Chapter - 2

\( T_{(1,2)^*} - \tilde{g} \)-SPACES

2.1 INTRODUCTION

Levine [37] introduced the notion of \( T_{1/2} \)-spaces which properly lies between \( T_1 \)-spaces and \( T_0 \)-spaces. Many authors studied properties of \( T_{1/2} \)-spaces: Dunham [24], Arenas et al. [3] etc. In this chapter, we introduce the notions called \( T_{(1,2)^*} - \tilde{g} \)-spaces, \( g T_{(1,2)^*} - \tilde{g} \)-spaces and \( \alpha T_{(1,2)^*} - \tilde{g} \)-spaces and obtain their properties and characterizations.

2.2 PRELIMINARIES

Definition 2.2.1

A subset \( A \) of a bitopological space \( X \) is called a \((1,2)^*\)-preopen set [49] if \( A \subseteq \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A)) \).

The complement of \((1,2)^*\)-pre-open set is called \((1,2)^*\)-preclosed.

The \((1,2)^*\)-preclosure [49] of a subset \( A \) of \( X \), denoted by \((1,2)^*\text{-pcl}(A) \) is defined to be the intersection of all \((1,2)^*\)-preclosed sets of \( X \) containing \( A \). It is known that \((1,2)^*\text{-pcl}(A) \) is a preclosed set. For any subset \( A \) of an arbitrarily chosen bitopological space, the \((1,2)^*\)-semi-interior [59] (resp. \((1,2)^*\)-\( \alpha \)-interior [49], \((1,2)^*\)-preinterior [49]) of \( A \), denoted by \((1,2)^*\text{-sint}(A) \) (resp. \((1,2)^*\)-\( \alpha \) int(A), \((1,2)^*\)-pint(A)), is defined to be the union of all semi-open (resp. \((1,2)^*\)-\( \alpha \)-open, \((1,2)^*\)-preopen) sets of \( X \) contained in \( A \).
Remark 2.2.2

The collection of all \((1,2)^*\mathring{g}\)-open (resp. \((1,2)^*\omega\)-open, \((1,2)^*\alpha\)-open, \((1,2)^*\mathrm{gsp}\)-open, \((1,2)^*\mathrm{gs}\)-open, \((1,2)^*\alpha\)-open, \((1,2)^*\mathrm{g}^*\mathrm{p}\)-open) sets is denoted by \((1,2)^*G\mathring{O}(X)\) (resp. \((1,2)^*\omega\mathring{O}(X), (1,2)^*\alpha\mathring{O}(X), (1,2)^*\mathrm{gsp}\mathring{O}(X), (1,2)^*\mathrm{gs}\mathring{O}(X), (1,2)^*\alpha\mathring{O}(X), (1,2)^*\mathrm{g}^*\mathring{P}\mathring{O}(X))
.

We denote the power set of \(X\) by \(P(X)\).

Definition 2.2.3 [64]

A bitopological space \(X\) is called

(i) \((1,2)^*\mathrm{T}_{1/2}\)-space if every \((1,2)^*\mathrm{g}\)-closed subset of \(X\) is \(\tau_{1,2}\)-closed in \(X\).

(ii) \((1,2)^*\mathrm{T}_b\)-space if every \((1,2)^*\mathrm{gs}\)-closed subset of \(X\) is \(\tau_{1,2}\)-closed in \(X\).

Definition 2.2.4 [65]

Let \(X\) be a bitopological space and \(A \subseteq X\). We define the \((1,2)^*\mathrm{sg}\)-closure of \(A\) (briefly \((1,2)^*\mathrm{sg}\)-cl\((A)\)) to be the intersection of all \((1,2)^*\mathrm{sg}\)-closed sets containing \(A\).

2.3 PROPERTIES OF \(T_{(1,2)^*\mathring{g}}\)-SPACES

We introduce the following definition:

Definition 2.3.1

(i) A bitopological space \(X\) is called \((1,2)^*\mathrm{semi}\) generalized-\(R_0\) (briefly \((1,2)^*\mathrm{sg}\)-\(R_0\)) if and only if for each \((1,2)^*\mathrm{sg}\)-open set \(G\) and \(x \in G\) implies \((1,2)^*\mathrm{sg}\)-cl\(\{x\} \subseteq G\).
(ii) A subset $A$ of a bitopological space $X$ is called $(1,2)^*$-$g^*$-preclosed (briefly $(1,2)^*$-$g^*p$-closed) set if $(1,2)^*$-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $(1,2)^*$-$g$-open in $X$. The complement of $(1,2)^*$-$g^*p$-closed set is called $(1,2)^*$-$g^*p$-open set.

**Definition 2.3.2**

A bitopological space $X$ is called

(i) $(1,2)^*$-semi generalized $-T_0$ (briefly $(1,2)^*$-sg-$T_0$) if and only if to each pair of distinct points $x, y$ of $X$, there exists a $(1,2)^*$-sg-open set containing one but not the other.

(ii) $(1,2)^*$-semi generalized $-T_1$ (briefly $(1,2)^*$-sg-$T_1$) if and only if to each pair of distinct points $x, y$ of $X$, there exist a pair of $(1,2)^*$-sg-open sets, one containing $x$ but not $y$, and the other containing $y$ but not $x$.

**Definition 2.3.3**

A bitopological space $X$ is called

(i) $(1,2)^*$-$\alpha T_b$-space if every $(1,2)^*$-$\alpha g$-closed subset of $X$ is $\tau_{1,2}$-closed in $X$.

(ii) $(1,2)^*$-$T_\omega$-space if every $(1,2)^*$-$\omega$-closed subset of $X$ is $\tau_{1,2}$-closed in $X$.

(iii) $(1,2)^*$-$T_p$-space if every $(1,2)^*$-$g^*p$-closed subset of $X$ is $\tau_{1,2}$-closed in $X$.

(iv) $(1,2)^*$-$*T_p$-space if every $(1,2)^*$-$gsp$-closed subset of $X$ is $(1,2)^*$-$g^*p$-closed in $X$.

(v) $(1,2)^*$-$\alpha T_d$-space if every $(1,2)^*$-$\alpha g$-closed subset of $X$ is $(1,2)^*$-$g$-closed in $X$.

(vi) $(1,2)^*$-$\alpha$-space if every $(1,2)^*$-$\alpha$-closed subset of $X$ is $\tau_{1,2}$-closed in $X$.

**Theorem 2.3.4**

For a bitopological space $X$, each of the following statement is equivalent:
(i) X is a (1,2)*-sg-T1.

(ii) Each one point set is (1,2)*-sg-closed set in X.

**Definition 2.3.5**

A bitopological space X is called a T$_{(1,2)^*}$-$g$-space if every (1,2)*- $g$-closed subset of X is $\tau_{1,2}$-closed in X.

**Example 2.3.6**

Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{b\}\}$ and $\tau_2 = \{\phi, X\}$. Then the sets in $\{\phi, X, \{b\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a, c\}\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^* - G C(X) = \{\phi, \{a, c\}, X\}$. Thus X is a T$_{(1,2)^*}$-$g$-space.

**Example 2.3.7**

Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a, c\}\}$ and $\tau_2 = \{\phi, X\}$. Then the sets in $\{\phi, X, \{a, c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{b\}\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^* - G C(X) = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$. Thus X is not a T$_{(1,2)^*}$-$g$-space.

**Proposition 2.3.8**

Every $(1,2)^* - T_{1/2}$-space is T$_{(1,2)^*}$-$g$-space but not conversely.

**Proof**

Follows from Proposition 1.3.14.

The converse of Proposition 2.3.8 need not be true as seen from the following example.

**Example 2.3.9**

In the Example 2.3.6, $(1,2)^* - G C(X) = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Thus X is not a $(1,2)^* - T_{1/2}$-space.
Proposition 2.3.10

Every \((1,2)^*\)-\(T_{\omega}\)-space is \(T_{(1,2)^*}\)-\(g\)-space but not conversely.

Proof

Follows from Proposition 1.3.8.

The converse of Proposition 2.3.10 need not be true as seen from the following example.

Example 2.3.11

Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, X, \{a, c\}\}\) and \(\tau_2 = \{\emptyset, X, \{b\}\}\). Then the sets in \(\{\emptyset, X, \{b\}, \{a, c\}\}\) are called \(\tau_{1,2}\)-open and the sets in \(\{\emptyset, X, \{b\}, \{a, c\}\}\) are called \(\tau_{1,2}\)-closed. Then \((1,2)^*\)-\(\omega C(X) = P(X)\) and \((1,2)^*\)-\(\tilde{G} C(X) = \{\emptyset, \{b\}, \{a, c\}, X\}\). Thus \(X\) is an \(T_{(1,2)^*}\)-\(g\)-space but not an \((1,2)^*\)-\(T_\omega\)-space.

Proposition 2.3.12

Every \((1,2)^*\)-\(\alpha\) \(T_b\)-space is \(T_{(1,2)^*}\)-\(g\)-space but not conversely.

Proof

Follows from Proposition 1.3.18.

The converse of Proposition 2.3.12 need not be true as seen from the following example.

Example 2.3.13

In the Example 2.3.6, \((1,2)^*\)-\(\alpha G C(X) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}\). Thus \(X\) is not an \((1,2)^*\)-\(\alpha\) \(T_b\)-space.
Proposition 2.3.14

Every \((1,2)*_S\text{-}T_p\)-space and \((1,2)*_T_p\)-space is \(T_{(1,2)^*}\bar{g}\)-space but not conversely.

Proof

Follows from Proposition 1.3.33 and Definition 2.3.3 (iii) and (vi).

The converse of Proposition 2.3.14 need not be true as seen from the following example.

Example 2.3.15

In the Example 2.3.6, \((1,2)*\text{-}GSP\ C(X) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X}\) and \((1,2)*\text{-}G^*P\ C(X) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}\). Thus \(X\) is neither an \((1,2)*_S\text{-}T_p\)-space nor an \((1,2)*_T_p\)-space.

Proposition 2.3.16

Every \((1,2)*\text{-}T_b\)-space is \(T_{(1,2)^*}\bar{g}\)-space but not conversely.

Proof

Follows from Proposition 1.3.20.

The converse of Proposition 2.3.16 need not be true as seen from the following example.

Example 2.3.17

In the Example 2.3.6, \((1,2)*\text{-}GSC\ C(X) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}\). Thus \(X\) is not an \((1,2)*\text{-}T_b\)-space.
Remark 2.3.18

We conclude from the next two examples that $T_{(1,2)g}$-spaces and $(1,2)*-\alpha$-spaces are independent.

Example 2.3.19

In the Example 2.3.6, $(1,2)*-\alpha C(X) = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Thus X is an $T_{(1,2)g}$-space but not an $(1,2)*-\alpha$-space.

Example 2.3.20

In the Example 2.3.7, $(1,2)*-\alpha C(X) = \{\phi, \{b\}, X\}$. Thus X is an $(1,2)*-\alpha$-space but not an $T_{(1,2)g}$-space.

Theorem 2.3.21

For a bitopological space X the following properties are equivalent:

(i) $X$ is a $T_{(1,2)g}$-space.

(ii) Every singleton subset of $X$ is either $(1,2)*$-sg-closed or $\tau_{1,2}$-open.

Proof

(i) $\Rightarrow$ (ii). Assume that for some $x \in X$, the set $\{x\}$ is not a $(1,2)*$-sg-closed in $X$. Then the only $(1,2)*$-sg-open set containing $\{x\}$ is $X$ and so $\{x\}^c$ is $(1,2)*-\bar{g}$-closed in $X$. By assumption $\{x\}^c$ is $\tau_{1,2}$-closed in $X$ or equivalently $\{x\}$ is $\tau_{1,2}$-open.

(ii) $\Rightarrow$ (i). Let $A$ be a $(1,2)*-\bar{g}$-closed subset of $X$ and let $x \in \tau_{1,2}$-cl$(A)$. By assumption $\{x\}$ is either $(1,2)*$-sg-closed or $\tau_{1,2}$-open.

Case (a) Suppose that $\{x\}$ is $(1,2)*$-sg-closed. If $x \notin A$, then $\tau_{1,2}$-cl$(A) - A$ contains a nonempty $(1,2)*$-sg-closed set $\{x\}$, which is a contradiction to Theorem 1.4.7. Therefore $x \in A$. 

31
Case (b) Suppose that \( \{x\} \) is \( \tau_{1,2}\)-open. Since \( x \in \tau_{1,2}\text{-cl}(A) \), \( \{x\} \cap A \neq \emptyset \) and so \( x \in A \). Thus in both case, \( x \in A \) and therefore \( \tau_{1,2}\text{-cl}(A) \subseteq A \) or equivalently \( A \) is a \( \tau_{1,2}\)-closed set of \( X \).

**Theorem 2.3.22**

For a bitopological space \( X \) the following properties hold:

(i) If \( X \) is \( (1,2)^*\)-sg-T\(_1\), then it is \( T_{(1,2)^*\text{-}g} \).

(ii) If \( X \) is \( T_{(1,2)^*\text{-}g} \), then it is \( (1,2)^*\)-sg-T\(_0\).

**Proof**

(i) The proof is obvious from Theorem 2.3.4.

(ii) Let \( x \) and \( y \) be two distinct elements of \( X \). Since the space \( X \) is \( T_{(1,2)^*\text{-}g} \), we have that \( \{x\} \) is \( (1,2)^*\)-sg-closed or \( \tau_{1,2}\)-open. Suppose that \( \{x\} \) is \( \tau_{1,2}\)-open. Then the singleton \( \{x\} \) is a \( (1,2)^*\)-sg-open set such that \( x \in \{x\} \) and \( y \notin \{x\} \). Also, if \( \{x\} \) is \( (1,2)^*\)-sg-closed, then \( X \setminus \{x\} \) is \( (1,2)^*\)-sg-open such that \( y \in X \setminus \{x\} \) and \( x \notin X \setminus \{x\} \). Thus, in the above two cases, there exists a \( (1,2)^*\)-sg-open set \( U \) of \( X \) such that \( x \in U \) and \( y \notin U \) or \( x \notin U \) and \( y \in U \). Thus, the space \( X \) is \( (1,2)^*\)-sg-T\(_0\).

**Theorem 2.3.23**

For a \( (1,2)^*\)-sg-R\(_0\) bitopological space \( X \) the following properties are equivalent:

(i) \( X \) is \( (1,2)^*\)-sg-T\(_0\).

(ii) \( X \) is \( T_{(1,2)^*\text{-}g} \).

(iii) \( X \) is \( (1,2)^*\)-sg-T\(_1\).
Proof

It suffices to prove only (i) $\Rightarrow$ (iii). Let $x \neq y$ and since $X$ is $(1,2)^*\text{-sg-T}_0$, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $(1,2)^*\text{-sg-open}$ set $U$. Then $x \in X \setminus (1,2)^*\text{-sg-cl}(\{y\})$ and $X \setminus (1,2)^*\text{-sg-cl}(\{y\})$ is $(1,2)^*\text{-sg-open}$. Since $X$ is $(1,2)^*\text{-sg-R}_0$, we have $(1,2)^*\text{-sg-cl}(\{x\}) \subseteq X \setminus (1,2)^*\text{-sg-cl}(\{y\}) \subseteq X \setminus \{y\}$ and hence $y \not\in (1,2)^*\text{-sg-cl}(\{x\})$. There exists $(1,2)^*\text{-sg-open}$ set $V$ such that $y \in V \subseteq X \setminus \{x\}$ and $X$ is $(1,2)^*\text{-sg-T}_1$.

2.4 $\text{	extit{g}}T_{(1,2)^*}$-SPACES

Definition 2.4.1

A bitopological space $X$ is called an $\text{	extit{g}}T_{(1,2)^*}$-space if every $(1,2)^*\text{-g-closed}$ set in it is $(1,2)^*\text{-}\text{	extit{g}}$-closed.

Example 2.4.2

In the Example 2.3.7, $X$ is an $\text{	extit{g}}T_{(1,2)^*}$-space and the space $X$ in the Example 2.3.6, is not an $\text{	extit{g}}T_{(1,2)^*}$-space.

Proposition 2.4.3

Every $(1,2)^*\text{-T}_{1/2}$-space is $\text{	extit{g}}T_{(1,2)^*}$-space but not conversely.

Proof

Follows from Proposition 1.3.2.

The converse of Proposition 2.4.3 need not be true as seen from the following example.
Example 2.4.4

In the Example 2.3.7, X is a $g T_{(1,2)^*}$-space but not an $(1,2)^*\cdot T_{1/2}$-space.

Remark 2.4.5

$T_{(1,2)^*}$-space and $g T_{(1,2)^*}$-space are independent.

Example 2.4.6

The space X in the Example 2.3.7, is a $g T_{(1,2)^*}$-space but not an $T_{(1,2)^*}$-space and the space X in the Example 2.3.6, is an $T_{(1,2)^*}$-space but not an $g T_{(1,2)^*}$-space.

Theorem 2.4.7

If X is a $g T_{(1,2)^*}$-space, then every singleton subset of X is either $(1,2)^*$-g-closed or $(1,2)^*$-g-open.

Proof

Assume that for some $x \in X$, the set $\{x\}$ is not a $(1,2)^*$-g-closed in X. Then $\{x\}$ is not a $\tau_{1,2}$-closed set, since every $\tau_{1,2}$-closed set is a $(1,2)^*$-g-closed set. So $\{x\}^c$ is not $\tau_{1,2}$-open and the only $\tau_{1,2}$-open set containing $\{x\}^c$ is X itself. Therefore $\{x\}^c$ is trivially a $(1,2)^*$-g-closed set and by assumption, $\{x\}^c$ is an $(1,2)^*$-g-closed set or equivalently $\{x\}$ is $(1,2)^*$-g-open.

The converse of Theorem 2.4.7 need not be true as seen from the following example.
Example 2.4.8

In the Example 2.3.6, the sets \{a\} and \{c\} are \(1,2^*\)-g-closed in \(X\) and the set \{b\} is \(1,2^*\)-\(g\)-open. But the space \(X\) is not an \(gT_{(1,2)^*}\)-space.

Theorem 2.4.9

A space \(X\) is \((1,2)^*\)-\(T\frac{1}{2}\) if and only if it is both \(T_{(1,2)^*}\)-\(g\) and \(gT_{(1,2)^*}\)-\(g\).

Proof

Necessity. Follows from Propositions 2.3.8 and 2.4.3.

Sufficiency. Assume that \(X\) is both \(T_{(1,2)^*}\)-\(g\) and \(gT_{(1,2)^*}\)-\(g\). Let \(A\) be a \((1,2)^*\)-g-closed subset of \(X\). Then \(A\) is \((1,2)^*\)-\(g\)-closed, since \(X\) is a \(gT_{(1,2)^*}\)-\(g\). Again since \(X\) is a \(T_{(1,2)^*}\)-\(g\), \(A\) is a \(\tau_{1,2}\)-closed set in \(X\) and so \(X\) is a \((1,2)^*\)-\(T\frac{1}{2}\).

2.5 \(\alpha T_{(1,2)^*}\)-SPACES

Definition 2.5.1

A bitopological space \(X\) is called a \(\alpha T_{(1,2)^*}\)-\(g\)-space if every \((1,2)^*\)-\(\alpha g\)-closed subset of \(X\) is \((1,2)^*\)-\(g\)-closed in \(X\).

Example 2.5.2

In the Example 2.3.7, \(X\) is an \(\alpha T_{(1,2)^*}\)-\(g\)-space and the space \(X\) in the Example 2.3.6, is not an \(\alpha T_{(1,2)^*}\)-\(g\)-space.

Proposition 2.5.3

Every \((1,2)^*\)-\(\alpha T_b\)-space is \(\alpha T_{(1,2)^*}\)-\(g\)-space but not conversely.
Proof

Follows from Proposition 1.3.2.

The converse of Proposition 2.5.3 need not be true as seen from the following example.

Example 2.5.4

In the Example 2.3.7, $X$ is an $\alpha T_{(1,2)\ast\ast}g$-space but not an $(1,2)\ast\ast\alpha T_b$-space.

Proposition 2.5.5

Every $\alpha T_{(1,2)\ast\ast}g$-space is a $(1,2)\ast\ast\alpha T_d$-space but not conversely.

Proof

Let $X$ be an $\alpha T_{(1,2)\ast\ast}g$-space and let $A$ be an $(1,2)\ast\ast\alpha g$-closed set of $X$. Then $A$ is a $(1,2)\ast\ast g$-closed subset of $X$ and by Proposition 1.3.14, $A$ is $(1,2)\ast g$-closed. Therefore $X$ is an $(1,2)\ast\ast\alpha T_d$-space.

The converse of Proposition 2.5.5 need not be true as seen from the following example.

Example 2.5.6

In the Example 2.3.6, $X$ is an $(1,2)\ast\ast\alpha T_d$-space but not an $\alpha T_{(1,2)\ast\ast}g$-space.

Theorem 2.5.7

If $(X, \tau)$ is a $\alpha T_{(1,2)\ast\ast}g$-space, then every singleton subset of $X$ is either $(1,2)\ast\ast\alpha g$-closed or $(1,2)\ast\ast g$-open.
Proof

Similar to Theorem 2.4.7.

The converse of Theorem 2.5.7 need not be true as seen from the following example.

Example 2.5.8

In the Example 2.3.6, the sets \{a\} and \{c\} are $(1,2)^*-\alpha$ g-closed in $X$ and the set \{b\} is $(1,2)^* \check{g}$-open. But the space $X$ is not an $\alpha T_{(1,2)^* \check{g}}$-space.