Chapter - 6

(1,2)*-\(\tilde{g}\) -LOCALLY CLOSED SETS

6.1 INTRODUCTION

The first step of locally closedness was done by Bourbaki [10]. He defined a set \(A\) to be locally closed if it is the intersection of an open set and a closed set. In literature many general topologists introduced the studies of locally closed sets. Extensive research on locally closedness and generalizing locally closedness were done in recent years. Stone [77] used the term FG for a locally closed set. Ganster and Reilly used locally closed sets in [28] to define LC-continuity and LC-irresoluteness. Balachandran et al [7] introduced the concept of generalized locally closed sets. Veera Kumar [88] (Sheik John [76]) introduced \(\tilde{g}\)-locally closed sets (= \(\omega\)-locally closed sets) respectively.

In this chapter, we introduce three forms of (1,2)*-locally closed sets called (1,2)*-\(\tilde{g}\) -locally closed sets, (1,2)*-\(\tilde{g}\) -lc\(^*\) sets and (1,2)*-\(\tilde{g}\) -lc\(^{**}\) sets. Properties of these new concepts are studied as well as their relations to the other classes of (1,2)*-locally closed sets will be investigated.

6.2 PRELIMINARIES

Definition 6.2.1

A subset \(A\) of a bitopological space \(X\) is called

(i) regular (1,2)*-open [62] if \(A = \tau_{1,2}\)-int(\(\tau_{1,2}\)-cl(\(A\))).

(ii) (1,2)*-regular generalized closed (briefly, (1,2)*-rg-closed) set [66] if \(\tau_{1,2}\)-cl(\(A\)) \(\subseteq\) \(U\) whenever \(A \subseteq U\) and \(U\) is regular (1,2)*-open in \(X\). The complement of (1,2)*-rg-closed set is called (1,2)*-rg-open set.
Remark 6.2.2

The collection of all \((1,2)^{-}\text{rg}\)-closed sets in \(X\) is denoted by \((1,2)^{-} \text{RG} C(X)\).

The collection of all \((1,2)^{-}\text{rg}\)-open sets in \(X\) is denoted by \((1,2)^{-} \text{RG} O(X)\).

We denote the power set of \(X\) by \(P(X)\).

Corollary 6.2.3

If \(A\) is a \((1,2)^{-}\text{rg}\)-closed set and \(F\) is a \(\tau_{1,2}\)-closed set, then \(A \cap F\) is a \((1,2)^{-}\text{rg}\)-closed set.

6.3 \((1,2)^{-}\text{rg}\)-LOCALLY CLOSED SETS

We introduce the following definition.

Definition 6.3.1

A subset \(S\) of a bitopological space \(X\) is called

(i) \((1,2)^{-}\text{-locally closed (briefly, (1,2)^{-}-lc)}\) if \(S = U \cap F\), where \(U\) is \(\tau_{1,2}\)-open and \(F\) is \(\tau_{1,2}\)-closed in \(X\).

(ii) \((1,2)^{-}\text{-generalized locally closed (briefly, (1,2)^{-}-glc)}\) if \(S = U \cap F\), where \(U\) is \((1,2)^{-}\text{-g-open}\) and \(F\) is \((1,2)^{-}\text{-g-closed}\) in \(X\).

(iii) \((1,2)^{-}\text{-semi-generalized locally closed (briefly, (1,2)^{-}-sglc)}\) if \(S = U \cap F\), where \(U\) is \((1,2)^{-}\text{-sg-open}\) and \(F\) is \((1,2)^{-}\text{-sg-closed}\) in \(X\).

(iv) \((1,2)^{-}\text{-regular-generalized locally closed (briefly, (1,2)^{-}-rg-lc)}\) if \(S = U \cap F\), where \(U\) is \((1,2)^{-}\text{-rg-open}\) and \(F\) is \((1,2)^{-}\text{-rg-closed}\) in \(X\).

(v) \((1,2)^{-}\text{-generalized locally (1,2)^{-}-semi-closed (briefly, (1,2)^{-}-glsc)}\) if \(S = U \cap F\), where \(U\) is \((1,2)^{-}\text{-g-open}\) and \(F\) is \((1,2)^{-}\text{-semi-closed}\) in \(X\).
(vi) $(1,2)^*$-locally semi-closed (briefly, $(1,2)^*$-lsc) if $S = U \cap F$, where $U$ is $\tau_{1,2}$-open and $F$ is $(1,2)^*$-semi-closed in $X$.

(vii) $(1,2)^*$-$\alpha$-locally closed (briefly, $(1,2)^*$-$\alpha$-lc) if $S = U \cap F$, where $U$ is $(1,2)^*$-$\alpha$-open and $F$ is $(1,2)^*$-$\alpha$-closed in $X$.

(viii) $(1,2)^*$-$\omega$-locally closed (briefly, $(1,2)^*$-$\omega$-lc) if $S = U \cap F$, where $U$ is $(1,2)^*$-$\omega$-open and $F$ is $(1,2)^*$-$\omega$-closed in $X$.

(ix) $(1,2)^*$-$sglc^*$ if $S = U \cap F$, where $U$ is $(1,2)^*$-sg-open and $F$ is $\tau_{1,2}$-closed in $X$.

The class of all $(1,2)^*$-locally closed (resp. $(1,2)^*$-generalized locally closed, $(1,2)^*$-generalized locally semi-closed, $(1,2)^*$-locally semi-closed, $(1,2)^*$-$\omega$-locally closed) sets in $X$ is denoted by $(1,2)^*$-$LC(X)$ (resp. $(1,2)^*$-$GLC(X)$, $(1,2)^*$-$GLSC(X)$, $(1,2)^*$-$LSC(X)$, $(1,2)^*$-$\omega$-$LC(X)$).

**Definition 6.3.2**

A subset of a bitopological space $X$ is called $(1,2)^*$-$\ddot{g}$-locally closed (briefly, $(1,2)^*$-$\ddot{g}$-lc) if $A = S \cap G$, where $S$ is $(1,2)^*$-$\ddot{g}$-open and $G$ is $(1,2)^*$-$\ddot{g}$-closed in $X$.

The class of all $(1,2)^*$-$\ddot{g}$-locally closed sets in $X$ is denoted by $(1,2)^*$-$\ddot{GLC}(X)$.

**Proposition 6.3.3**

Every $(1,2)^*$-$\ddot{g}$-closed (resp. $(1,2)^*$-$\ddot{g}$-open) set is $(1,2)^*$-$\ddot{g}$-lc set but not conversely.

**Proof**

It follows from Definition 6.3.2.
Example 6.3.4

Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{b\}\} \) and \( \tau_2 = \{\phi, X\} \). Then the sets in \( \{\phi, X, \{b\}\} \) are called \( \tau_{1,2} \)-open and the sets in \( \{\phi, X, \{a, c\}\} \) are called \( \tau_{1,2} \)-closed. Then the set \( \{b\} \) is \( (1,2)^*\)-\( g \)-lc set but it is not \( (1,2)^*\)-\( g \)-closed and the set \( \{a, c\} \) is \( (1,2)^*\)-\( g \)-lc set but it is not \( (1,2)^*\)-\( g \)-open in \( X \).

Proposition 6.3.5

Every \( (1,2)^*\)-lc set is \( (1,2)^*\)-\( g \)-lc set but not conversely.

Proof

It follows from Proposition 1.3.2.

Example 6.3.6

Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{b, c\}\} \) and \( \tau_2 = \{\phi, X\} \). Then the sets in \( \{\phi, X, \{b, c\}\} \) are called \( \tau_{1,2} \)-open and the sets in \( \{\phi, X, \{a\}\} \) are called \( \tau_{1,2} \)-closed. Then the set \( \{b\} \) is \( (1,2)^*\)-\( g \)-lc set but it is not \( (1,2)^*\)-lc set in \( X \).

Proposition 6.3.7

Every \( (1,2)^*\)-\( g \)-lc set is a (i) \( (1,2)^*\)-\( \omega \)-lc set, (ii) \( (1,2)^*\)-g lc set and (iii) \( (1,2)^*\)-sglc set. However the separate converses are not true.

Proof

It follows from Propositions 1.3.8, 1.3.14 and 1.3.22.

Example 6.3.8

Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{a\}\} \) and \( \tau_2 = \{\phi, X\} \). Then the sets in \( \{\phi, X, \{a\}\} \) are called \( \tau_{1,2} \)-open and the sets in \( \{\phi, X, \{b, c\}\} \) are called \( \tau_{1,2} \)-closed. Then
the set \{b\} is (1,2)*-g-lc set but it is not (1,2)*-\ddot{g}-lc set in X. Moreover, the set \{c\} is (1,2)*-sg-lc set but it is not (1,2)*-\ddot{g}-lc set in X.

**Example 6.3.9**

Let \(X = \{a, b, c\}, \tau_1 = \{\phi, X, \{b\}\}\) and \(\tau_2 = \{\phi, X, \{a, c\}\}\). Then the sets in \{\phi, X, \{b\}, \{a, c\}\} are called \(\tau_{1,2}\)-open and the sets in \{\phi, X, \{b\}, \{a, c\}\} are called \(\tau_{1,2}\)-closed. Then the set \{a\} is (1,2)*-\(\omega\)-lc set but it is not (1,2)*-\(\ddot{g}\)-lc set in X.

**Remark 6.3.10**

The concepts of (1,2)*-\(\alpha\)-lc sets and (1,2)*-\(\ddot{g}\)-lc sets are independent of each other.

**Example 6.3.11**

The set \{b, c\} in Example 6.3.4 is (1,2)*-\(\alpha\)-lc set but it is not a (1,2)*-\(\ddot{g}\)-lc set in X and the set \{a, b\} in Example 6.3.6 is (1,2)*-\(\ddot{g}\)-lc set but it is not an (1,2)*-\(\alpha\)-lc set in X.

**Remark 6.3.12**

The concepts of (1,2)*-lsc sets and (1,2)*-\(\ddot{g}\)-lc sets are independent of each other.

**Example 6.3.13**

The set \{a\} in Example 6.3.4 is (1,2)*-lsc set but it is not a (1,2)*-\(\ddot{g}\)-lc set in X and the set \{a, b\} in Example 6.3.6 is (1,2)*-\(\ddot{g}\)-lc set but it is not a (1,2)*-lsc set in X.
Remark 6.3.14

The concepts of \((1,2)^*\)-\(\bar{g}\)-lc sets and \((1,2)^*\)-glsc sets are independent of each other.

Example 6.3.15

The set \(\{b, c\}\) in Example 6.3.4 is \((1,2)^*\)-glsc set but it is not a \((1,2)^*\)-\(\bar{g}\)-lc set in \(X\) and the set \(\{a, b\}\) in Example 6.3.6 is \((1,2)^*\)-\(\bar{g}\)-lc set but it is not a \((1,2)^*\)-glsc set in \(X\).

Remark 6.3.16

The concepts of \((1,2)^*\)-\(\bar{g}\)-lc sets and \((1,2)^*\)-sglc* sets are independent of each other.

Example 6.3.17

The set \(\{b, c\}\) in Example 6.3.4 is \((1,2)^*\)-sglc* set but it is not a \((1,2)^*\)-\(\bar{g}\)-lc set in \(X\) and the set \(\{a, b\}\) in Example 6.3.6 is \((1,2)^*\)-\(\bar{g}\)-lc set but it is not a \((1,2)^*\)-sglc* set in \(X\).

Theorem 6.3.18

For a \(T_{(1,2)^*}\)-\(\bar{g}\)-space \(X\), the following properties hold:

(i) \((1,2)^*\)-\(\bar{G}LC\) (\(X\)) = \((1,2)^*\)-\(LC\) (\(X\)).

(ii) \((1,2)^*\)-\(\bar{G}LC\) (\(X\)) \(\subseteq\) \((1,2)^*\)-GLC (\(X\)).

(iii) \((1,2)^*\)-\(\bar{G}LC\) (\(X\)) \(\subseteq\) \((1,2)^*\)-GLSC (\(X\)).

(iv) \((1,2)^*\)-\(\bar{G}LC\) (\(X\)) \(\subseteq\) \((1,2)^*\)-\(\omega\)-\(LC\) (\(X\)).
Proof

(i) Since every \((1,2)*-\tilde{g}\)-open set is \(\tau_{1,2}\)-open and every \((1,2)*-\tilde{g}\)-closed set is \(\tau_{1,2}\)-closed in \(X\), \((1,2)*-\tilde{G}LC(X) \subseteq (1,2)*-LC(X)\) and hence \((1,2)*-\tilde{G}LC(X) = (1,2)*-LC(X)\).

(ii), (iii) and (iv) follows from (i), since for any space \((X, \tau)\), \((1,2)*-LC(X) \subseteq (1,2)*-GLC(X)\), \((1,2)*-LC(X) \subseteq (1,2)*-GLSC(X)\) and \((1,2)*-LC(X) \subseteq (1,2)*-\omega-LC(X)\).

Corollary 6.3.19

If \((1,2)*-G O(X) = (1,2)*-O(X)\) where \((1,2)*-O(X)\) is the collection of all \(\tau_{1,2}\)-open subsets of \(X\), then \((1,2)*-\tilde{G}LC(X) \subseteq (1,2)*-GLSC(X) \subseteq (1,2)*-LSC(X)\).

Proof

\((1,2)*-G O(X) = (1,2)*-O(X)\) implies that \(X\) is a \(T_{(1,2)*-\tilde{g}}\)-space and hence by Theorem 6.3.18, \((1,2)*-\tilde{G}LC(X) \subseteq (1,2)*-GLSC(X)\). Let \(A \in (1,2)*-GLSC(X)\). Then \(A = U \cap F\), where \(U\) is \((1,2)*-g\)-open and \(F\) is \((1,2)*\)-semi-closed. By hypothesis, \(U\) is \(\tau_{1,2}\)-open and hence \(A\) is a \((1,2)*\)-lsc set and so \(A \in (1,2)*-LSC(X)\).

Definition 6.3.20

A subset \(A\) of a bitopological space \(X\) is called

(i) \((1,2)*-\tilde{g}-lc^*\) set if \(A = S \cap G\), where \(S\) is \((1,2)*-\tilde{g}\)-open in \(X\) and \(G\) is \(\tau_{1,2}\)-closed in \(X\).

(ii) \((1,2)*-\tilde{g}-lc^{**}\) set if \(A = S \cap G\), where \(S\) is \(\tau_{1,2}\)-open in \(X\) and \(G\) is \((1,2)*-\tilde{g}\)-closed in \(X\).
The class of all \((1,2)^*\)-lc set (resp. \((1,2)^*\)-lc set) sets in a bitopological space \(X\) is denoted by \((1,2)^*\)-GLC (resp. \((1,2)^*\)-GLC) (X).

**Proposition 6.3.21**

Every \((1,2)^*\)-lc set is \((1,2)^*\)-lc set but not conversely.

**Proof**

It follows from Definitions 6.3.1 (i) and 6.3.20 (i).

**Example 6.3.22**

The set \(\{b\}\) in Example 6.3.6 is \((1,2)^*\)-lc set but it is not a \((1,2)^*\)-lc set in \(X\).

**Proposition 6.3.23**

Every \((1,2)^*\)-lc set is \((1,2)^*\)-lc set but not conversely.

**Proof**

It follows from Definitions 6.3.1 (i) and 6.3.20 (ii).

**Example 6.3.24**

The set \(\{a, c\}\) in Example 6.3.6 is \((1,2)^*\)-lc set but it is not a \((1,2)^*\)-lc set in \(X\).

**Proposition 6.3.25**

Every \((1,2)^*\)-lc set is \((1,2)^*\)-lc set but not conversely.

**Proof**

It follows from Definitions 6.3.2 and 6.3.20 (i).
Example 6.3.26

The set \{a, b\} in Example 6.3.6 is \((1,2)^*\)-\(\mathcal{g}\)-lc set but it is not a \((1,2)^*\)-\(\mathcal{g}\)-lc* set in X.

Proposition 6.3.27

Every \((1,2)^*\)-\(\mathcal{g}\)-lc** set is \((1,2)^*\)-\(\mathcal{g}\)-lc set but not conversely.

Proof

It follows from Definitions 6.3.2 and 6.3.20 (ii).

Remark 6.3.28

The concepts of \((1,2)^*\)-\(\mathcal{g}\)-lc* sets and \((1,2)^*\)-lsc sets are independent of each other.

Example 6.3.29

The set \{c\} in Example 6.3.6 is \((1,2)^*\)-\(\mathcal{g}\)-lc* set but it is not a \((1,2)^*\)-lsc set in X and the set \{a\} in Example 6.3.4 is \((1,2)^*\)-lsc set but it is not a \((1,2)^*\)-\(\mathcal{g}\)-lc* set in X.

Remark 6.3.30

The concepts of \((1,2)^*\)-\(\mathcal{g}\)-lc** sets and \((1,2)^*\)-\(\alpha\)-lc sets are independent of each other.

Example 6.3.31

The set \{a, b\} in Example 6.3.6 is \((1,2)^*\)-\(\mathcal{g}\)-lc** set but it is not a \((1,2)^*\)-\(\alpha\)-lc set in X and the set \{a, b\} in Example 6.3.4 is \((1,2)^*\)-\(\alpha\)-lc set but it is not a \((1,2)^*\)-\(\mathcal{g}\)-lc** set in X.
Remark 6.3.32

From the above discussions we have the following implications where $A \Rightarrow B$ (resp. $A \Leftarrow \Rightarrow B$) represents $A$ implies $B$ but not conversely (resp. $A$ and $B$ are independent of each other).

\[
\begin{align*}
(1,2)^*-\overset{\cdot}{g}^-\text{lc} & \quad \leftrightarrow \quad (1,2)^*\overset{\cdot}{g}^\leftrightarrow\text{lc} \\
(1,2)^*\text{lsc} & \quad \leftrightarrow \quad (1,2)^*\overset{\cdot}{g}^-\text{lc} \\
(1,2)^*\text{glsc} & \quad \leftrightarrow \quad (1,2)^*\text{glc} \\
(1,2)^*\omega^-\text{lc} & \quad \leftrightarrow \quad (1,2)^*\text{sglc} \\
(1,2)^*\alpha^-\text{lc} & \quad \leftrightarrow \quad (1,2)^*\text{sglc}^\leftrightarrow
\end{align*}
\]

Proposition 6.3.33

If $(1,2)^*G\text{O}(X) = (1,2)^*\overset{\cdot}{G}\text{O}(X)$, then $(1,2)^*\overset{\cdot}{G}\text{LC}(X) = (1,2)^*\overset{\cdot}{G}\text{LC}^\leftrightarrow(X) = (1,2)^*\overset{\cdot}{G}\text{LC}^{\leftrightarrow\leftrightarrow}(X)$.

Proof

Since $(1,2)^*\overset{\cdot}{G}\text{O}(X) \subseteq (1,2)^*G\text{O}(X) = (1,2)^*\overset{\cdot}{G}\text{O}(X)$, therefore by hypothesis, $X$ is a $T_{(1,2)^*\overset{\cdot}{g}^-}\text{-space}$ and hence $(1,2)^*\overset{\cdot}{G}\text{LC}(X) = (1,2)^*\overset{\cdot}{G}\text{LC}^\leftrightarrow(X) = (1,2)^*\overset{\cdot}{G}\text{LC}^{\leftrightarrow\leftrightarrow}(X)$.

Remark 6.3.34

The converse of Proposition 6.3.33 need not be true.
For the bitopological space $X$ in Example 6.3.4, $(1,2)^\ast-\bar{G}LC \ (X) = (1,2)^\ast-\bar{G}LC^\ast \ (X)$. However $(1,2)^\ast-G \ O(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \ X\} \neq (1,2)^\ast-O(X)$.

**Proposition 6.3.35**

Let $X$ be a bitopological space. If $(1,2)^\ast-G \ O(X) \subseteq (1,2)^\ast-LC \ (X)$, then $(1,2)^\ast-\bar{G}LC \ (X) = (1,2)^\ast-\bar{G}LC^\ast \ (X)$.

**Proof**

Let $A \in (1,2)^\ast-\bar{G}LC \ (X)$. Then $A = S \cap G$ where $S$ is $(1,2)^\ast-\bar{g}$-open and $G$ is $(1,2)^\ast-\bar{g}$-closed. Since $(1,2)^\ast-\bar{G}O(X) \subseteq (1,2)^\ast-\ G \ O(X)$ and by hypothesis $(1,2)^\ast-G \ O(X) \subseteq (1,2)^\ast-LC \ (X)$, $S$ is $(1,2)^\ast$-locally closed. Then $S = P \cap Q$, where $P$ is $\tau_{1,2}$-open and $Q$ is $\tau_{1,2}$-closed. Therefore, $A = P \cap (Q \cap G)$. By Corollary 6.2.3, $Q \cap G$ is $(1,2)^\ast-\bar{g}$-closed and hence $A \in (1,2)^\ast-\bar{G}LC^\ast \ (X)$. That is $(1,2)^\ast-\bar{G}LC \ (X) \subseteq (1,2)^\ast-\bar{G}LC^\ast \ (X)$. For any bitopological space, $(1,2)^\ast-\bar{G}LC^\ast \ (X) \subseteq (1,2)^\ast-\bar{G}LC \ (X)$ and so $(1,2)^\ast-\bar{G}LC \ (X) = (1,2)^\ast-\bar{G}LC^\ast \ (X)$.

**Remark 6.3.36**

The converse of Proposition 6.3.35 need not be true in general.

For the bitopological space $X$ in Example 6.3.4, then $(1,2)^\ast-\bar{G}LC \ (X) = (1,2)^\ast-\bar{G}LC^\ast \ (X) = \{\phi, \{b\}, \{a, c\}, \ X\}$. But $(1,2)^\ast-G \ O(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \ X\} \subseteq (1,2)^\ast-LC \ (X) = \{\phi, \{b\}, \{a, c\}, \ X\}$.

**Corollary 6.3.37**

Let $X$ be a bitopological space. If $(1,2)^\ast-\omega \ O(X) \subseteq (1,2)^\ast-LC \ (X)$, then $(1,2)^\ast-\bar{G}LC \ (X) = (1,2)^\ast-\bar{G}LC^\ast \ (X)$. 

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Proof

It follows from the fact that \((1,2)^*\omega O(X) \subseteq (1,2)^*G O(X)\) and Proposition 6.3.35.

Remark 6.3.38

The converse of Corollary 6.3.37 need not be true in general.

For the bitopological space \(X\) in Example 6.3.6, then \((1,2)^*\tilde{G}LC (X) = (1,2)^*\tilde{G}LC^{**} (X) = P(X)\). But \((1,2)^*\omega O(X) = P(X) \subseteq (1,2)^*LC (X) = \{\emptyset, \{a\}, \{b, c\}, X\}\).

The following theorems are exploring the characterizations of \((1,2)^*\tilde{g} - lc\) sets, \((1,2)^*\tilde{g} - lc^*\) sets and \((1,2)^*\tilde{g} - lc^{**}\) sets.

Theorem 6.3.39

For a subset \(A\) of \(X\) the following statements are equivalent:

(i) \(A \in (1,2)^*\tilde{G}LC (X)\),

(ii) \(A = S \cap (1,2)^*\tilde{g} - cl(A)\) for some \((1,2)^*\tilde{g} - open\) set \(S\),

(iii) \((1,2)^*\tilde{g} - cl(A) - A\) is \((1,2)^*\tilde{g} - closed\),

(iv) \(A \cup ((1,2)^*\tilde{g} - cl(A))^c\) is \((1,2)^*\tilde{g} - open\),

(v) \(A \subseteq (1,2)^*\tilde{g} - int( A \cup ((1,2)^*\tilde{g} - cl(A))^c)\).

Proof

(i) \(\Rightarrow\) (ii). Let \(A \in (1,2)^*\tilde{G}LC (X)\). Then \(A = S \cap G\) where \(S\) is \((1,2)^*\tilde{g} - open\) and \(G\) is \((1,2)^*\tilde{g} - closed\). Since \(A \subseteq G\), \((1,2)^*\tilde{g} - cl(A) \subseteq G\) and so \(S \cap (1,2)^*\tilde{g} - cl(A) \subseteq \epsilon G\).
A. Also $A \subseteq S$ and $A \subseteq (1,2)^* - \bar{g} - \text{cl}(A)$ implies $A \subseteq S \cap (1,2)^* - \bar{g} - \text{cl}(A)$ and therefore $A = S \cap (1,2)^* - \bar{g} - \text{cl}(A)$.

(ii) $\Rightarrow$ (iii). $A = S \cap (1,2)^* - \bar{g} - \text{cl}(A)$ implies $(1,2)^* - \bar{g} - \text{cl}(A) - A = (1,2)^* - \bar{g} - \text{cl}(A) \cap S^c$ which is $(1,2)^* - \bar{g}$ -closed since $S^c$ is $(1,2)^* - \bar{g}$ - closed and $(1,2)^* - \bar{g} - \text{cl}(A)$ is $(1,2)^* - \bar{g}$ -closed.

(iii) $\Rightarrow$ (iv). $A \cup ((1,2)^* - \bar{g} - \text{cl}(A))^c = ((1,2)^* - \bar{g} - \text{cl}(A) - A)^c$ and by assumption, $((1,2)^* - \bar{g} - \text{cl}(A) - A)^c$ is $(1,2)^* - \bar{g}$ -open and so is $A \cup ((1,2)^* - \bar{g} - \text{cl}(A))^c$.

(iv) $\Rightarrow$ (v). By assumption, $A \cup ((1,2)^* - \bar{g} - \text{cl}(A))^c = (1,2)^* - \bar{g} - \text{int}(A \cup ((1,2)^* - \bar{g} - \text{cl}(A))^c)$ and hence $A \subseteq (1,2)^* - \bar{g} - \text{int}(A \cup ((1,2)^* - \bar{g} - \text{cl}(A))^c)$.

(v) $\Rightarrow$ (i). By assumption and since $A \subseteq (1,2)^* - \bar{g} - \text{cl}(A)$, $A = (1,2)^* - \bar{g} - \text{int}(A \cup ((1,2)^* - \bar{g} - \text{cl}(A))^c) \cap (1,2)^* - \bar{g} - \text{cl}(A)$. Therefore, $A \in (1,2)^* - \bar{G}LC(X)$.

**Theorem 6.3.40**

For a subset $A$ of $X$, the following statements are equivalent:

(i) $A \in (1,2)^* - \bar{G}LC^* (X)$,

(ii) $A = S \cap \tau_{1,2} - \text{cl}(A)$ for some $(1,2)^* - \bar{g}$ -open set $S$,

(iii) $\tau_{1,2} - \text{cl}(A) - A$ is $(1,2)^* - \bar{g}$ -closed,

(iv) $A \cup (\tau_{1,2} - \text{cl}(A))^c$ is $(1,2)^* - \bar{g}$ -open.

**Proof**

(i) $\Rightarrow$ (ii). Let $A \in (1,2)^* - \bar{G}LC^* (X)$. There exist an $(1,2)^* - \bar{g}$ -open set $S$ and a $\tau_{1,2}$-closed set $G$ such that $A = S \cap G$. Since $A \subseteq S$ and $A \subseteq \tau_{1,2} - \text{cl}(A)$, $A \subseteq S \cap \tau_{1,2} - \text{cl}(A)$. Also since $\tau_{1,2} - \text{cl}(A) \subseteq G$, $S \cap \tau_{1,2} - \text{cl}(A) \subseteq S \cap G = A$. Therefore $A = S \cap \tau_{1,2} - \text{cl}(A)$. 

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(ii) ⇒ (i). Since $S$ is $(1,2)*$-\(\bar{g}\)$-open and $\tau_{1,2}$-cl$(A)$ is a $\tau_{1,2}$-closed set, $A = S \cap \tau_{1,2}$-cl$(A) \in (1,2)*$-\(\bar{G}LC\)$(X)$.

(ii) ⇒ (iii). Since $\tau_{1,2}$-cl$(A) - A = \tau_{1,2}$-cl$(A) \cap S^c$, $\tau_{1,2}$-cl$(A) - A$ is $(1,2)*$-\(\bar{g}\)$-closed by Corollary 6.2.3.

(iii) ⇒ (ii). Let $S = (\tau_{1,2}$-cl$(A) - A)^c$. Then by assumption $S$ is $(1,2)*$-\(\bar{g}\)$-open in $(X, \tau)$ and $A = S \cap \tau_{1,2}$-cl$(A)$.

(iii) ⇒ (iv). Let $G = \tau_{1,2}$-cl$(A) - A$. Then $G^c = A \cup (\tau_{1,2}$-cl$(A))^c$ and $A \cup (\tau_{1,2}$-cl$(A))^c$ is $(1,2)*$-\(\bar{g}\)$-open.

(iv) ⇒ (iii). Let $S = A \cup (\tau_{1,2}$-cl$(A))^c$. Then $S^c$ is $(1,2)*$-\(\bar{g}\)$-closed and $S^c = \tau_{1,2}$-cl$(A) - A$ and so $\tau_{1,2}$-cl$(A) - A$ is $(1,2)*$-\(\bar{g}\)$ closed.

**Theorem 6.3.41**

Let $A$ be a subset of $X$. Then $A \in (1,2)*$-\(\bar{G}LC\)$(X)$ if and only if $A = S \cap (1,2)*$-\(\bar{g}\)$-cl$(A)$ for some $\tau_{1,2}$-open set $S$.

**Proof**

Let $A \in (1,2)*$-\(\bar{G}LC\)$(X)$. Then $A = S \cap G$ where $S$ is $\tau_{1,2}$-open and $G$ is $(1,2)*$-\(\bar{g}\)$-closed. Since $A \subseteq G$, $(1,2)*$-\(\bar{g}\)$-cl$(A) \subseteq G$. We obtain $A = A \cap (1,2)*$-\(\bar{g}\)$-cl$(A) = S \cap G \cap (1,2)*$-\(\bar{g}\)$-cl$(A) = S \cap (1,2)*$-\(\bar{g}\)$-cl$(A)$.

Converse part is trivial.

**Corollary 6.3.42**

Let $A$ be a subset of $X$. If $A \in (1,2)*$-\(\bar{G}LC\)$(X)$, then $(1,2)*$-\(\bar{g}\)$-cl$(A) - A$ is $(1,2)*$-\(\bar{g}\)$-closed and $A \cup ((1,2)*$-\(\bar{g}\)$-cl$(A))^c$ is $(1,2)*$-\(\bar{g}\)$-open.
Proof

Let $A \in (1,2)^*\-\mathcal{G}^\ast (X)$. Then by Theorem 6.3.41, $A = S \cap (1,2)^*\-\mathcal{G} \- \text{cl}(A)$ for some $\tau_{1,2}$-open set $S$ and $(1,2)^*\-\mathcal{G} \- \text{cl}(A) - A = (1,2)^*\-\mathcal{G} \- \text{cl}(A) \cap S^c$ is $(1,2)^*\-\mathcal{G}$-closed in $X$. If $G = (1,2)^*\-\mathcal{G} \- \text{cl}(A) - A$, then $G^c = A \cup ((1,2)^*\-\mathcal{G} \- \text{cl}(A))^c$ and $G^c$ is $(1,2)^*\-\mathcal{G}$-open and so is $A \cup (\mathcal{G} \- \text{cl}(A))^c$.

4. $(1,2)^*\-\mathcal{G}$-DENSE SETS AND $(1,2)^*\-\mathcal{G}$-SUBMAXIMAL SPACES

We introduce the following definition.

Definition 6.4.1

A subset $A$ of a space $X$ is called $(1,2)^*\-\mathcal{G}$-dense if $(1,2)^*\-\mathcal{G} \- \text{cl}(A) = X$.

Example 6.4.2

Consider the bitopological space $X$ in Example 6.3.6. Then the set $A = \{b, c\}$ is $(1,2)^*\-\mathcal{G}$-dense in $X$.

Recall that a subset $A$ of a space $X$ is called $(1,2)^*$-dense if $\tau_{1,2} \- \text{cl}(A) = X$.

Proposition 6.4.3

Every $(1,2)^*\-\mathcal{G}$-dense set is $(1,2)^*$-dense.

Proof

Let $A$ be an $(1,2)^*\-\mathcal{G}$-dense set in $X$. Then $(1,2)^*\-\mathcal{G} \- \text{cl}(A) = X$. Since $(1,2)^*\-\mathcal{G} \- \text{cl}(A) \subseteq \tau_{1,2} \- \text{cl}(A)$, we have $\tau_{1,2} \- \text{cl}(A) = X$ and so $A$ is $(1,2)^*$-dense.

The converse of Proposition 6.4.3 need not be true as can be seen from the following example.
Example 6.4.4

The set \{a, c\} in Example 6.3.6 is a \((1,2)^*\)-dense in \(X\) but it is not \((1,2)^*\)-\(\tilde{g}\)-dense in \(X\).

Definition 6.4.5

A bitopological space \(X\) is called

(i) \((1,2)^*\)-submaximal if every \((1,2)^*\)-dense subset is \(\tau_{1,2}\)-open.

(ii) \((1,2)^*\)-\(\tilde{g}\) (or \((1,2)^*\)-\(\omega\))-submaximal if every \((1,2)^*\)-dense subset is \((1,2)^*\)-\(\omega\)-open.

(iii) \((1,2)^*\)-\(g\)-submaximal if every \((1,2)^*\)-dense subset is \((1,2)^*\)-\(g\)-open.

(iv) \((1,2)^*\)-\(rg\)-submaximal if every \((1,2)^*\)-dense subset is \((1,2)^*\)-\(rg\)-open.

Proposition 6.4.6

Let \(X\) be a bitopological space.

(i) If \(X\) is \((1,2)^*\)-submaximal, then \(X\) is \((1,2)^*\)-\(\tilde{g}\)-submaximal.

(ii) If \(X\) is \((1,2)^*\)-\(\tilde{g}\)-submaximal, then \(X\) is \((1,2)^*\)-\(g\)-submaximal.

(iii) If \(X\) is \((1,2)^*\)-\(g\)-submaximal, then \(X\) is \((1,2)^*\)-\(rg\)-submaximal.

(iv) The respective converses of the above need not be true in general.

Definition 6.4.7

A bitopological space \(X\) is called \((1,2)^*\)-\(\tilde{g}\)-submaximal if every \((1,2)^*\)-dense subset in \(X\) is \((1,2)^*\)-\(\tilde{g}\)-open in \(X\).

Proposition 6.4.8

Every \((1,2)^*\)-submaximal space is \((1,2)^*\)-\(\tilde{g}\)-submaximal.
Proof

Let X be a (1,2)*-submaximal space and A be a (1,2)*-dense subset of X. Then A is $\tau_{1,2}$-open. But every $\tau_{1,2}$-open set is (1,2)*-$\tilde{g}$-open and so A is (1,2)*-$\tilde{g}$-open. Therefore X is (1,2)*-$\tilde{g}$-submaximal.

The converse of Proposition 6.4.8 need not be true as can be seen from the following example.

Example 6.4.9

For the bitopological space X of Example 6.3.6, every (1,2)*-dense subset is (1,2)*-$\tilde{g}$-open and hence X is (1,2)*-$\tilde{g}$-submaximal. However, the set $A = \{a, b\}$ is (1,2)*-dense in X, but it is not $\tau_{1,2}$-open in X. Therefore X is not (1,2)*-submaximal.

Proposition 6.4.10

Every (1,2)*-$\tilde{g}$-submaximal space is (1,2)*-$\omega$-submaximal.

Proof

Let X be an (1,2)*-$\tilde{g}$-submaximal space and A be a (1,2)*-dense subset of X. Then A is (1,2)*-$\tilde{g}$-open. But every (1,2)*-$\tilde{g}$-open set is (1,2)*-$\omega$-open [Remark 3.3.2 (iv)] and so A is (1,2)*-$\omega$-open. Therefore is X is (1,2)*-$\omega$-submaximal.

The converse of Proposition 6.4.10 need not be true as can be seen from the following example.

Example 6.4.11

Consider the bitopological space X in Example 6.3.10. Then X is (1,2)*-$\omega$-submaximal but it is not (1,2)*-$\tilde{g}$-submaximal, because the set $A = \{b, c\}$ is a (1,2)*-dense set in X but it is not (1,2)*-$\tilde{g}$-open in X.
Remark 6.4.12

From Propositions 6.4.6, 6.4.8 and 6.4.10, we have the following diagram:

\[(1,2)^*\text{-submaximal} \rightarrow (1,2)^*\text{-}g\text{-submaximal} \rightarrow (1,2)^*\text{-}\omega\text{-submaximal}\]

\[(1,2)^*\text{-rg-submaximal} \leftarrow (1,2)^*\text{-g-submaximal}\]

Theorem 6.4.13

A space \((X, \tau)\) is \((1,2)^*\text{-}g\text{-submaximal}\) if and only if \(P(X) = (1,2)^*\text{-}\bar{GLC}^* (X)\).

Proof

Necessity. Let \(A \in P(X)\) and let \(V = A \cup (\tau_{1,2}\text{-cl}(A))^c\). This implies that \(\tau_{1,2}\text{-cl}(V) = \tau_{1,2}\text{-cl}(A) \cup (\tau_{1,2}\text{-cl}(A))^c = X\). Hence \(\tau_{1,2}\text{-cl}(V) = X\). Therefore \(V\) is a \((1,2)^*\text{-dense}\) subset of \(X\). Since \(X\) is \((1,2)^*\text{-}g\text{-submaximal}\), \(V\) is \((1,2)^*\text{-}g\text{-open}\). Thus \(A \cup (\tau_{1,2}\text{-cl}(A))^c\) is \((1,2)^*\text{-}g\text{-open}\) and by Theorem 6.3.40, we have \(A \in (1,2)^*\text{-}\bar{GLC}^* (X)\).

Sufficiency. Let \(A\) be a \((1,2)^*\text{-dense}\) subset of \(X\). This implies \(A \cup (\tau_{1,2}\text{-cl}(A))^c = A \cup X^c = A \cup \phi = A\). Now \(A \in (1,2)^*\text{-}\bar{GLC}^* (X)\) implies that \(A = A \cup (\tau_{1,2}\text{-cl}(A))^c\) is \((1,2)^*\text{-}g\text{-open}\) by Theorem 6.3.40. Hence \(X\) is \((1,2)^*\text{-}g\text{-submaximal}\).

Proposition 6.4.14

Assume that \((1,2)^*\text{-}\bar{GC} (X)\) forms a topology. For subsets \(A\) and \(B\) in \(X\), the following are true:

(i) If \(A, B \in (1,2)^*\text{-}\bar{GLC} (X)\), then \(A \cap B \in (1,2)^*\text{-}\bar{GLC} (X)\).

(ii) If \(A, B \in (1,2)^*\text{-}\bar{GLC}^* (X)\), then \(A \cap B \in (1,2)^*\text{-}\bar{GLC}^* (X)\).

(iii) If \(A, B \in (1,2)^*\text{-}\bar{GLC}^{**} (X)\), then \(A \cap B \in (1,2)^*\text{-}\bar{GLC}^{**} (X)\).
(iv) If $A \in (1,2)^* \mathcal{GLC}(X)$ and $B$ is $(1,2)^* - \mathcal{g}$-open (resp. $(1,2)^* - \mathcal{g}$-closed), then $A \cap B \in (1,2)^* \mathcal{GLC}(X)$.

(v) If $A \in (1,2)^* \mathcal{GLC}^*(X)$ and $B$ is $(1,2)^* - \mathcal{g}$-open (resp. $\tau_{1,2}$-closed), then $A \cap B \in (1,2)^* \mathcal{GLC}^*(X)$.

(vi) If $A \in (1,2)^* \mathcal{GLC}^{**}(X)$ and $B$ is $(1,2)^* - \mathcal{g}$-closed (resp. $\tau_{1,2}$-open), then $A \cap B \in (1,2)^* \mathcal{GLC}^{**}(X)$.

(vii) If $A \in (1,2)^* \mathcal{GLC}^*(X)$ and $B$ is $(1,2)^* - \mathcal{g}$-closed, then $A \cap B \in (1,2)^* \mathcal{GLC}(X)$.

(viii) If $A \in (1,2)^* \mathcal{GLC}^{**}(X)$ and $B$ is $(1,2)^* - \mathcal{g}$-open, then $A \cap B \in (1,2)^* \mathcal{GLC}(X)$.

(ix) If $A \in (1,2)^* \mathcal{GLC}^{**}(X)$ and $B \in (1,2)^* \mathcal{GLC}^*(X)$, then $A \cap B \in (1,2)^* \mathcal{GLC}(X)$.

**Proof**

By Corollary 6.2.3 (i) to (viii) hold.

(ix). Let $A = S \cap G$ where $S$ is $\tau_{1,2}$-open and $G$ is $(1,2)^* - \mathcal{g}$ -closed and $B = P \cap Q$ where $P$ is $(1,2)^* - \mathcal{g}$ -open and $Q$ is $\tau_{1,2}$-closed. Then $A \cap B = (S \cap P) \cap (G \cap Q)$ where $S \cap P$ is $(1,2)^* - \mathcal{g}$ -open and $G \cap Q$ is $(1,2)^* - \mathcal{g}$ -closed, by Corollary 6.2.3. Therefore $A \cap B \in (1,2)^* \mathcal{GLC}(X)$.

**Remark 6.4.15**

Union of two $(1,2)^* - \mathcal{g}$-lc sets (resp. $(1,2)^* - \mathcal{g}^*$- sets, $(1,2)^* - \mathcal{g} - lc^*$ sets) need not be an $(1,2)^* - \mathcal{g}$-lc set (resp. $(1,2)^* - \mathcal{g}^*$ set, $(1,2)^* - \mathcal{g} - lc^*$ set) as can be seen from the following examples.
Example 6.4.16

Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{a, b\}\}$. Then the sets in $\{\phi, X, \{a\}, \{a, b\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{c\}, \{b, c\}\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^*\cdot \tilde{GLC} (X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then the sets $\{a\}$ and $\{c\}$ are $(1,2)^*\cdot \tilde{g} -lc$ sets, but their union $\{a, c\} \notin (1,2)^*\cdot \tilde{GLC} (X)$.

Example 6.4.17

Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{b\}\}$ and $\tau_2 = \{\phi, X, \{a, b\}\}$. Then the sets in $\{\phi, X, \{b\}, \{a, b\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{c\}, \{a, c\}\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^*\cdot \tilde{GLC}^* (X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then the sets $\{b\}$ and $\{c\}$ are $(1,2)^*\cdot \tilde{g}^* -lc^*$ sets, but their union $\{b, c\} \notin (1,2)^*\cdot \tilde{GLC}^* (X)$.

Example 6.4.18

Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{b\}\}$ and $\tau_2 = \{\phi, X, \{a, c\}\}$. Then the sets in $\{\phi, X, \{b\}, \{a, c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, X, \{a\}, \{a, c\}\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^*\cdot \tilde{GLC}^{**} (X) = \{\phi, \{a\}, \{b\}, \{a, c\}, \{b, c\}, X\}$. Then the sets $\{a\}$ and $\{b\}$ are $(1,2)^*\cdot \tilde{g}^{**} -lc^{**}$ sets, but their union $\{a, b\} \notin (1,2)^*\cdot \tilde{GLC}^{**} (X)$.

We introduce the following definition.

Definition 6.4.19

Let $A$ and $B$ be subsets of $X$. Then $A$ and $B$ are said to be $(1,2)^*\cdot \tilde{g}^*$ -separated if $A \cap (1,2)^*\cdot \tilde{g}^* -cl(B) = \phi$ and $(1,2)^*\cdot \tilde{g}^* -cl(A) \cap B = \phi$.

Example 6.4.20

For the bitopological space $X$ of Example 6.3.6. Let $A = \{b\}$ and let $B = \{c\}$. Then $(1,2)^*\cdot \tilde{g}^* -cl(A) = \{a, b\}$ and $(1,2)^*\cdot \tilde{g}^* -cl(B) = \{a, c\}$ and so the sets $A$ and $B$ are $(1,2)^*\cdot \tilde{g}^*$ -separated.
Proposition 6.4.21

For a bitopological space X, the followings are true:

(i) Let $A, B \in (1,2)^*\tilde{g}LC \,(X)$. If A and B are $(1,2)^*\tilde{g}$-separated then $A \cup B \in (1,2)^*\tilde{g}LC \,(X)$.

(ii) Let $A, B \in (1,2)^*\tilde{GL}c^* \,(X)$. If A and B are separated (i.e., $A \cap \tau_{1,2}\cl(B) = \phi$ and $\tau_{1,2}\cl(A) \cap B = \phi$), then $A \cup B \in (1,2)^*\tilde{GL}c^* \,(X)$.

(iii) Let $A, B \in (1,2)^*\tilde{GL}c^{**} \,(X)$. If A and B are $(1,2)^*\tilde{g}$-separated then $A \cup B \in (1,2)^*\tilde{GL}c^{**} \,(X)$.

Proof

(i) Since $A, B \in (1,2)^*\tilde{g}LC \,(X)$, by Theorem 6.3.39, there exist $(1,2)^*\tilde{g}$-open sets U and V of $(X, \tau)$ such that $A = U \cap (1,2)^*\tilde{g}\cl(A)$ and $B = V \cap (1,2)^*\tilde{g}\cl(B)$. Now $G = U \cap (X - (1,2)^*\tilde{g}\cl(B))$ and $H = V \cap (X - (1,2)^*\tilde{g}\cl(A))$ are $(1,2)^*\tilde{g}$-open subsets of $(X, \tau)$. Since $A \cap (1,2)^*\tilde{g}\cl(B) = \phi$, $A \subseteq ((1,2)^*\tilde{g}\cl(B))^c$. Now $A = U \cap (1,2)^*\tilde{g}\cl(A)$ becomes $A \cap ((1,2)^*\tilde{g}\cl(B))^c = G \cap (1,2)^*\tilde{g}\cl(A)$. Then $A = G \cap (1,2)^*\tilde{g}\cl(A)$. Similarly $B = H \cap (1,2)^*\tilde{g}\cl(B)$. Moreover $G \cap (1,2)^*\tilde{g}\cl(B) = \phi$ and $H \cap (1,2)^*\tilde{g}\cl(A) = \phi$. Since G and H are $(1,2)^*\tilde{g}$-open sets of $(X, \tau)$, $G \cup H$ is $(1,2)^*\tilde{g}$-open. Therefore $A \cup B = (G \cup H) \cap (1,2)^*\tilde{g}\cl(A \cup B)$ and hence $A \cup B \in (1,2)^*\tilde{g}LC \,(X)$.

(ii) and (iii) are similar to (i), using Theorems 6.3.40 and 6.3.41.

Remark 6.4.22

The assumption that A and B are $(1,2)^*\tilde{g}$-separated in (i) of Proposition 6.4.21 cannot be removed. In the bitopological space X in Example 6.4.16, the sets {a} and {c} are not $(1,2)^*\tilde{g}$-separated and their union {a, c} $\notin (1,2)^*\tilde{g}LC \,(X)$. 

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