CHAPTER 2

INTUITIONISTIC FUZZY SETS AND ITS EXTENSIONS

2.1 FUZZY SETS

In 1965, Professor L.A. Zadeh introduced the Fuzzy set theory to represent data and information possessing non-statistical uncertainties. It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems.

The first publication in Fuzzy set theory by Zadeh (1965) and Goguen (1967, 1969) shows the intension of the authors to generalize the classical set.

In classical set theory, a subset \( A \) of a set \( X \) can be defined by its characteristic function \( \chi_A \) as a mapping from the elements of the Universal set \( X \) to the values of the set \( \{0, 1\} \):

\[
\chi_A: X \rightarrow \{0, 1\}.
\]

The mapping may be represented as a set of ordered pairs \( \{(x, \chi_A(x))\} \) with exactly one ordered pair present for each element of \( X \).
The first element of the ordered pair is an element of the set $X$ and the second is its value in $\{0,1\}$. The value ‘0’ is used to represent non-membership and the value ‘1’ is used to represent membership of the element of $A$. The truth or falsity of the statement “$x$ is in $A$” is determined by the ordered pair. The statement is true, if the second element of the ordered pair is ‘1’, and the statement is false, if it is ‘0’.

Similarly, a fuzzy set $A$ of a set $X$ can be defined as a set of ordered pairs $\{(x, \mu_A(x)) : x \in X\}$, each with the first element from $X$ and the second element from the interval $[0, 1]$ with exactly one ordered pair present for each element of $X$. This defines a mapping, $\mu_A$ between elements of the set $X$ and values in the interval $[0, 1]$: $\mu_A : X \rightarrow [0, 1]$.

The value ‘0’ is used to represent complete non-membership, the value ‘1’ is used to represent complete membership and values in between are used to represent intermediate degrees of membership.

The set $X$ is referred to as the Universe of discourse for the fuzzy set $A$. Frequently, the mapping $\mu_A$ is described as a function, the membership function of $A$, the degree to which the statement “$x$ is in $A$” is true, is determined by finding the ordered pair $(x, \mu_A(x))$. The degree of truth of the statement is the second element of the ordered pair.
Definition 2.1.1 [33]

Let $X$ be a non-empty set. A *fuzzy set* (FS) $A$ in $X$ is characterized by its membership function $\mu_A : X \rightarrow [0, 1]$ and $\mu_A (x)$ is interpreted as the degree of membership of element $x$ in fuzzy set $A$, for each $x \in X$.

It is clear that $A$ is completely determined by the set of tuples,

$$A = \{(x, \mu_A (x)) | x \in X\}.$$

Example: 2.1.2

Let $A$ = “real numbers close to 10”

Then, $A=\left\{(x, \mu_A (x)) | \mu_A (x)=\left(1+(x-10)^2\right)^{-1}\right\}$

is a fuzzy set, which is characterized by this membership function is shown in figure 2-1.

Figure: 2-1 Real numbers close to 10
2.2 Intuitionistic Fuzzy Sets (IFS) and its Extensions

After the introduction of the concept of fuzzy sets which determine only degree of membership by Zadeh, several research were conducted on the generalizations of the notion of fuzzy set. The idea of Intuitionistic Fuzzy Set was first published by Krassimir T. Atanassov, a Bulgarian Professor. He introduced a new component which determines the degree of non-membership also in defining the IFS theory.

He defined the well known operations like $\cup$, $\cap$, $+$ and $-$ over the new sets and the modal operators necessity and possibility.
**Definition: 2.2.1. [1]**

Let $X$ be a non empty set. An *IFS* $A$ in $X$ is defined as an object of the form $A=\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X \}$, where the functions $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ define the degree of membership and degree of non-membership of the element $x \in X$ respectively, and for every $x \in X$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

**Remark: 2.2.2** An ordinary fuzzy set can also be written as

$$\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : x \in X \}$$

i.e. all the fuzzy sets are IFSs

**Definition: 2.2.3 [1]**

The value of $\Pi_A(x)=1-\mu_A(x)-\nu_A(x)$ is called the degree of non determinacy or (uncertainty) of the element $x \in X$ to the IFS $A$.

Clearly, in the case of ordinary fuzzy sets $\Pi_A(x)=0$ for every $x \in X$
Figure 2.2 represents the standard geometric interpretation of the IFS.
2.3 Extension of IFS

In this section, we present the following four extensions of the IFSs.

- Intuitionistic L-fuzzy sets,
- IFSs over different universes,
- Temporal IFSs, and
- IFSs of second type.

An intuitionistic L-fuzzy set (ILFSs) is a generalization of the notions of an L-fuzzy set and an IFS.

Let \( (L, \leq) \) be a complete lattice with an involutive order reversing operation \( N : L \to L \). Let a set \( X \) be fixed.

**Definition: 2.3.1 [1]**

An intuitionistic L-fuzzy set (ILFS) \( A^* \) in \( X \) is defined as an object of the form:

\[
A^* = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \},
\]

where the functions \( \mu_A : X \to L \) and \( \nu_A : X \to L \) defined, the degree of membership and the degree of non-membership of the elements \( x \in X \), and for every \( x \in X \),

\[
\mu_A(x) \leq N(\nu_A(x)).
\]
**Definition 2.3.2 [1]**

Let $X$ and $Y$ be two different universes, and let $A_X$ and $B_Y$ be IFSs over $X$ and $Y$ respectively, i.e.

$$A_X = \{ \{ x, \mu_A(x), \nu_A(x) \} \mid x \in X \}$$

$$B_Y = \{ \{ x, \mu_B(x), \nu_B(x) \} \mid x \in Y \},$$

We will call an IFS $A$ defined over the universe $X$ "an $X$-IFS"

**Definition 2.3.3 [1]**

Let $X$ be an universe and $T$ be a non-empty set. We will call the elements of $T$ “time-moments”. A *temporal intuitionistic fuzzy set (TIFS)* is defined as.

$$A(T) = \{ \{ x, \mu_A(x,t), \nu_A(x,t) \} \mid \langle x, t \rangle \in X \times T \},$$

where

(i) $A \subseteq X$ is a fixed set,

(ii) $\mu_A(x,t) + \nu_A(x,t) \leq 1$ for every $\langle x, t \rangle \in X \times T$

(iii) $\mu_A(x,t)$ and $\nu_A(x,t)$ are the degrees of membership and degree of non-membership respectively, of the element $x \in X$ at the time-moment $t \in T$
Definition: 2.3.4 [1]

Let $X$ be a non-empty set. An Intuitionistic Fuzzy Set of Second Type (IFSST) $A$ in $X$ is defined as an object of the form

$$A = \left\{ \left( x, \mu_A(x), \nu_A(x) \right) : x \in X \right\}$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and degree of non-membership of the elements $x \in X$, and for every $x \in X$, $0 \leq [\mu_A(x)]^2 + [\nu_A(x)]^2 \leq 1$

Remark: 2.3.5

It is obvious that for all real numbers $a, b \in [0,1]$ if $0 \leq a + b \leq 1$ then

$0 \leq a^2 + b^2 \leq 1$

Hence all IFSs are IFSST.