INTRODUCTION


The present thesis mainly deals with fuzzy join subsemilattice, fuzzy join semi L-ideal, homomorphism on fuzzy join semi L-ideal and fuzzy join prime semi L-ideal. Diagrams have been used throughout the thesis to prove various counter examples and also to convey the ideas of the proofs of the theorems and propositions in a simple way.

In chapter I, under the heading “Preliminaries”, definitions of fuzzy join subsemilattice, fuzzy join semi L-ideal, homomorphism on fuzzy join semi L-ideal and fuzzy join prime semi L-ideal are given. The relations between them are established. Some more theorems which are useful to the thesis are also derived.

Chapter II - Fuzzy join subsemilattice

This chapter introduces the concept of fuzzy join subsemilattice. Concept of fuzzy join subsemilattice and fuzzy level join subsemilattice are defined. Theorems on fuzzy join subsemilattice and fuzzy join subsemilattice homomorphism are established.

Definition:

Let A be a fuzzy join semilattice. A fuzzy join subset \( S(\mu) : A \to [0, 1] \) of a fuzzy join semilattice A is called a fuzzy join subsemilattice of A, if \( \forall x, y \in A \),

\[ S[ \mu(x \lor y) ] \geq \min \{ S[\mu(x)], S[\mu(y)] \}. \]
Definition:

Let $S(\mu)$ be any fuzzy join subsemilattice of a fuzzy join semilattice $A$ and let $t \in [0,1]$. Then $S(\mu_t) = \{ x \in A / S[\mu(x)] \geq t \}$ is called a fuzzy level join subsemilattice of $S(\mu)$.

Definition:

Let $S(\mu_1)$ and $S(\mu_2)$ be any two fuzzy join subsemilattices of $A$. Then $S(\mu_1)$ is said to be contained in $S(\mu_2)$ if $S[\mu_1(x)] \leq S[\mu_2(x)]$, $\forall x \in A$ and is denoted by $S(\mu_1) \subseteq S(\mu_2)$.

Definition:

Let $S(\mu_1)$ and $S(\mu_2)$ be any two fuzzy join subsemilattices of $A$. If $S[\mu_1(x)] = S[\mu_2(x)]$, $\forall x \in A$, then $S(\mu_1)$ and $S(\mu_2)$ are said to be equal and it is written as $S(\mu_1) = S(\mu_2)$.

Definition:

The complement of a fuzzy join subsemilattice $S(\mu)$ of $A$ symbolized $\sim S(\mu)$ is defined by $\sim S[\mu(x)] = 1 - S[\mu(x)]$, $\forall x \in A$.

Definition:

The union of two fuzzy join subsemilattices $S(\mu_1)$ and $S(\mu_2)$ of $A$ is defined as $[S(\mu_1) \cup S(\mu_2)](x) = \max \{ S[\mu_1(x)], S[\mu_2(x)] \}$, $\forall x \in A$. 
The intersection of two fuzzy join subsemilattices $S(\mu_1)$ and $S(\mu_2)$ of $A$ is defined as \[
S(\mu_1) \cap S(\mu_2) = \min \{ S[\mu_1(x)], S[\mu_2(x)] \}, \forall x \in A.\]

**Definition:**

Let $S(\mu)$ be a fuzzy join subsemilattice of $A$. The fuzzy level join subsemilattices are defined by,

\[
S(\mu_t) = \{ x \in A / S[\mu(x)] \geq t \}
\]

\[
S(\mu_s) = \{ x \in A / S[\mu(x)] \geq s \}
\]

Clearly, $S(\mu_t) \subseteq S(\mu_s)$ whenever $t > s$.

**Theorem:**

Two fuzzy join subsemilattices $S(\mu)$ and $S(\theta)$ of $A$ such that \(\text{card Im } S(\mu) < \infty\) are equal iff $\text{Im } S(\mu) = \text{Im } S(\theta)$ and $F_{S(\mu)} = F_{S(\theta)}$, where $F_{S(\mu)} = \{ S(\mu_t) / S(\mu_t) \text{ is a fuzzy level join subsemilattice of } A \text{ for all } t \in \text{Im } S(\mu) \}$ and $F_{S(\theta)} = \{ S(\theta_t) / S(\theta_t) \text{ is a fuzzy level join subsemilattice of } A \text{ for all } t \in \text{Im } S(\theta) \}$.

**Theorem:**

If $B$ is any fuzzy join semilattice of $A$, $B \neq A$, then the fuzzy join subsemilattice $S(\mu)$ of $A$ is defined by

\[
S[\mu(x)] = \begin{cases} 
s, & \text{if } x \in B \\
t, & \text{if } x \in A \sim B\end{cases}
\]

where $s, t \in [0, 1]$, $s > t$ is a fuzzy join subsemilattice of $A$. 
Theorem:

Let $S(\theta)$ be any fuzzy join subsemilattice of $A$ such that $\text{Im } S(\theta) = \{ t \}$, where $t \in [0, 1]$. If $S(\theta) = S(\mu) \cup S(\sigma)$, where $S(\mu)$ and $S(\sigma)$ are fuzzy join subsemilattices of $A$, then either $S(\mu) \subseteq S(\sigma)$ or $S(\sigma) \subseteq S(\mu)$.

Theorem:

Let $S(\theta)$ be any fuzzy join subsemilattice of $A$ such that $\text{Im } S(\theta) = \{0,t\}$, where $t \in [0,1]$. If $S(\theta) = S(\mu) \cup S(\sigma)$, where $S(\mu)$ and $S(\sigma)$ are fuzzy join subsemilattices of $A$, then either $S(\mu) \subseteq S(\sigma)$ or $S(\sigma) \subseteq S(\mu)$.

Theorem:

Let $A$ be a fuzzy join semilattice. If $S(\mu) : A \rightarrow [0,1]$ is a fuzzy join subsemilattice then the fuzzy level join subset $S(\mu_t)$, $t \in \text{Im } S(\mu)$ is fuzzy level join subsemilattice of $A$.

Chapter III - Fuzzy join semi L-ideal

In this chapter the concept of fuzzy join semi L-ideal. Concept of fuzzy join semi L-ideal and fuzzy level join semi L-ideal are defined. Theorems on fuzzy join semi L-ideal and fuzzy level join semi L-ideal are established.

Definition:

Let $A$ be a fuzzy join semilattice. A fuzzy join subset $S(\mu) : A \rightarrow [0,1]$ of a fuzzy join subsemilattice is called a fuzzy join semi L-ideal of $A$ if $\forall x, y \in A$

$$S[ \mu(x \ y)] \geq \max \{ S[ \mu(x)], S[ \mu(y)] \}.$$
Definition:

Let \( S(\mu_1) \) and \( S(\mu_2) \) be any two fuzzy join semi L-ideals of a fuzzy join semilattice \( A \). \( S(\mu_1) \) is said to be contained in \( S(\mu_2) \) if \( S(\mu_1(x)) \leq S(\mu_2(x)) \), \( \forall x \in A \) and is denoted by \( S(\mu_1) \subseteq S(\mu_2) \).

Definition:

Let \( S(\mu_1) \) and \( S(\mu_2) \) be any two fuzzy join semi L-ideals of \( A \). If \( S(\mu_1(x)) = S(\mu_2(x)) \), \( \forall x \in A \), then \( S(\mu_1) \) and \( S(\mu_2) \) are said to be equal and it is written as \( S(\mu_1) = S(\mu_2) \).

Definition:

The complement of a fuzzy join semi L-ideal \( S(\mu) \) of a fuzzy join semilattice \( A \) symbolized by \( \sim S(\mu) \) is defined by \( \sim S(\mu(x)) = 1 - S(\mu(x)) \), \( \forall x \in A \).

Theorem:

If \( I \) is any fuzzy join semi L-ideal of a fuzzy join semilattice \( A \), \( I \neq A \), then the fuzzy join semi L-ideal \( S(\mu) \) of \( A \) is defined by

\[
S(\mu(x)) = \begin{cases} 
  s, & \text{if } x \in I \\
  t, & \text{if } x \in A \sim I
\end{cases}
\]

where \( s, t \in [0, 1] \), \( s > t \) is a fuzzy join semi L-ideal of \( A \).
Definition:

Let $S(\mu)$ and $S(\sigma)$ be any two fuzzy join semi L-ideals of a fuzzy join semilattice $A$. $S(\mu) \lor S(\sigma)$ is defined by $[ S(\mu) \lor S(\sigma) ] (x) = \max \{ \min \{ S[ \mu(y) ], S[ \sigma(z) ] \} \}$, where $x, y, z \in A$.

Theorem:

If $S(\mu)$ and $S(\sigma)$ are any two fuzzy join semi L-ideals of a fuzzy join semilattice $A$, then $[ S(\mu) \lor S(\sigma) ] (x \lor y) \geq \max \{ S[ \mu(x) ], S[ \sigma(y) ] \}$.

Definition:

Let $S(\mu)$ be any fuzzy join semi L-ideal of a fuzzy join semilattice $A$ and let $t \in [0,1]$. Then $S(\mu_t) = \{ x \in A / S[ \mu(x) ] \geq t \}$ is called fuzzy level join semi L-ideal of $S(\mu)$.

Definition:

Let $S(\mu)$ be a fuzzy join semi L-ideal of a fuzzy join semilattice $A$. The fuzzy level join semi L-ideals are defined by

$S(\mu_t) = \{ x \in A / S[ \mu(x) ] \geq t \}$

$S(\mu_s) = \{ x \in A / S[ \mu(x) ] \geq s \}$

Clearly, $S(\mu_t) \subseteq S(\mu_s)$, whenever $t > s$. 

Theorem:

Two fuzzy level join semi L-ideals \( S(\mu_s) \) and \( S(\mu_t) \) (with \( s < t \)) of a fuzzy join semi L-ideal \( S(\mu) \) of a fuzzy join semilattice \( A \) are equal iff there is no \( x \in A \) such that \( s \leq S[\mu(x)] < t \).

Theorem:

A fuzzy join subsemilattice \( S(\mu) \) of a fuzzy join semilattice \( A \) is a fuzzy join semi L-ideal of \( A \) iff the fuzzy level join subsemilattice \( S(\mu_t) \), \( \forall t \in \text{Im } S(\mu) \) is fuzzy level join semi L-ideal of \( A \).

Chapter IV - Homomorphism on fuzzy join semi L-ideal

In this chapter the concept of homomorphism on fuzzy join semi L-ideal. Concept of homomorphism on fuzzy join semi L-ideal and homomorphism on fuzzy level join semi L-ideal are defined. Theorems on homomorphism of fuzzy join semi L-ideal and homomorphism on fuzzy level join semi L-ideal are established.

Definition:

Let \((A, \lor)\) and \((A', \lor)\) be two fuzzy join semilattices. Let \( f \) be a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of \( A \) onto a fuzzy join semi L-ideal of \( A' \). If \( S(\mu) \) and \( S(\sigma) \) are fuzzy join semi L-ideals of \( A \) then the following is true :

\[
f[ S(\mu) \lor S(\sigma) ] = f[ S(\mu) ] \lor f[ S(\sigma) ], \forall S(\mu), S(\sigma) \in A.
\]
**Definition:**

A one-one and onto fuzzy join semi L-ideal homomorphism is called a fuzzy join semi L-ideal isomorphism.

**Theorem:**

Let \( f \) be a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of \( A \) onto a fuzzy join semi L-ideals of \( A \). If \( S(\mu) \) and \( S(\sigma) \) are fuzzy join semi L-ideals of \( A \), then the following are true:

\[
\begin{align*}
\text{(i)} & \quad f \left[ S(\mu) \lor S(\sigma) \right] = f \left[ S(\mu) \right] \lor f \left[ S(\sigma) \right] \\
& \quad \text{and} \\
\quad f \left[ S(\mu) \land S(\sigma) \right] \subseteq f \left[ S(\mu) \right] \land f \left[ S(\sigma) \right], \text{ with equality if atleast one of } S(\mu) \text{ or } S(\sigma) \text{ is } \ f\text{-invariant.}
\end{align*}
\]

**Theorem:**

If \( f \) is a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of \( A \) onto a fuzzy join semi L-ideal of \( A' \) then for each fuzzy join semi L-ideal \( S(\mu) \) of \( A \), \( f \left[ S(\mu) \right] \) is a fuzzy join semi L-ideal of \( A' \) and for each fuzzy join semi L-ideal \( S(\mu') \) of \( A' \), \( f^{-1}[S(\mu')] \) is a fuzzy join semi L-ideal of \( A \).

**Definition:**

Let \( f \) be any function from a fuzzy join semi L-ideal of \( A \) onto a fuzzy join semi L-ideal of \( A' \). Then \( S(\mu) \) is called \( f\)-invariant if \( f \left( S[\mu(x)] \right) = f( S[\mu(y)] ) \) then \( S[\mu(x)] = S[\mu(y)] \) where \( x, y \in A \).
**Theorem:**

Let $f$ be a fuzzy join semi L-ideal isomorphism from a fuzzy join semi L-ideal of $A$ onto a fuzzy join semi L-ideal of $A'$. Let $S(\mu)$ and $S(\mu')$ be fuzzy join semi L-ideals of $A$ and $A'$ respectively and let $S(\mu)$ be $f$-invariant. Let $t = S[\mu(x)] = S[\mu'f(x)]$. Then the following statements are true:

(i) $\text{F}_{f(S(\mu))] = \{ f[S(\mu_t)] / t \in \text{Im } S(\mu) \}$ and

(ii) $\text{F}^{-1}_{f(S(\mu))] = \{ f^{-1}[S(\mu_{s})] / s \in \text{Im } S(\mu') \}$

**Theorem:**

Let $f$ be a join semi L-ideal homomorphism from a fuzzy join semi L-ideal of $A$ onto a fuzzy join semi L-ideal of $A'$. If $S(\mu')$ and $S(\theta')$ are any two fuzzy join semi L-ideal of $A'$, then $[S(\mu'f^{-1}) \lor S(\theta'f^{-1})] \subseteq [S(\mu') \lor S(\theta')] (f^{-1})$

**Definition:**

Let $S(\mu)$ be any fuzzy join semi L-ideal of a fuzzy join semi L-ideal of $A$. Then the fuzzy join semi L-ideal $S(\mu_x^*)$ of $A$, where $x \in A$ defined by $S[\mu_x^*(y)] = S[\mu(y \lor x)]$, for all $y \in A$ is termed as the fuzzy join semi L-quotient ideal determined by $x$ and $S(\mu)$.

**Theorem:**

Let $S(\mu)$ be any fuzzy join semi L-ideal of a fuzzy join semilattice $A$. Then $S(\mu_x^*)$, for all $x \in A$, the fuzzy join semi L-quotient ideal $S(\mu)$ of $A$ is also a fuzzy join semi L-ideal of $A$. 

16
**Theorem:**

Let $f$ be a fuzzy join semi L-ideal homomorphism from a fuzzy join semi L-ideal of $A$ onto a fuzzy join semi L-ideal of $A'$ and let $S(\mu)$ be any $f$-invariant fuzzy join semi L-ideal of $A$, then $A_{S(\mu)} \cong A' \ f[ S(\mu) ]$.

**In chapter V - Fuzzy join prime semi L-ideal**

In this chapter the concept of fuzzy join prime semi L-ideal and fuzzy level join prime semi L-ideal are defined. Theorems on fuzzy join prime semi L-ideal and homomorphism on fuzzy join prime semi L-ideal are established.

**Definition:**

A fuzzy join semi L-ideal $S(\mu)$ of a fuzzy join semilattice of $A$ is said to be a fuzzy join prime semi L-ideal of $A$ if

(i) $S(\mu)$ is not a constant function and

(ii) For any two fuzzy join semi L-ideals $S(\sigma)$ and $S(\theta)$ in $A$ if $S(\sigma) \lor S(\theta) \subseteq S(\mu)$, then either $S(\sigma) \subseteq S(\mu)$ or $S(\theta) \subseteq S(\mu)$.

**Definition:**

A fuzzy join prime semi L-ideal $S(\mu)$ of a fuzzy join semilattice of $A$ is called a fuzzy level join semi L-prime if the fuzzy join semi L-ideal $S( \mu_t )$, where $t = S[ \mu(\theta) ]$ is a fuzzy join prime semi L-ideal of $A$. 
Proposition:

Let $S(\mu)$ be any fuzzy join prime semi L-ideal of a fuzzy join semilattice $A$ such that each fuzzy level join semi L-ideal $S(\mu_t)$, $t \in \text{Im } S(\mu)$ is prime. If $S[\mu(x)] < S[\mu(y)]$ for some $x, y \in A$, then $S[\mu(x \lor y)] = S[\mu(y)]$.

Theorem:

Let $S(\mu)$ be a fuzzy join prime semi L-ideal of a fuzzy join semilattice $A$ then $\text{Card } \text{Im } S(\mu) = 2$.

Theorem:

Let $S(\mu)$ be any fuzzy join prime semi L-ideal of a fuzzy join semilattice, such that $1 \in \text{Im } S(\mu)$. Let $S(\theta)$ be any fuzzy join prime semi L-ideal of $A$. Then $S(\mu) \cap S(\theta)$ is a fuzzy join prime semi L-ideal of the fuzzy level join semi L-ideal $S(\mu_t) = \{x \in A / S[\mu(x)] = 1\}$.

Theorem:

If $\{S(\mu_i) / i \in \mathbb{Z}_+\}$ is any collection of non constant fuzzy join prime semi L-ideals of a fuzzy join semilattice $A$ such that $S(\mu_1) \subseteq S(\mu_2) \subseteq S(\mu_3) \subseteq \ldots \subseteq S(\mu_n) \subseteq \ldots$, then the following statements are true:

a. $\bigcup S(\mu_i)$ is a fuzzy join prime semi L-ideal of $A$.

b. $\bigcap S(\mu_i)$ is a fuzzy join prime semi L-ideal of $A$. 
Theorem:

Let $A$ be a fuzzy join semilattice and let $S(\mu)$ be a fuzzy join prime semi $L$-ideal of $A$. Then $S[\mu(0)] = 1$.

Theorem:

Let $A$ be a fuzzy join semilattice and let $S(\mu)$ be a fuzzy join prime semi $L$-ideal of $A$ such that $\text{Card } \text{Im } S(\mu) = 2$, $S[\mu(0)] = 1$ and the set $S(\mu_0) = \{ x \in A / S[\mu(x)] = S[\mu(0)] \}$ is a fuzzy join prime semi $L$-ideal of $A$. Then $S(\mu)$ is a fuzzy join prime semi $L$-ideal of $A$.

Theorem:

If $f$ is a fuzzy join prime semi $L$-ideal homomorphism from a fuzzy join semilattice $A$ onto a fuzzy join semilattice $A'$ and $S(\mu)$ is any $f$-invariant fuzzy join prime semi $L$-ideal of $A$, then $f[ S(\mu) ]$ is a fuzzy join prime semi $L$-ideal of $A'$.

Theorem:

If $f$ is a fuzzy join prime semi $L$-ideal homomorphism from a fuzzy join subsemilattice $A$ onto a fuzzy join subsemilattice $A'$ and $S(\mu')$ is any fuzzy join prime semi $L$-ideal of $A'$, then $f^{-1}[ S(\mu') ]$ is a fuzzy join prime semi $L$-ideal of $A$. 

19
CHAPTER- I

PRELIMINARIES
CHAPTER-I

PRELIMINARIES

Definitions, results, lemmas, propositions and theorems which are used throughout the thesis are listed in this chapter. The symbols ≤, ∨ will denote inclusion, join( least upper bound ) in a subsemilattice, while symbols ⊆, ∪, ∩, ∈, ∉, φ will refer to set inclusion, union, intersection, membership, non membership and empty set. Small letters a, b, c… will denote elements of a subsemilattice.

Definition: 1.1 [ 18 ]

A non-empty subset J of a lattice L is said to be an ideal if and only if

(i) x ∈ J, y ∈ J ⇒ x ∨ y ∈ J
(ii) x ∈ J, t ∈ J, t ≤ x ⇒ t ∈ J

Definition: 1.2 [ 18 ]

Let ‘a’ be an element of a lattice L. Then the set { x ∈ L / x ≤ a} form an ideal of L and it is called the principal ideal generated by ‘a’ and is denoted by (a).
Theorem: 1.3 [18]

Let \( I(L) \) be the set of all ideals of a lattice \( L \). Then \( I(L) \) is a lattice with respect to the following:

\[
\begin{align*}
(i) & \quad J_1 \leq J_2 \iff J_1 \subseteq J_2 \\
(ii) & \quad J_1 \lor J_2 = \{ x \in L \mid x \leq x_1 \lor x_2, \text{ for some } x_1 \in J_1 \} \\
(iii) & \quad J_1 \lor J_2 = J_1 \cap J_2 = \{ x \in L \mid x \in J_1 \text{ and } x \in J_2 \}
\end{align*}
\]

Definition: 1.4 [18]

A join semilattice or semilattice is a nonempty set \( S \) with binary operation \( \lor \) defined on it and satisfies the following:

\[
\begin{align*}
(i) & \quad \text{Idempotent Law:} \\
& \quad a \lor a = a, \text{ for all } a \in S. \\
(ii) & \quad \text{Commutative Law:} \\
& \quad a \lor b = b \lor a, \text{ for all } a, b \in S. \\
(iii) & \quad \text{Associative Law:} \\
& \quad a \lor (b \lor c) = (a \lor b) \lor c, \text{ for all } a, b, c \in S.
\end{align*}
\]

Definition: 1.5 [18]

A semilattice is a nonempty set \( S \) with binary relation \( \leq \) defined on it and satisfies the following:

\[
\begin{align*}
(i) & \quad \text{\( \leq \) is reflexive:}
\end{align*}
\]
\( a \leq a \) for all \( a \in S \)

(ii) ‘\( \leq \)’ is antisymmetric:

\[ a \leq b \text{ and } b \leq a \Rightarrow a = b \text{ for all } a, b \in S \]

(iii) ‘\( \leq \)’ is transitive:

\[ a \leq b \text{ and } b \leq c \Rightarrow a \leq c \text{ for all } a, b, c \in S \]

(iv) Any two elements in \( S \) have a least upper bound.

**Result: 1.6 [18]**

Two definitions for semilattice \( S \) are equivalent with respect to the following

\( (i) \quad a \leq b \iff a \lor b = b \)

(iii) \( a \lor b = \) least upper bound of \( a \) and \( b \) where \( a, b \in S \).

**Definition: 1.7 [18]**

Let \( S \) be a semilattice and \( I, \) a nonempty subset of \( S \). Then \( I \) is called an ideal of \( S \) if

(i) \( x \in I, y \in I \Rightarrow x \lor y \in I \)

(ii) \( x \in I, t \in s \) and \( t \leq x \Rightarrow t \in I \)

**Theorem: 1.8 [18]**

If \( I(S) \) denote set of all ideals of a semilattice \( S \) then \( I(S) \) is a lattice with respect to the following,

(i) \( I_1 \leq I_2 \iff I_1 \subseteq I_2 \).

(ii) \( I_1 \lor I_2 = \{ x \in S / x \leq x_1 \lor x_2 \text{ for some } x_1 \in I_1, x_2 \in I_2 \} \).
Theorem: 1.9 [18]

If S is a semilattice and a, b in S then \((a \vee b) = (a) \vee (b)\).

Definition: 1.10 [14]

Let \(\mu\) be any fuzzy subset of a set S and let \(t \in [0, 1]\). The set \(\mu_t = \{x \in X / \mu(x) \geq t\}\) is called a level subset of \(\mu\).

Clearly \(\mu_t \subseteq \mu_s\), where \(t > s\).

Definition: 1.11 [14]

Let \(\mu\) and \(\sigma\) be any fuzzy subsets of a set S. Then \(\mu\) is said to be contained in \(\sigma\), denoted by \(\mu \subseteq \sigma\), if \(\mu(x) \subseteq \sigma(x)\) for all \(x \in S\).

If \(\mu(x) = \sigma(x)\), for all \(x \in S\), then \(\mu\) and \(\sigma\) are said to be equal and is written as \(\mu = \sigma\).

Definition: 1.12 [14]

The complement of a fuzzy subset \(\mu\) of a set S, symbolized by \(\sim\mu\), is a fuzzy subset of S defined by \(\sim\mu(x) = 1 - \mu(x)\), for all \(x \in S\).

Definition: 1.13 [14]

The union of two fuzzy subsets \(\mu\) and \(\sigma\) of a set S, denoted by \(\mu \cup \sigma\) is a fuzzy subset of S defined by \((\mu \cup \sigma) = \max \{\mu(x), \sigma(x)\}\), for all \(x \in S\).

Remark: 1.14 [14]

24
The union of $\mu$ and $\sigma$ is the smallest fuzzy set containing both $\mu$ and $\sigma$.

**Definition: 1.15 [14]**

The intersection of $\mu$ and $\sigma$ symbolized by $\mu \cap \sigma$ is a fuzzy subset of $S$, defined by

$$( \mu \cap \sigma ) (x) = \min \{ \mu(x), \sigma(x) \}, \text{ for all } x \in S.$$ 

**Remark: 1.16 [14]**

The intersection of two fuzzy sets $\mu$ and $\sigma$ is the largest fuzzy set which is contained in both $\mu$ and $\sigma$.

**Remark: 1.17 [14]**

The union and intersection of any family $\{ \mu_i / i \in \Omega \}$ of fuzzy subsets of a set $S$ are defined by

$$\bigcup \mu_i (x) = \sup_{i \in \Omega} \mu_i (x), \text{ for all } x \in S$$

$$\bigcap \mu_i (x) = \inf_{i \in \Omega} \mu_i (x), \text{ for all } x \in S$$

Clearly,

$$\mu \cap \chi_\emptyset = \chi_\emptyset, \mu \cup \chi_\emptyset = \mu, \mu \cap \chi_\emptyset = \mu, \alpha \vee \delta \mu \cup \chi_\emptyset = \chi_\sigma$$

**Definition: 1.18 [14]**

Let $f$ be any function from a set $S$ to a set $T$, and let $\mu$ be any fuzzy subset of $S$.

Then $\mu$ is called $f$-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, where $x, y \in S$. 

25
Lemma: 1.19 [14]

Let $f$ be any function from a set $S$ to a set $S'$; $\mu, \theta$ be any two fuzzy subsets of $S$; and $\mu', \theta'$ be any two fuzzy subsets of $S'$. Then the following statements are true:

(i) $f^{-1}(\mu') = \mu'$, $\mu \subseteq f^{-1}[f(\mu)]$

(ii) $f^{-1}[f(\mu)] = \mu$, provided that $\mu$ is $f$-invariant

(iii) $\mu \subseteq \theta \Rightarrow f(\mu) \subseteq f(\theta)$ and

(iv) $\mu' \subseteq \theta' \Rightarrow f^{-1}(\mu') \subseteq f^{-1}(\theta')$

Definition: 1.20 [14]

A fuzzy subset $\mu$ of a set $S$ is said to have sup property if, for any subset $A$ of $S$, there exists $a_0 \in A$, such that $\mu(a_0) = \sup_A \mu(a)$, $a \in A$. For example, a fuzzy subset $\mu$ with $\text{card}\text{Im}\mu < \infty$, has the sup property.

Definition: 1.21 [14]

Two level subgroups $\mu_s$ and $\mu_t$ (with $s < t$) of a fuzzy subgroup $\mu$ of a group $G$ are equal if and only if, there is no $x$ in $G$ such that $s \leq \mu(x) \leq t$.

Theorem: 1.22 [14]

Two fuzzy subgroups $\mu$ and $\theta$ such that $\text{card}\text{Im}\mu < \infty$ and $\text{card}\text{Im}\theta < \infty$ of a group $G$ are equal iff $\text{Im}\mu = \text{Im}\theta$ and $F_{\mu} = F_{\theta}$.

Definition: 1.23 [10]

A fuzzy ideal $P$ of a ring $R$ is said to be a fuzzy prime ideal of $R$ if
(i) P is not a constant function and

(ii) For any fuzzy ideals A, B in R if A o B ⊆ P, then either A ⊆ P or B ⊆ P.

**Theorem: 1.24 [5]**

For any two elements ‘a’ and ‘b’ of a lattice L, (a] ∨ (b].

**Theorem: 1.25 [6]**

Let I be an ideal of a lattice L. For any ideal J of L, I ∨ J = { i ∨ j / i ∈ I, j ∈ J }. 