Chapter - 5

(1,2)*-g-CLOSED AND (1,2)*-g*-OPEN MAPS

5.1 INTRODUCTION

Malghan [40] introduced the concept of generalized closed maps in topological spaces. Devi [18] introduced and studied sg-closed maps and gs-closed maps. Recently, Sheik John [76] defined $\omega$-closed maps and studied some of their properties. In this chapter, we introduce (1,2)*-g-closed maps, (1,2)*-g*-open maps, (1,2)*-g*-closed maps and (1,2)*-g*-open maps in bitopological spaces and obtain certain characterizations of these classes of maps. In last section, we introduce (1,2)*-g*-homeomorphisms and prove that the set of all (1,2)*-g*-homeomorphisms forms a group under the operation composition of functions.

5.2 PRELIMINARIES

Definition 5.2.1

A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called

(i) (1,2)*-g-closed [68] if $f(V)$ is (1,2)*-g-closed in $Y$, for every $\tau_{1,2}$-closed set $V$ of $X$.

(ii) (1,2)*-sg-closed [65] if $f(V)$ is (1,2)*-sg-closed in $Y$, for every $\tau_{1,2}$-closed set $V$ of $X$.

(iii) (1,2)*-gs-closed [65] if $f(V)$ is (1,2)*-gs-closed in $Y$, for every $\tau_{1,2}$-closed set $V$ of $X$.

(iv) (1,2)*-$\psi$-closed [51] if $f(V)$ is (1,2)*-$\psi$-closed in $Y$, for every $\tau_{1,2}$-closed set $V$ of $X$. 
5.3 (1,2)*-\textit{g} -CLOSED MAPS

**Definition 5.3.1 [Definition 3.2.1]**

A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be (1,2)*-\textit{g} -closed if the image of every \( \tau_{1,2} \)-closed set in \( X \) is (1,2)*-\textit{g} -closed in \( Y \).

**Example 5.3.2**

Let \( X = Y = \{a, b, c\} \), \( \tau_1 = \{\emptyset, X, \{a\}\} \) and \( \tau_2 = \{\emptyset, X, \{b\}\} \). Then the sets in \( \{\emptyset, X, \{a\}, \{b\}\} \) are called \( \tau_{1,2} \)-open and the sets in \( \{\emptyset, X, \{c\}, \{a, c\}, \{b, c\}\} \) are called \( \tau_{1,2} \)-closed. Let \( \sigma_1 = \{\emptyset, Y\} \) and \( \sigma_2 = \{\emptyset, Y, \{a, b\}\} \). Then the sets in \( \{\emptyset, Y, \{a, b\}\} \) are called \( \sigma_{1,2} \)-open in \( Y \) and the sets in \( \{\emptyset, Y, \{c\}\} \) are called \( \sigma_{1,2} \)-closed in \( Y \). Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be the identity map. Then \( f \) is an (1,2)*-\textit{g} -closed map.

**Proposition 5.3.3**

A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is (1,2)*-\textit{g} -closed if and only if (1,2)*-\textit{g} -\textit{cl}(f(A)) \subseteq f(\tau_{1,2}-\textit{cl}(A)) for every subset \( A \) of \( X \).

**Proof**

Suppose that \( f \) is (1,2)*-\textit{g} -closed and \( A \subseteq X \). Then \( \tau_{1,2}-\textit{cl}(A) \) is \( \tau_{1,2} \)-closed in \( X \) and so \( f(\tau_{1,2}-\textit{cl}(A)) \) is (1,2)*-\textit{g} -closed in \( Y \). We have \( f(A) \subseteq f(\tau_{1,2}-\textit{cl}(A)) \) and by Propositions 1.6.9 and 1.6.10, (1,2)*-\textit{g} -\textit{cl}(f(A)) \subseteq (1,2)*-\textit{g} -\textit{cl}(f(\tau_{1,2}-\textit{cl}(A))) = f(\tau_{1,2}-\textit{cl}(A)).

Conversely, let \( A \) be any \( \tau_{1,2} \)-closed set in \( X \). Then \( A = \tau_{1,2}-\textit{cl}(A) \) and so \( f(A) = f(\tau_{1,2}-\textit{cl}(A)) \supseteq (1,2)*-\textit{g} -\textit{cl}(f(A)) \), by hypothesis. We have \( f(A) \subseteq (1,2)*-\textit{g} -\textit{cl}(f(A)) \). Therefore \( f(A) = (1,2)*-\textit{g} -\textit{cl}(f(A)) \). That is \( f(A) \) is (1,2)*-\textit{g} -closed by Proposition 1.6.9 and hence \( f \) is (1,2)*-\textit{g} -closed.
Proposition 5.3.4

Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a map such that \( (1,2)^*\mathcal{g}\text{-cl}(f(A)) \subseteq f(\tau_{1,2}\text{-cl}(A)) \) for every subset \( A \subseteq X \). Then the image \( f(A) \) of a \( \tau_{1,2}\)-closed set \( A \) in \( X \) is \( (1,2)^*\mathcal{g}\text{-closed} \) in \( Y \).

Proof

Let \( A \) be a \( \tau_{1,2}\)-closed set in \( X \). Then by hypothesis \( (1,2)^*\mathcal{g}\text{-cl}(f(A)) \subseteq f(\tau_{1,2}\text{-cl}(A)) = f(A) \) and so \( (1,2)^*\mathcal{g}\text{-cl}(f(A)) = f(A) \). Therefore \( f(A) \) is \( (1,2)^*\mathcal{g}\text{-closed} \) in \( Y \).

Theorem 5.3.5

A map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \( (1,2)^*\mathcal{g}\text{-closed} \) if and only if for each subset \( S \) of \( Y \) and each \( \tau_{1,2}\)-open set \( U \) containing \( f^{-1}(S) \) there is an \( (1,2)^*\mathcal{g}\text{-open} \) set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Proof

Suppose \( f \) is \( (1,2)^*\mathcal{g}\text{-closed} \). Let \( S \subseteq Y \) and \( U \) be an \( \tau_{1,2}\)-open set of \( X \) such that \( f^{-1}(S) \subseteq U \). Then \( V = (f(U^c))^c \) is an \( (1,2)^*\mathcal{g}\text{-open} \) set containing \( S \) such that \( f^{-1}(V) \subseteq U \).

For the converse, let \( F \) be a \( \tau_{1,2}\)-closed set of \( X \). Then \( f^{-1}((f(F))^c) \subseteq F^c \) and \( F^c \) is \( \tau_{1,2}\)-open. By assumption, there exists an \( (1,2)^*\mathcal{g}\text{-open} \) set \( V \) in \( Y \) such that \( (f(F))^c \subseteq V \) and \( f^{-1}(V) \subseteq F^c \) and so \( F \subseteq (f^{-1}(V))^c \). Hence \( V^c \subseteq f(F) \subseteq f(f^{-1}(V))^c \subseteq V^c \) which implies \( f(F) = V^c \). Since \( V^c \) is \( (1,2)^*\mathcal{g}\text{-closed} \), \( f(F) \) is \( (1,2)^*\mathcal{g}\text{-closed} \) and therefore \( f \) is \( (1,2)^*\mathcal{g}\text{-closed} \).
Proposition 5.3.6

If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is (1,2)*-sg-irresolute \((1,2)*-\bar{g}\text{-closed and } A\) is an \((1,2)*-\bar{g}\text{-closed subset of } X\), then \( f(A) \) is \((1,2)*-\bar{g}\text{-closed in } Y\).

Proof

Let \( U \) be an \((1,2)*\text{-sg-open set in } Y\) such that \( f(A) \subseteq U\). Since \( f \) is \((1,2)*\text{-sg-irresolute, } f^{-1}(U) \) is an \((1,2)*\text{-sg-open set containing } A\). Hence \( \tau_{1,2}\text{-cl}(A) \subseteq f^{-1}(U) \) as \( A \) is \((1,2)*-\bar{g}\text{-closed in } X\). Since \( f \) is \((1,2)*-\bar{g}\text{-closed, } f(\tau_{1,2}\text{-cl}(A)) \) is an \((1,2)*-\bar{g}\text{-closed set contained in the } (1,2)*\text{-sg-open set } U\), which implies that \( \tau_{1,2}\text{-cl}(f(\tau_{1,2}\text{-cl}(A))) \subseteq U \) and hence \( \tau_{1,2}\text{-cl}(f(A)) \subseteq U \). Therefore, \( f(A) \) is \((1,2)*-\bar{g}\text{-closed set in } Y\).

The following example shows that the composition of two \((1,2)*-\bar{g}\text{-closed maps need not be a } (1,2)*-\bar{g}\text{-closed.}

Example 5.3.7

Let \( X, Y \) and \( f \) be as in Example 5.3.2. Let \( Z = \{a, b, c\} \) and \( \eta_1 = \{\phi, Z, \{c\}\} \) and \( \eta_2 = \{\phi, Z, \{a, b\}\} \). Then the sets in \( \{\phi, Z, \{c\}, \{a, b\}\} \) are called \( \eta_{1,2}\text{-open and the sets in } \{\phi, Z, \{c\}, \{a, b\}\} \) are called \( \eta_{1,2}\text{-closed. Let } g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be the identity map. Then both \( f \) and \( g \) are \((1,2)*-\bar{g}\text{-closed maps but their composition } g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is not an \((1,2)*-\bar{g}\text{-closed map, since for the } \tau_{1,2}\text{-closed set } \{b, c\} \text{ in } X, (g \circ f)(\{b, c\}) = \{b, c\} \text{, which is not an } (1,2)*-\bar{g}\text{-closed set in } Z.\)
**Corollary 5.3.8**

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $(1,2)^*\cdot \tilde{g}$-closed and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be $(1,2)^*\cdot \tilde{g}$-closed and $(1,2)^*\cdot \text{sg}$-irresolute, then their composition $g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)$ is $(1,2)^*\cdot \tilde{g}$-closed.

**Proof**

Let $A$ be a $\tau_{1,2}$-closed set of $X$. Then by hypothesis $f(A)$ is an $(1,2)^*\cdot \tilde{g}$-closed set in $Y$. Since $g$ is both $(1,2)^*\cdot \tilde{g}$-closed and $(1,2)^*\cdot \text{sg}$-irresolute by Proposition 5.3.6, $g(f(A)) = (g \circ f)(A)$ is $(1,2)^*\cdot \tilde{g}$-closed in $Z$ and therefore $g \circ f$ is $(1,2)^*\cdot \tilde{g}$-closed.

**Proposition 5.3.9**

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be $(1,2)^*\cdot \tilde{g}$-closed maps where $Y$ is a $T_{(1,2)^*\cdot \tilde{g}}$-space. Then their composition $g \circ f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(1,2)^*\cdot \tilde{g}$-closed.

**Proof**

Let $A$ be a $\tau_{1,2}$-closed set of $X$. Then by assumption $f(A)$ is $(1,2)^*\cdot \tilde{g}$-closed in $Y$. Since $Y$ is a $T_{(1,2)^*\cdot \tilde{g}}$-space, $f(A)$ is $\sigma_{1,2}$-closed in $Y$ and again by assumption $g(f(A))$ is $(1,2)^*\cdot \tilde{g}$-closed in $Z$. That is $(g \circ f)(A)$ is $(1,2)^*\cdot \tilde{g}$-closed in $Z$ and so $g \circ f$ is $(1,2)^*\cdot \tilde{g}$-closed.

**Proposition 5.3.10**

If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(1,2)^*\cdot \tilde{g}$-closed, $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is $(1,2)^*\cdot \tilde{g}$-closed (resp. $(1,2)^*\cdot \text{g}$-closed, $(1,2)^*\cdot \psi$-closed, $(1,2)^*\cdot \text{sg}$-closed and $(1,2)^*\cdot \text{gs}$-closed) and $Y$ is a $T_{(1,2)^*\cdot \tilde{g}}$-space, then their composition $g \circ f : (X, \tau_1, \tau_2)$
\( (Z, \eta_1, \eta_2) \) is \((1,2)^*\)-\(g\)-closed (resp. \((1,2)^*-g\)-closed, \((1,2)^*-\psi\)-closed, \((1,2)^*-sg\)-closed and \((1,2)^*-gs\)-closed).

**Proof**

Similar to Proposition 5.3.9.

**Proposition 5.3.11**

Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a \((1,2)^*\)-closed map and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be an \((1,2)^*-g\)-closed map, then their composition \( g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is \((1,2)^*-g\)-closed.

**Proof**

Similar to Proposition 5.3.9.

**Remark 5.3.12**

If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is an \((1,2)^*-g\)-closed and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) is \((1,2)^*-\tilde{\psi}\)-closed, then their composition need not be an \((1,2)^*-\tilde{g}\)-closed map as seen from the following example.

**Example 5.3.13**

Let \( X, Y \) and \( f \) be as in Example 5.3.2. Let \( Z = \{a, b, c\} \) and \( \eta_1 = \{\phi, Z, \{a\}\} \) and \( \eta_2 = \{\phi, Z, \{a, b\}\} \). Then the sets in \( \{\phi, Z, \{a\}\} \) are called \( \eta_{1,2}\)-open and the sets in \( \{\phi, Z, \{c\}, \{b, c\}\} \) are called \( \eta_{1,2}\)-closed. Let \( g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be the identity map. Then \( f \) is an \((1,2)^*-\tilde{g}\)-closed map and \( g \) is a \((1,2)^*-\tilde{\psi}\)-closed map. But their composition \( g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is not an \((1,2)^*-\tilde{g}\)-closed map, since for the \( \tau_{1,2}\)-closed set \( \{a, c\} \) in \( X \), \( (g \circ f)(\{a, c\}) = \{a, c\} \), which is not an \((1,2)^*-\tilde{g}\)-closed set in \( Z \).
Theorem 5.3.14

Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) be two maps such that their composition \( g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is an \((1,2)^*\)-\( \tilde{g} \)-closed map. Then the following statements are true.

(i) If \( f \) is \((1,2)^*\)-continuous and surjective, then \( g \) is \((1,2)^*\)-\( \tilde{g} \)-closed.

(ii) If \( g \) is \((1,2)^*\)-\( \tilde{g} \)-irresolute and injective, then \( f \) is \((1,2)^*\)-\( \tilde{g} \)-closed.

(iii) If \( f \) is \((1,2)^*\)-\( \tilde{g} \)-continuous, surjective and \((X, \tau)\) is a \((1,2)^*\)-\( T_\omega \)-space, then \( g \) is \((1,2)^*\)-\( \tilde{g} \)-closed.

(iv) If \( g \) is strongly \((1,2)^*\)-\( \tilde{g} \)-continuous and injective, then \( f \) is \((1,2)^*\)-closed.

Proof

(i) Let \( A \) be a \( \sigma_{1,2} \)-closed set of \( Y \). Since \( f \) is \((1,2)^*\)-continuous, \( f^1(A) \) is \( \tau_{1,2} \)-closed in \( X \) and since \( g \circ f \) is \((1,2)^*\)-\( \tilde{g} \)-closed, \( (g \circ f)(f^1(A)) \) is \((1,2)^*\)-\( \tilde{g} \)-closed in \( Z \). That is \( g(A) \) is \((1,2)^*\)-\( \tilde{g} \)-closed in \( Z \), since \( f \) is surjective.

Therefore \( g \) is an \((1,2)^*\)-\( \tilde{g} \)-closed map.

(ii) Let \( B \) be a \( \tau_{1,2} \)-closed set of \( X \). Since \( g \circ f \) is \((1,2)^*\)-\( \tilde{g} \)-closed, \( (g \circ f)(B) \) is \((1,2)^*\)-\( \tilde{g} \)-closed in \( Z \). Since \( g \) is \((1,2)^*\)-\( \tilde{g} \)-irresolute, \( g^{-1}((g \circ f)(B)) \) is \((1,2)^*\)-\( \tilde{g} \)-closed set in \( Y \). That is \( f(B) \) is \((1,2)^*\)-\( \tilde{g} \)-closed in \( Y \), since \( g \) is injective.

Thus \( f \) is an \((1,2)^*\)-\( \tilde{g} \)-closed map.

(iii) Let \( C \) be a \( \sigma_{1,2} \)-closed set of \( Y \). Since \( f \) is \((1,2)^*\)-\( \tilde{g} \)-continuous, \( f^1(C) \) is \((1,2)^*\)-\( \tilde{g} \)-closed in \( X \). Since \( X \) is a \((1,2)^*\)-\( T_\omega \)-space, \( f^1(C) \) is \( \tau_{1,2} \)-closed in \( X \) and so as in (i), \( g \) is an \((1,2)^*\)-\( \tilde{g} \)-closed map.
(iv) Let D be a $\tau_{1,2}$-closed set of X. Since $g \circ f$ is $(1,2)^*\cdot \bar{g}$-closed, $(g \circ f)(D)$ is $(1,2)^*\cdot \bar{g}$-closed in Z. Since $g$ is strongly $(1,2)^*\cdot \bar{g}$-continuous, $g^{-1}((g \circ f)(D))$ is $\sigma_{1,2}$-closed in Y. That is $f(D)$ is $\sigma_{1,2}$-closed set in Y, since $g$ is injective. Therefore $f$ is a $(1,2)^*$-closed map.

In the next theorem we show that $(1,2)^*$-normality is preserved under $(1,2)^*$-continuous $(1,2)^*\cdot \bar{g}$-closed maps.

**Theorem 5.3.15**

If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(1,2)^*$-continuous, $(1,2)^*\cdot \bar{g}$-closed map from a $(1,2)^*$-normal space $X$ onto a space $Y$, then $Y$ is $(1,2)^*$-normal.

**Proof**

Let $A$ and $B$ be two disjoint $\sigma_{1,2}$-closed subsets of $Y$. Since $f$ is $(1,2)^*$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\tau_{1,2}$-closed sets of $X$. Since $X$ is $(1,2)^*$-normal, there exist disjoint $\tau_{1,2}$-open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is $(1,2)^*\cdot \bar{g}$-closed, by Theorem 5.3.5, there exist disjoint $(1,2)^*\cdot \bar{g}$-open sets $G$ and $H$ in $Y$ such that $A \subseteq G$, $B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since $U$ and $V$ are disjoint, $\sigma_{1,2}$-int$(G)$ and $\sigma_{1,2}$-int$(H)$ are disjoint $\sigma_{1,2}$-open sets in $Y$. Since $A$ is $\sigma_{1,2}$-closed, $A$ is $(1,2)^*$-sg-closed and therefore we have by Theorem 3.3.3, $A \subseteq \sigma_{1,2}$-int$(G)$. Similarly $B \subseteq \sigma_{1,2}$-int$(H)$ and hence $Y$ is $(1,2)^*$-normal.

Analogous to an $(1,2)^*\cdot \bar{g}$-closed map, we have defined an $(1,2)^*\cdot \bar{g}$-open map as follows:

**Definition 5.3.16 [Definition 3.2.1]**

A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be an $(1,2)^*\cdot \bar{g}$-open map if the image $f(A)$ is $(1,2)^*\cdot \bar{g}$-open in $Y$ for each $\tau_{1,2}$-open set $A$ in $X$.  

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Proposition 5.3.17

For any bijection \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), the following statements are equivalent:

(i) \( f^{-1} : (Y, \sigma_1, \sigma_2) \rightarrow (X, \tau_1, \tau_2) \) is \((1,2)^*\)-\(g\)-continuous.

(ii) \( f \) is \((1,2)^*\)-\(g\)-open map.

(iii) \( f \) is \((1,2)^*\)-\(g\)-closed map.

Proof

(i) \(\Rightarrow\) (ii). Let \( U \) be an \( \tau_{1,2} \)-open set of \( X \). By assumption, \((f^{-1})^{-1}(U) = f(U)\) is \((1,2)^*\)-\(g\)-open in \( Y \) and so \( f \) is \((1,2)^*\)-\(g\)-open.

(ii) \(\Rightarrow\) (iii). Let \( F \) be a \( \tau_{1,2} \)-closed set of \( X \). Then \( F^c \) is \( \tau_{1,2} \)-open set in \( X \). By assumption, \( f(F^c) \) is \((1,2)^*\)-\(g\)-open in \( Y \). That is \( f(F^c) = (f(F))^c \) is \((1,2)^*\)-\(g\)-open in \( Y \) and therefore \( f(F) \) is \((1,2)^*\)-\(g\)-closed in \( Y \). Hence \( f \) is \((1,2)^*\)-\(g\)-closed.

(iii) \(\Rightarrow\) (i). Let \( F \) be a \( \tau_{1,2} \)-closed set of \( X \). By assumption, \( f(F) \) is \((1,2)^*\)-\(g\)-closed in \( Y \). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is \((1,2)^*\)-\(g\)-continuous.

In the next two theorems, we obtain various characterizations of \((1,2)^*\)-\(g\)-open maps.

Theorem 5.3.18

Assume that the collection of all \((1,2)^*\)-\(g\)-open sets of \( Y \) is closed under arbitrary union. Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a map. Then the following statements are equivalent:

(i) \( f \) is an \((1,2)^*\)-\(g\)-open map.
(ii) For a subset \( A \) of \( X \), \( f(\tau_{1,2}\text{-int}(A)) \subseteq (1,2)^*\text{-}g\text{-int}(f(A)) \).

(iii) For each \( x \in X \) and for each \( \tau_{1,2}\text{-neighborhood} U \) of \( x \) in \( X \), there exists an \((1,2)^*\text{-}g\text{-neighborhood} W \) of \( f(x) \) in \( Y \) such that \( W \subseteq f(U) \).

**Proof**

(i) \( \Rightarrow \) (ii). Suppose \( f \) is \((1,2)^*\text{-}g\text{-open}\). Let \( A \subseteq X \). Then \( \tau_{1,2}\text{-int}(A) \) is \( \tau_{1,2}\text{-open} \) in \( X \) and so \( f(\tau_{1,2}\text{-int}(A)) \) is \((1,2)^*\text{-}g\text{-open} \) in \( Y \). We have \( f(\tau_{1,2}\text{-int}(A)) \subseteq f(A) \). Therefore by Proposition 1.5.3, \( f(\tau_{1,2}\text{-int}(A)) \subseteq (1,2)^*\text{-}g\text{-int}(f(A)) \).

(ii) \( \Rightarrow \) (iii). Suppose (ii) holds. Let \( x \in X \) and \( U \) be an arbitrary \( \tau_{1,2}\text{-neighborhood} \) of \( x \) in \( X \). Then there exists an \( \tau_{1,2}\text{-open set} \) \( G \) such that \( x \in G \subseteq U \). By assumption, \( f(G) = f(\tau_{1,2}\text{-int}(G)) \subseteq (1,2)^*\text{-}g\text{-int}(f(G)) \). This implies \( f(G) = (1,2)^*\text{-}g\text{-int}(f(G)) \). By Proposition 1.5.3, we have \( f(G) \) is \((1,2)^*\text{-}g\text{-open} \) in \( Y \). Further, \( f(x) \in f(G) \subseteq f(U) \) and so (iii) holds, by taking \( W = f(G) \).

(iii) \( \Rightarrow \) (i). Suppose (iii) holds. Let \( U \) be any \( \tau_{1,2}\text{-open set in} \ X \), \( x \in U \) and \( f(x) = y \). Then \( y \in f(U) \) and for each \( y \in f(U) \), by assumption there exists an \((1,2)^*\text{-}g\text{-neighborhood} \) \( W_y \) of \( y \) in \( Y \) such that \( W_y \subseteq f(U) \). Since \( W_y \) is an \((1,2)^*\text{-}g\text{-neighborhood} \) of \( y \), there exists an \((1,2)^*\text{-}g\text{-open set} \) \( V_y \) in \( Y \) such that \( y \in V_y \subseteq W_y \). Therefore, \( f(U) = \bigcup \{ V_y : y \in f(U) \} \) is an \((1,2)^*\text{-}g\text{-open set in} \ Y \). Thus \( f \) is an \((1,2)^*\text{-}g\text{-open map} \).

**Theorem 5.3.19**

A map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \((1,2)^*\text{-}g\text{-open} \) if and only if for any subset \( S \) of \( Y \) and for any \( \tau_{1,2}\text{-closed set} \) \( F \) containing \( f^{-1}(S) \), there exists an \((1,2)^*\text{-}g\text{-closed set} \) \( K \) of \( Y \) containing \( S \) such that \( f^{-1}(K) \subseteq F \).
Proof

Similar to Theorem 5.3.5.

Corollary 5.3.20

A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((1,2)^*\bar{g}\)-open if and only if
\[
f^{-1}((1,2)^*\bar{g}\text{-}\text{cl}(B)) \subseteq \tau_{1,2}\text{-}\text{cl}(f^{-1}(B))
\]
for each subset \( B \) of \( Y \).

Proof

Suppose that \( f \) is \((1,2)^*\bar{g}\)-open. Then for any \( B \subseteq Y \), \( f^{-1}(B) \subseteq \tau_{1,2}\text{-}\text{cl}(f^{-1}(B)) \).

By Theorem 5.3.19, there exists an \((1,2)^*\bar{g}\)-closed set \( K \) of \( Y \) such that \( B \subseteq K \) and \( f^{-1}(K) \subseteq \tau_{1,2}\text{-}\text{cl}(f^{-1}(B)) \). Therefore, \( f^{-1}((1,2)^*\bar{g}\text{-}\text{cl}(B)) \subseteq (f^{-1}(K)) \subseteq \tau_{1,2}\text{-}\text{cl}(f^{-1}(B)) \), since \( K \) is an \((1,2)^*\bar{g}\)-closed set in \( Y \).

Conversely, let \( S \) be any subset of \( Y \) and \( F \) be any \( \tau_{1,2}\)-closed set containing \( f^{-1}(S) \). Put \( K = (1,2)^*\bar{g}\text{-}\text{cl}(S) \). Then \( K \) is an \((1,2)^*\bar{g}\)-closed set and \( S \subseteq K \). By assumption, \( f^{-1}(K) = f^{-1}((1,2)^*\bar{g}\text{-}\text{cl}(S)) \subseteq \tau_{1,2}\text{-}\text{cl}(f^{-1}(S)) \subseteq F \) and therefore by Theorem 5.3.19, \( f \) is \((1,2)^*\bar{g}\)-open.

Finally in this section, we define another new class of maps called \((1,2)^*\bar{g}^\ast\)-closed maps which are stronger than \((1,2)^*\bar{g}\)-closed maps.

Definition 5.3.21

A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be \((1,2)^*\bar{g}^\ast\)-closed if the image \( f(A) \) is \((1,2)^*\bar{g}\)-closed in \( Y \) for every \((1,2)^*\bar{g}\)-closed set \( A \) in \( X \).

For example the map \( f \) in Example 5.3.2 is an \((1,2)^*\bar{g}^\ast\)-closed map.
Remark 5.3.22

Since every $\tau_{1,2}$-closed set is an $(1,2)^*-\tilde{g}$-closed set we have $(1,2)^*-\tilde{g}^*$-closed map is an $(1,2)^*-\tilde{g}$-closed map. The converse is not true in general as seen from the following example.

Example 5.3.23

Let $X = Y = \{a, b, c\}$ $\tau_1 = \{\emptyset, X, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X\}$. Then the sets in $\{\emptyset, X, \{a, b\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{c\}\}$ are called $\tau_{1,2}$-closed.

Let $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y, \{a, b\}\}$. Then the sets in $\{\emptyset, Y, \{a\}\}$, $\{a, b\}$ are called $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{c\}\}$, $\{b, c\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be the identity map. Then $f$ is an $(1,2)^*\tilde{g}$-closed but not $(1,2)^*\tilde{g}^*$-closed map. Since $\{a, c\}$ is $(1,2)^*\tilde{g}$-closed set in $X$, but its image under $f$ is $\{a, c\}$ which is not $(1,2)^*\tilde{g}$-closed set in $Y$.

Proposition 5.3.24

A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(1,2)^*\tilde{g}^*$-closed if and only if $(1,2)^*\tilde{g} - \text{cl}(f(A)) \subseteq f((1,2)^*\tilde{g} - \text{cl}(A))$ for every subset $A$ of $X$.

Proof

Similar to Proposition 5.3.3.

Analogous to $(1,2)^*\tilde{g}^*$-closed map we can also define $(1,2)^*\tilde{g}^*$-open map.

Proposition 5.3.25

For any bijection $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:
(i) \( f^{-1} : (Y, \sigma_1, \sigma_2) \rightarrow (X, \tau_1, \tau_2) \) is \( \tilde{g} \)-irresolute.

(ii) \( f \) is \((1,2)^*\)-\( \tilde{g}^* \)-open map.

(iii) \( f \) is \((1,2)^*\)-\( \tilde{g}^* \)-closed map.

**Proof**

Similar to Proposition 5.3.17.

**Proposition 5.3.26**

If \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((1,2)^*\)-sg-irresolute and \((1,2)^*\)-\( \tilde{g} \)-closed, then it is an \((1,2)^*\)-\( \tilde{g}^* \)-closed map.

**Proof**

The proof follows from Proposition 5.3.6.

**5.4 \((1,2)^*\)-\( \tilde{g}^* \)-HOMEOMORPHISMS**

The notion of \((1,2)^*\)-homeomorphisms plays a very important role in bitopological spaces. By definition, an \((1,2)^*\)-homeomorphism between two bitopological spaces \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) is a bijective map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) when \( f \) and \( f^{-1} \) are \((1,2)^*\)-continuous.

We introduce the following definition:

**Definition 5.4.1**

A bijection \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be

(i) \((1,2)^*\)-\( \tilde{g} \)-homeomorphism if \( f \) is both \((1,2)^*\)-\( \tilde{g} \)-continuous and \((1,2)^*\)-\( \tilde{g} \)-open.

(ii) \((1,2)^*\)-\( \tilde{g}^* \)-homeomorphism if both \( f \) and \( f^{-1} \) are \((1,2)^*\)-\( \tilde{g} \)-irresolute.
We denote the family of all \((1,2)^*\)-homeomorphisms of a bitopological space \((X, \tau_1, \tau_2)\) onto itself by \((1,2)^*\)-h(X).

**Theorem 5.4.2**

Let \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be a bijective \((1,2)^*\)-continuous map. Then the following are equivalent:

(i) \(f\) is an \((1,2)^*\)-open map.

(ii) \(f\) is an \((1,2)^*\)-homeomorphism.

(iii) \(f\) is an \((1,2)^*\)-closed map.

**Proof**

Follows from Proposition 5.3.17.

**Proposition 5.4.3**

If \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) and \(g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)\) are \((1,2)^*\)-homeomorphisms, then their composition \(g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)\) is also \((1,2)^*\)-homeomorphism.

**Proof**

Let \(U\) be \((1,2)^*\)-open set in \((Z, \eta_1, \eta_2)\). Now, \((g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)\), where \(V = g^{-1}(U)\). By hypothesis, \(V\) is \((1,2)^*\)-open in \(Y\) and so again by hypothesis, \(f^{-1}(V)\) is \((1,2)^*\)-open in \(X\). Therefore, \(g \circ f\) is \((1,2)^*\)-irresolute.

Also for an \((1,2)^*\)-open set \(G\) in \(X\), we have \((g \circ f)(G) = g(f(G)) = g(W)\), where \(W = f(G)\). By hypothesis \(f(G)\) is \((1,2)^*\)-open in \(Y\) and so again by hypothesis, \(g(f(G))\) is \((1,2)^*\)-open in \(Z\). That is \((g \circ f)(G)\) is \((1,2)^*\)-open in \(Z\).
and therefore \((g \circ f)^{-1}\) is \((1,2)^*-\tilde{g}^*\)-irresolute. Hence \(g \circ f\) is a \((1,2)^*-\tilde{g}^*\)-homeomorphism.

**Theorem 5.4.4**

The set \((1,2)^*-\tilde{g}^*\)-h(X) is a group under the composition of maps.

**Proof**

Define a binary operation \(*\) : \((1,2)^*-\tilde{g}^*\)-h(X) \times \((1,2)^*-\tilde{g}^*\)-h(X) \to \((1,2)^*-\tilde{g}^*\)-h(X) by \(f \ast g = g \circ f\) for all \(f, g \in (1,2)^*-\tilde{g}^*\)-h(X) and \(\circ\) is the usual operation of composition of maps. Then by Proposition 5.4.3, \(g \circ f \in (1,2)^*-\tilde{g}^*\)-h(X). We know that the composition of maps is associative and the identity map \(I : (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2)\) belonging to \((1,2)^*-\tilde{g}^*\)-h(X) serves as the identity element. If \(f \in (1,2)^*-\tilde{g}^*\)-h(X), then \(f^{-1} \in (1,2)^*-\tilde{g}^*\)-h(X) such that \(f \circ f^{-1} = f^{-1} \circ f = I\) and so inverse exists for each element of \((1,2)^*-\tilde{g}^*\)-h(X). Therefore, \(((1,2)^*-\tilde{g}^*\)-h(X), \(\circ\)) is a group under the operation of composition of maps.

**Theorem 5.4.5**

Let \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be an \((1,2)^*-\tilde{g}^*\)-homeomorphism. Then \(f\) induces an \((1,2)^*\)-isomorphism from the group \((1,2)^*-\tilde{g}^*\)-h(X) on to the group \((1,2)^*-\tilde{g}^*\)-h(Y).

**Proof**

Using the map \(f\), we define a map \(\theta_f : (1,2)^*-\tilde{g}^*\)-h(X) \to (1,2)^*-\tilde{g}^*\)-h(Y) by \(\theta_f(h) = f \circ h \circ f^{-1}\) for every \(h \in (1,2)^*-\tilde{g}^*\)-h(X). Then \(\theta_f\) is a bijection. Further, for all \(h_1, h_2 \in (1,2)^*-\tilde{g}^*\)-h(X), \(\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1})\)
\[ \theta_1(h_1) \circ \theta_2(h_2) \]. Therefore, \( \theta_f \) is a \((1,2)^*\)-homomorphism and so it is an \((1,2)^*\)-isomorphism induced by \( f \).

**Theorem 5.4.6**

\((1,2)^*\)-\( \tilde{g}^* \)-homeomorphism is an equivalence relation in the collection of all bitopological spaces.

**Proof**

Reflexivity and symmetry are immediate and transitivity follows from Proposition 5.4.3.