}$ by $S(\mu' \mu_x^*) = S[\mu(x)]$, $\forall x \in A$, where $S(\mu')$ is the fuzzy join subsemilattice of $A_{S(\mu)}$.

Theorem: 2.3.12

If $S(\mu)$ is any fuzzy join subsemilattice of $A$ then the fuzzy subset $S(\mu')$ of $A_{S(\mu)}$ defined by $S(\mu' \mu_x^*) = S[\mu(x)]$, where $x \in A$ is a fuzzy join subsemilattice of $A_{S(\mu)}$.

Proof:

Given that $S(\mu)$ is a fuzzy join subsemilattice of $A$.

Now we show that the fuzzy join subset $S(\mu')$ of $A_{S(\mu)}$ defined by $S(\mu' \mu_x^*) = S[\mu(x)]$, where $x \in A$ is a fuzzy join subsemilattice $A$.

For this, let $x, y \in A$.

Then $[S(\mu' \mu_x^*) \lor S(\mu' \mu_y^*)] \geq \min \{ S(\mu' \mu_x^*), S(\mu' \mu_y^*) \}$.
\[
S(\mu(x)) = \min \{ S(\mu(x)), S(\mu(x)) \}
\]

Therefore, \(S(\mu')\) is a fuzzy join subsemilattice of \(A_{S(\mu)}\).

**Theorem: 2.3.13**

If \(S(\mu)\) is any fuzzy join subsemilattice of \(A\) then each fuzzy join subsemilattice of fuzzy join subsemi L-quotient \(A_{S(\mu)}\) corresponds in a natural way to a fuzzy join subsemilattice of \(A\).

**Proof:**

Let \(S(\mu')\) be any fuzzy join subsemilattice of \(A_{S(\mu)}\).

Define the fuzzy join subset \(S(\theta)\) of \(A\) by

\[
S(\theta(x)) = [ S(\mu'(\mu'_{x}^{*}) ], \forall x \in A.
\]

**Claim:** \(S(\theta)\) is a fuzzy join subsemilattice of \(A\).

\[
S(\theta(x \vee y)) \geq \max \{ S(\theta(x)), S(\theta(y)) \}
\]

\[
= \max \{ S(\mu'(\mu'_{x}^{*})), S(\mu'(\mu'_{y}^{*})) \}
\]

Therefore, \(S(\theta)\) is a fuzzy join subsemilattice of \(A\).

**Lemma: 2.3.14**

If \(S(\mu)\) is any fuzzy join subsemilattice of \(A\), then \(S(\mu(x)) = S(\mu(0)) \Leftrightarrow S(\mu(x)^{*}) = S(\mu_{x}^{*})\), where \(x \in A\).
Proof:

Let \( S[\mu(x)] = S[\mu(0)] \)\(^{(1)}\).

\( \forall y \in A, S[\mu(y)] \leq S[\mu(0)] \)\(^{(2)}\).

(1) and (2) imply \( S[\mu(y)] \leq S[\mu(x)] \)

Case (i):

If \( S[\mu(x)] < S[\mu(y)] \) then

\( S[\mu(y \lor x)] \leq \max\{S[\mu(y)], S[\mu(x)]\} \)

\[ = S[\mu(x)]. \]

Case (ii):

If \( S[\mu(y)] = S[\mu(x)] \), then \( x, y \in S(\mu_t) \), where \( t = S[\mu(0)] \).

Hence, \( S[\mu(y \lor x)] \leq \max\{S[\mu(y)], S[\mu(x)]\} \)

\[ = S[\mu(x)] \]

\[ = S[\mu(0)]. \]

Therefore, \( S[\mu(y \lor x)] = S[\mu(0)] = S[\mu(y)] = S[\mu(x)] \)

Thus in either case,

\( S[\mu(y \lor x)] = S[\mu(x)], \forall y \in A \)

(i.e.) \( S[\mu_x^*(y)] = S[\mu(x)] = S[\mu_x^*(0)]. \)

Therefore, \( S(\mu_x^*) = S(\mu_0^*). \)
The converse is straight forward.

**Theorem: 2.3.15**

If $S(\mu)$ be any fuzzy join subsemilattice of $A$ then $A/S(\mu) \cong A_{S(\mu)}$, where $t=S[\mu(0)]$.

**Proof:**

To prove $f: A \rightarrow A_{S(\mu)}$ is a map defined by $f(x) = S(\mu_x^*)$, for all $x \in A$ is an onto homomorphism.

(i.e.) To prove $f(x \lor y) = S(\mu_{x \lor y}^*)$

\[
= S[\mu_{x \lor y}^*(z)]
\]

\[
= S[\mu(x \lor z)] \lor S[\mu(y \lor z)]
\]

\[
= S[\mu(x \lor z)] \lor S[\mu(y \lor z)]
\]

\[
= S(\mu_x^*) \lor S(\mu_y^*)
\]

Therefore, $f$ is an onto fuzzy join subsemilattice homomorphism.

Now, $f(x) = S(\mu_x^*) \iff S(\mu_x^*) = S(\mu_0^*)$

\[
\iff S[\mu(x)] = S[\mu(0)], \text{ by lemma 2.3.14}
\]

This shows that kernel of $f$ equal to $S(\mu)$.

Therefore, $A / S(\mu) \cong A_{S(\mu)}$.
Theorem: 2.3.16

Let \( f \) be a fuzzy join subsemilattice homomorphism from fuzzy join subsemilattice of \( A \) onto a fuzzy join subsemilattice of \( A' \). Let \( K_f = \{ x \in A / f(x) = 0' \in A' \} \).

Let \( S(\mu) \) be any fuzzy join subsemilattice \( L \)-coset of \( A \) such that \( S(\mu) \subseteq K_f \), where \( t = S[\mu(0)] \).

Then there exists a unique fuzzy join subsemilattice homomorphism \( f \) from \( A_{S(\mu)} \) onto \( A' \) such that \( f = f' \circ g \), where \( g(x) = S(\mu_x^+) \), \( \forall \ x \in A \).

Proof:

Define a function \( f'(s) : A_{S(\mu)} \rightarrow A' \) by \( f'(\mu_x^+) = f(x), \forall \ x \in A \).

Now, \( S(\mu_x^+) = S(\mu_y^+) \Rightarrow S(\mu_{x\lor y}^+) = S(\mu_0^+) \).

\( \Rightarrow S[\mu(x\lor y)] = S[\mu(0)], \) by lemma 2.3.14

\( \Rightarrow x\lor y \in S(\mu) \subseteq K_f \)

\( \Rightarrow f(x) = f(y) \)

\( \Rightarrow f'[S(\mu_x^+)] = f'[S(\mu_y^+)] \)

Therefore, \( f' \) is well defined.

Also, Since \( f \) is onto, \( f' \) is onto.

A routine computation establishes that \( f' \) is a fuzzy join subsemilattice homomorphism.

Also the diagram given below is commutative.

Because \( f(x) = f'[S(\mu_x^+)] = f[g(x)] = [f' \circ g] (x), \forall \ x \in A \).
Finally we have to show that this factorization of \( f \) is unique.

For this, Suppose that \( f = h \circ g \) for some function \( h : A_{S(\mu')} \rightarrow A' \)

Then, \( f' [ S(\mu_x^*)] = f(x) = (h \circ g) (x) = h [ g(x) ] \)

\[ = h [ S(\mu_x^*)], \forall x \in A. \]

Hence there is a unique fuzzy join subsemilattice homomorphism \( f' \) from \( A_{S(\mu)} \) onto \( A' \), with the property that \( f = f' \circ g \), where \( g(x) = S(\mu_x^*) \), \( \forall x \in A. \)

**Corollary: 2.3.17**

The induced \( f' \) is a fuzzy join subsemilattice isomorphism iff \( S(\mu) \) is \( f \)-invariant.

**Proof:**

Assume \( f' \) is one-one.

**Claim:** \( S(\mu) \) is \( f \)-invariant.

Now \( f(x) = f(y) \).

\[ \Rightarrow f' [ S(\mu_x^*)] = f' [ S(\mu_y^*)] \]
$\Rightarrow S(\mu_x^*) = S(\mu_y^*)$, Since $f'$ is one-one.

$\Rightarrow S(\mu_{x\vee y}^*) = S(\mu_0^*)$.

$\Rightarrow S[\mu(x\vee y)] = S[\mu(0)]$, by lemma 2.3.14

$\Rightarrow S[\mu(x)] = S[\mu(y)]$.

On the other hand, let $S(\mu)$ be $f$-invariant.

**Claim:** $f'$ is 1-1.

$f'(S(\mu_x^*)) = f'(S(\mu_0^*))$

$\Rightarrow f(x) = f(0)$

$\Rightarrow S[\mu(x)] = S[\mu(0)]$, Since $f$ is invariant.

$\Rightarrow S(\mu_x^*) = S(\mu_0^*)$, by lemma 2.3.14

**Theorem: 2.3.18**

Let $f$ be a fuzzy join subsemilattice homomorphism from a fuzzy join subsemilattice of $A$ onto a fuzzy join subsemilattice of $A'$ and let $S(\mu)$ be any $f$-invariant fuzzy join subsemilattice of $A$. Then $A_{S(\mu)} \cong A'_{[S(\mu)]}$.

**Proof:**

Since $S(\mu)$ is $f$-invariant, $K_\mu \subseteq S(\mu)$, where $t = S[\mu(0)]$.

Now, $f(S[\mu(0)]) = t$, because

$f(S[\mu(0)]) = \sup S[\mu(x)]$

$x \in f'(0')$
\[ S[\mu(0)] = S[\mu(x)] \leq S[\mu(0)], \quad \forall x \in A. \]

Next, \( f[S(\mu)] = f[S(\mu)] \), Since

\[ f(x) \in f[S(\mu)] \iff f(S[\mu f(x)]) \geq t \]

\[ \iff f^{-1}f[S(\mu)] \geq t \]

\[ \iff S[\mu(x)] \geq t, \text{ as } f^{-1}f[S(\mu)] = S(\mu) \]

\[ \iff x \in S(\mu) \]

\[ \iff f(x) \in f[S(\mu)], \text{ because } Kf \subseteq S(\mu) \]

Therefore by theorem 2.3.15,

\[ A_{S(\mu)} = A / S(\mu) \text{ and } A'f[S(\mu)] = A' / f[S(\mu)] \]

Also note that \( A / S(\mu) = A'f[S(\mu)] \)

From this,

\[ A_{S(\mu)} = A / S(\mu) \approx A'f[S(\mu)] \]

\[ \approx A'(s) / f[S(\mu)] \]

\[ \approx A'f[S(\mu)] \]

\[ \Rightarrow A_{S(\mu)} \approx A'S(\mu). \]