Chapter 2

Substitutable Inventory System with Partial Backlogging
2.1 Introduction

In multi-item continuous review inventory systems, Ballintfy [17] and Silver [217] have considered a can order policy which is represented by the triplet \((S, c, s)\), where the three parameters \(S_i, c_i\) and \(s_i\) are specified for each item \(i\) with \(s_i \leq c_i \leq S_i\), under the unit sized Poisson demand and constant lead time. Subsequently, many articles have appeared with models involving the above policy and another article of interest is Federgruen, Groenevelt and Tijms [71], which deals with the general case of compound Poisson demands and non-zero lead times. Gök et al. [88] studied the impacts of various factors on cost savings achieved by coordinating replenishments. By a heuristic approach, Atkins and Iyogun [16] proposed a lower bound for the optimal cost of the coordinated multi-item inventory systems. A review of various inventory models under joint replenishment is provided by Goyal and Satir [90].

For detailed study of joint replenishment problems in multi commodity inventory system we can refer the reader to the publications of Fung and Ma [81], Goyal [89], Kaspi and Rosenblatt [126], Nilsson et al. [167], Nilsson and Silver [168], Olsen [170], Silver [218] and Viswanathan ([241],[242]).

Kalpakam and Arivarignan [115] discussed an \((s, S)\) coordinated replenishment policy in which a single reorder level \(s\) is defined in terms of the total number of items in stock. Liming Liu and Xue-Ming Yuan [143] studied the two commodity can order policy with correlated demands.

Anbazhagan and Arivarignan [3] have considered a two commodity inventory system with Poisson demands and a joint reorder policy which placed fixed ordering quantities for both commodities whenever both inventory lev-
els are less than or equal to their respective reorder levels.

Anbazhagan and Arivarignan [5] have also analyzed the model with individual and joint ordering policy. For the individual reorder policy, the reorder level for $i$-th commodity is fixed as $r_i$ and whenever the inventory level of $i$-th commodity falls on $r_i$ an order for $P_i := S_i - r_i$ items is placed for that commodity irrespective of the inventory level of the other commodity. A joint reorder policy is used with prefixed reorder levels $s$. An order for $Q_1^1 (S_1 - x)$ and $Q_2^2 (S_2 - y)$ items is placed for both commodities by cancelling the previous orders, whenever both commodities have their inventory levels on or below the reorder level $s$, $(x + y = s)$.

Anbazhagan [6] has considered a two commodity substitutable inventory system, in which if no substitute is available, then a demand is backlogged. The backlogging is allowed up to the level $N_i$, $(i = 1, 2)$ for the $i$-th commodity. Whenever the inventory level reaches $N_i (i = 1, 2)$, an order for $N_i$ items are placed to get instantaneously. Roy and Chaudhuri [2007] introduced a can order-level inventory model for a deteriorating item, taking the demand to be dependent on the sale price of the item. This chapter is the extended work of Anbazhagan [6] by considering emergency order to be replenished instantaneously by local purchase up to their reorder levels.

In the classical coordinated replenishment dynamic lot-sizing problem, the primary motivation for coordination is in the presence of major and minor setup costs, but in most of the cases it may be the discount for quantity ordered. The mathematical programming formulation for the all unit discount and incremental discount problem are all NP hard. A more simple but viable model for two commodity inventory system is the major study
focussed in this chapter. This considerably reduced the complexity of problem, at the same time it capture the another demand level coordination (not correlation) namely substitution. This motivates the authors to explore the possibilities of satisfying the independent Poisson demands, by substitution and joint reorder is initiated to save cost. Branded items with less variation in price and usage are the prime factors to be taken into account for this kind of inventory system.

Most of the real world inventory system stock many items, not merely a single item. It is permissible to study each item individually only as long as there are no interactions among the items. There can be many interactions between them: the items may be partial substitutes for each other, warehouse capacity may be limited and the items are competing for floor space; there may be upper bound to the number of orders and the items are competing for these; or there may be an upper limit for the investment capacity in terms of money.

Recently more research has been focussed on substitutable items, because of production limitations and inventory reduction strategy. Through this multi item inventory management, considerable saving may be achieved by coordination of replenishment orders for group of items. This motivates the authors to study coordinated replenishment for substitutable items under one group, a unique study in the literature. The presence of major and minor set up cost or quantity discount is the prime reason for coordinated replenishment option.

This model does not fall into any of the above category, because this model is two commodity inventory system with substitutable items. Since
the quantity of substitutable items ordered for two items complement each other, an item which yield for discount, but occupy less floor space may be considered for higher volume purchase. Substitutable items model reduce the complexity of problem, in one sense at the same time it limits the area of application of the model in real life. This motivate the authors to explore the possibilities of modeling the systems and study its performance in a practical situation.

Organization of this chapter is as follows: The problem formulation and the notation are introduced in section 2. Section 3 deals with analysis part of the problem. The measures of system performance and total expected cost rate are computed in section 4. Some numerical examples are considered to illustrate the model description in section 5. The last section concludes the chapter with positive future research options.

2.2 Model Description

A two commodity stochastic inventory system with the maximum capacity $S_i$ units for $i$-th commodity ($i = 1, 2$) is considered. The demand for $i$-th commodity is of unit size and the time points of demand occurrences form independent Poisson processes each with parameter $\lambda_i$, ($i = 1, 2$). The joint-reorder level for the $i$-th commodity is fixed at $s_i$, ($1 \leq s_i \leq S_i$) and ordering quantity for $i$-th commodity is $Q_i$ ($= S_i - s_i > s_i + N_i$) items when both inventory levels are less than or equal to their respective reorder levels. The requirement $S_i - s_i > s_i + N_i$, ensures that after a replenishment the inventory levels of both commodities would always be above the respective reorder levels $s_i$, $i = 1, 2$. Otherwise it may not be possible to place reorder
in the next demand epochs which leads to perpetual shortage. The lead time is assumed to be distributed as negative exponential with parameter \( \mu(>0) \). The two commodities are assumed to be substitutable. That is, a customer switches from one type of commodity to the other when the commodity demanded is not available. If no substitute is available, then this demand is backlogged. The backlog is allowed upto the level \( N_i, (i = 1, 2) \) for the \( i \)-th commodity. Whenever the inventory level reaches \( N_1 \) or \( N_2 \), both inventory levels are consequently pulled back to their prespecified reorder levels \( s_1 \) and \( s_2 \) instantaneously by local purchase.

The following notations will be used for the infinitesimal generator.

Notations:

\[
0 \quad : \text{zero matrix} \\
1_N' \quad : (1, 1, \cdots, 1)_{1 \times N} \\
I_N \quad : \text{an identity matrix of order } N \\
\delta_{ij} \quad : \text{Kronecker delta.} \\
[A]_{ij} \quad : (i, j)-\text{th element of the matrix } A. \\
H(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases} \\
\sum_{i=1}^{k} a_i = 0, \text{ if } k = 0.
\]

2.3 Analysis

Let \( L_i(t) \) denote the inventory level of the \( i \)-th commodity at time \( t \). Then the vector valued stochastic process \( \{(L_1(t), L_2(t)), t \geq 0\} \) has the state space

\[
E = E_1 \cup E_2 \cup E_3,
\]
where

\[ E_1 = \{(i, j)|i = 1, 2, \ldots, S_1, j = 0, 1, \ldots, S_2\}, \]

\[ E_2 = \{(i, j)|i = 0, j = -(N_2 - 1), -(N_2 - 2), \ldots, 0, 1, \ldots, S_2\}, \]

and

\[ E_3 = \{(i, j)|i = -(N_1 - 1), -(N_1 - 2), \ldots, -1, j = -(N_2 - 1), -(N_2 - 2), \ldots, -1, 0\}. \]

From the assumptions made on demand and on replenishment processes, it follows that \{\{(L_1(t), L_2(t))|t \geq 0\}\} is a Markov process. The infinitesimal generator \( \tilde{A} = ((a((i, j), (k, l)))| (i, j), (k, l) \in E) \), can be conveniently expressed as a block partitioned matrix \((A_{ij})\),

where

\[ A_{ij} = \begin{cases} A_3 & j = i, \; i = -(N_1 - 1), -(N_1 - 2), \ldots, -1 \\ A_2 & j = i, \; i = 0 \\ A_1 & j = i, \; i = 1, 2, \ldots, s_1 \\ A & j = i, \; i = s_1 + 1, s_1 + 2, \ldots, S_1 \\ B_6 & j = s_1, \; i = -(N_1 - 1) \\ B_5 & j = s_1, \; i = -(N_1 - 2), \ldots, -1 \\ B_4 & j = s_1, \; i = 0 \\ B_3 & j = i - 1, \; i = -(N_1 - 2), \ldots, -1 \\ B_2 & j = i - 1, \; i = 0 \\ B_1 & j = i - 1, \; i = 1 \\ B & j = i - 1, \; i = 2, 3, \ldots, S_1 \\ C_2 & j = i + Q_1, \; i = -(N_1 - 1), -(N_1 - 2), \ldots, -1 \\ C_1 & j = i + Q_1, \; i = 0 \\ C & j = i + Q_1, \; i = 1, 2, \ldots, s_1 \\ 0 & \text{Otherwise.} \end{cases} \]

More explicitly the infinitesimal generator \( \tilde{A} \) of the process \{\{(L_1(t), L_2(t))|t \geq 0\}\}
0\} is:

\[
\tilde{A} = \begin{pmatrix}
S_1 & A & B \\
S_1 - 1 & A & B \\
\vdots & \ddots & \ddots & \ddots \\
s_1 + 1 & C & \cdots & A_1 & B \\
s_1 & C & \ddots & \ddots & \ddots \\
1 & C_1 & \cdots & A_1 & B_1 \\
0 & C_2 & B_4 & A_2 & B_2 \\
-1 & C_2 & B_5 & A_3 & B_3 \\
-2 & \ddots & \ddots & \ddots \\
\vdots & & & & \\
-(N_1 - 2) & C_2 & \cdots & B_5 & \\
-(N_1 - 1) & C_2 & B_6 & A_3 & B_3
\end{pmatrix}
\]

The detailed structure of the sub matrices of \( \tilde{A} \) are given in Appendix A.

It can be seen from the structure of \( \tilde{A} \) that the homogeneous Markov process \( \{ (L_1(t), L_2(t)), t \geq 0 \} \) on the state space \( E \) is irreducible. Hence the limiting distribution

\[
\Phi = (\phi(S_1), \phi(S_1-1), \ldots, \phi(0), \ldots, \phi-(N_1-1))
\]

with \( \phi(m) = \begin{cases} 
(\phi(m,S_2), \phi(m,S_2-1), \ldots, \phi(m,1), \phi(m,0)), & m = 1, 2, \ldots, S_1 \\
(\phi(m,S_2), \phi(m,S_2-1), \ldots, \phi(m,-(N_2-1))), & m = 0 \\
(\phi(m,-(N_2-1)), \ldots, \phi(m,-1), \phi(m,0)), & m = -(N_1 - 1), \ldots, -1 
\end{cases} \)

where \( \phi(i,j) \) denotes the steady state probability for the state \((i,j)\) of the
inventory level process, exists and is given by

$$\Phi \tilde{A} = 0 \quad \text{and} \sum_{(i,j) \in E} \phi^{(i,j)} = 1. \quad (2.1)$$

The first equation of the above yields the following set of equations:

$$\phi^{(i)} A_3 + \phi^{(i+1)} B_3 = 0, \quad i = -(N_1 - 1), -(N_1 - 2), \ldots, -2,$$

$$\phi^{(i)} A_3 + \phi^{(i+1)} B_2 = 0, \quad i = -1,$$

$$\phi^{(i)} A_2 + \phi^{(i+1)} B_1 = 0, \quad i = 0,$$

$$\phi^{(i)} A_1 + \phi^{(i+1)} B = 0, \quad i = 1, 2, \ldots, s_1 - 1,$$

$$\phi^{(i)} A_1 + \phi^{(i+1)} B + \phi^{(0)} B_4 + \sum_{j=1}^{N_1-2} \phi^{(-j)} B_5 + \phi^{(-(N_1-1))} B_6 = 0, \quad i = s_1, \quad (\ast)$$

$$\phi^{(i)} A + \phi^{(i+1)} B = 0, \quad i = s_1 + 1, \ldots, Q_1 - N_1$$

$$\phi^{(i)} A + \phi^{(i+1)} B + \phi^{(i-Q_1)} C_2 = 0, \quad i = Q_1 - N_1 + 1, \ldots, Q_1 - 1,$$

$$\phi^{(i)} A + \phi^{(i+1)} B + \phi^{(i-Q_1)} C_1 = 0, \quad i = Q_1$$
\[ \phi^{(i)} A + \phi^{(i+1)} B + \phi^{(i-Q_1)} C = 0, \quad i = Q_1 + 1, \ldots, S_1 - 1 \]

and
\[ \phi^{(S_1)} A + \phi^{(s_1)} C = 0. \]

After long simplifications, the above equations, except (*), yield
\[ \phi^{(i)}, i = S_1, S_1 - 1, \ldots, 0, -1, \ldots, -(N_1 - 1), \]
with the special reference that \( \phi^{(s_1)} \) can be obtained in a different method of solving recurrence relations with \( \Phi e = 1 \) (see appendix B).

### 2.4 System Performance Measures

The steady state probabilities of the system states \( \phi^{(i)}, i = S_1, S_1 - 1, \ldots, 0, -1, \ldots, -(N_1 - 1) \) gives raise to an easy computation of mean inventory level, reorder rate and backlogging level as system performance measures. These system performance measures can be swiftly used to get the total expected cost rate (total cost per unit time) of the inventory control system in force, by imposing a proper cost structure. This section describes crucial phase of our work in optimization and decision making.

#### 2.4.1 Mean Inventory Level

Let \( I_1 \) denote the expected inventory level of the first commodity in the steady state which is given by,
\[
I_1 = \sum_{i=1}^{S_1} i \left( \sum_{j=0}^{S_2} \phi^{(i,j)} \right)
\]
Let \( I_2 \) denote the expected inventory level of the second commodity in the steady state which is given by,

\[
I_2 = \sum_{j=1}^{S_2} \left( \sum_{i=0}^{S_1} \phi^{(i,j)} \right)
\]

### 2.4.2 Mean Reorder Rate

Let \( JR \) denote the mean joint reorder rate for both the commodity in the steady state which is given by,

\[
JR = \sum_{i=0}^{s_1} (\delta \lambda_1 + \lambda_2) \phi^{(i,s_2+1)} + \sum_{j=0}^{s_2} (\lambda_1 + \delta \lambda_2) \phi^{(s_1+1,j)}
\]

Let \( ER \) denote the mean emergency reorder rate for both the commodity in the steady state which is given by,

\[
ER = \sum_{i=-(N_1-1)}^{0} (\delta(-N_1) \lambda_1 + \lambda_2) \phi^{(i,-(N_2-1))} + \sum_{j=-(N_2-1)}^{0} (\lambda_1 + \delta(-N_2) \lambda_2) \phi^{(-(N_1-1),j)}
\]

### 2.4.3 Mean Backlogging

Let \( B_1 \) denote the mean backlogging of the first commodity in the steady state then we have,

\[
B_1 = \sum_{i=-(N_1-1)}^{-1} \sum_{j=-(N_2-1)}^{0} |i| \phi^{(i,j)}.
\]

Let \( B_2 \) denote the mean backlogging of the second commodity in the steady state then we have,
\[ B_2 = \sum_{j=-(N_2-1)}^{-1} \sum_{i=-(N_1-1)}^{0} |j| \phi^{(i,j)}. \]

**2.4.4 Total Expected Cost Rate**

The total expected cost rate (total expected cost per unit time), can be computed, by imposing the following cost structure on the inventory control system.

- \( h_1 \): the inventory carrying cost per item per unit time of first commodity.
- \( h_2 \): the inventory carrying cost per item per unit time of second commodity.
- \( c_s \): setup cost per order.
- \( c_{es} \): setup cost per emergency order.
- \( c_{b1} \): backlog cost per unit item of first commodity.
- \( c_{b2} \): backlog cost per unit item of second commodity.

Now the long-run total expected cost rate is given by

\[ TC(S_1, s_1, N_1, S_2, s_2, N_2) = h_1 I_1 + h_2 I_2 + c_s JR + c_{es} ER + c_{b1} B_1 + c_{b2} B_2. \]

Since the total expected cost rate is known only implicitly, the analytical properties such as convexity of the total expected cost rate cannot be proved in the present form. However considerable number of numerical examples demonstrate the work conveniently. To illustrate the existence of local optima of cost function, we present some of the best possible numerical examples in the next section.
2.5 Numerical Illustration

As the total expected cost rate is obtained in a complex form, the convexity of the total expected cost rate by the analytical methods cannot be studied. Hence, numerical search procedures are used to find the local optimal values for any two of the decision variables \((S_1, s_1, N_1, S_2, s_2, N_2)\) by considering a small set of integer values for these variables. With a large number of numerical examples, it is found that the total cost rate per unit time in the long run is either convex function of both variables or an increasing function of any one variable.

The total expected cost rate as a function of \(S_1\) and \(S_2\) by fixing constant values for the other variables and costs is given in Table 2.1. After obtaining the local optima, \(S^*_1\) and \(S^*_2\), of \(S_1\) and \(S_2\) respectively, we carried out the sensitivity analysis to see how the changes in \(S_1\) and \(S_2\) affect the total expected cost rate (refer figure 2.1). We have computed the values of

\[
\frac{TC(S_1, 6, 7, S_2, 8, 6)}{TC(S^*_1, 6, 7, S^*_2, 8, 6)}
\]

by fixing the parameters and costs as: \(N_1 = 7; N_2 = 6; \mu = 4; s_1 = 6; s_2 = 8; \lambda_1 = 11.8; \lambda_2 = 10; h_1 = 0.2; h_2 = 0.3; c_s = 10; c_{es} = 14; c_{b1} = 2.9; c_{b2} = 1.8\). Here \(S^*_1 = 26\) and \(S^*_2 = 31\) and \(TC(S^*_1, 6, 7, S^*_2, 8, 6) = 12.772448\). It appears that the total expected cost rate is more sensitive to the changes in \(S_1\) than to those in \(S_2\).

Next we fix the values for the parameters, except \(s_1\) and \(s_2\), and other costs and treat the total expected cost per unit time as a function of \(s_1\) and \(s_2\). After finding the local optimal values \(s^*_1\) and \(s^*_2\) of \(s_1\) and \(s_2\), respectively, by numerical search procedures, we present, in Table 2.2, the value
Figure 2.1: Effect of $S_1$ and $S_2$ on Total Expected Cost

of \( \frac{TC(30, s_1, 5, 31, s_2, 6)}{TC(30, s_1^*, 5, 31, s_2^*, 6)} \) for some fixed values of \((s_1, s_2)\) in the neighbourhood of \((s_1^*, s_2^*)\). The values assumed are: \( \mu = 4; \lambda_1 = 11.8; \lambda_2 = 10.0; h_1 = 0.2; h_2 = 0.3; c_s = 10; c_{cs} = 14; c_{b1} = 2.9; c_{b2} = 1.8 \). We also calculated as \( s_1^* = 10 \) and \( s_2^* = 7 \) and \( TC(30, s_1^*, 5, 31, s_2^*, 6) = 12.760333 \). We observe that the total expected cost rate is more sensitive to changes in \( s_1 \) than that of in \( s_2 \).

Fixing all parameters and cost values except \( N_1 \) and \( N_2 \) yield the total cost function \( TC(36, 10, N_1, 52, 12, N_2) \) and \( N_i^* \) is the local optimal value of \( N_i, i = 1, 2 \). In table 2.3, we present \( \frac{TC(36, 10, N_1, 52, 12, N_2)}{TC(36, 10, N_1^*, 52, 12, N_2^*)} \), for some values of \((N_1, N_2)\) in the neighbourhood of \((N_1^*, N_2^*)\), by fixing constant values
Table 2.1: Sensitivity of $S_1$ and $S_2$ on Total expected cost rate

<table>
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<th>$S_1$</th>
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<th>30</th>
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Table 2.2: Sensitivity of $s_1$ and $s_2$ on Total expected cost rate

<table>
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<th>6</th>
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</tbody>
</table>

for other parameters and costs, namely, $\mu = 5; \lambda_1 = 21.8; \lambda_2 = 20.1; h_1 = 0.2; h_2 = 0.19; c_s = 10; c_{es} = 0.3; c_{bl} = 0.9; c_{b2} = 0.8$. Here $N_1^* = 13, N_2^* = 6$ and $TC(36, 10, N_1^*, 52, 12, N_2^*) = 13.314727$. We note that the total expected cost rate is more sensitive to changes in $N_2$ than that of in $N_1$.

### 2.6 Conclusion

In this chapter we have studied an inventory control system maintaining two closely related (branded and substitutable) commodities, a substitutable inventory system with partial backlogging for two commodities. This model is most appropriate for the commodities which are substitutable in both ways. Also we considered a mixture of two kinds of ordering policies, coordinated policy and emergency ordering policy. Future work on these kinds of
Table 2.3: Sensitivity of $N_1$ and $N_2$ on Total expected cost rate

closely related (branded) but with perishable nature can be taken to explore the more complicated but down to earth model.
Appendix A: Structure of the sub matrices of infinitesimal generator.

The sub matrices in \((A_{ij})\), are of the form:

\[
[C_2]_{pq} = \begin{cases} 
\mu, & q = p + Q_2, \quad p = -(N_2 - 1), \ldots, -1, 0, \\
0, & \text{otherwise}
\end{cases}
\]

\[
[C_1]_{pq} = \begin{cases} 
\mu, & q = p + Q_2, \quad p = -(N_2 - 1), \ldots, -1, 0, 1, \ldots, s_2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
[C]_{pq} = \begin{cases} 
\mu, & q = p + Q_2, \quad p = 0, 1, \ldots, s_2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B_6]_{pq} = \begin{cases} 
\lambda_1, & q = s_2, \quad p = -(N_2 - 2), \ldots, -1, 0, \\
\lambda_1 + \lambda_2, & q = s_2, \quad p = -(N_2 - 1), \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B_5]_{pq} = \begin{cases} 
\lambda_2, & q = s_2, \quad p = -(N_2 - 1), \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B_4]_{pq} = \begin{cases} 
\lambda_2, & q = s_2, \quad p = -(N_2 - 1), \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B_3]_{pq} = \begin{cases} 
\lambda_1, & q = p, \quad p = -(N_2 - 1), -(N_2 - 2), \ldots, -1, 0, \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B_2]_{pq} = \begin{cases} 
\lambda_1, & q = p, \quad p = -(N_2 - 1), -(N_2 - 2), \ldots, -1, 0, \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B_1]_{pq} = \begin{cases} 
\lambda_1, & q = p, \quad p = 1, 2, \ldots, S_2, \\
\lambda_1 + \lambda_2, & q = p, \quad p = 0, \\
0, & \text{otherwise}
\end{cases}
\]

\[
[B]_{pq} = \begin{cases} 
\lambda_1, & q = p, \quad p = 1, 2, \ldots, S_2, \\
\lambda_1 + \lambda_2, & q = p, \quad p = 0, \\
0, & \text{otherwise}
\end{cases}
\]
\[
[A_3]_{pq} = \begin{cases} 
\lambda_2, & q = p - 1, \\
-(\lambda_1 + \lambda_2 + \mu), & q = p, \\
0, & \text{otherwise}
\end{cases}
\]

\[
[p = -(N_2 - 2), \ldots, -1, 0, \\
(p = -(N_2 - 1), \ldots, -1, 0, \\
(p = -(N_2 - 1), \ldots, s_2, \\
(p = s_2 + 1, \ldots, S_2, \\
(p = 0, 1, \ldots, s_2, \\
\text{otherwise}
\end{cases}
\]

\[
[A_2]_{pq} = \begin{cases} 
\lambda_1 + \lambda_2, & q = p - 1, \\
\lambda_2, & q = p - 1, \\
-(\lambda_1 + \lambda_2 + \mu), & q = p, \\
-(\lambda_1 + \lambda_2), & q = p, \\
0, & \text{otherwise}
\end{cases}
\]

\[
\begin{cases} 
\lambda_2, & q = p - 1, \\
-(\lambda_1 + \lambda_2), & q = p, \\
-(\lambda_1 + \lambda_2 + \mu), & q = p, \\
0, & \text{otherwise}
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\[
[A_1]_{pq} = \begin{cases} 
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\end{cases}
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\[
[A]_{pq} = \begin{cases} 
\lambda_2, & q = p - 1, \\
-(\lambda_1 + \lambda_2), & q = p, \\
0, & \text{otherwise}
\end{cases}
\]

It may be noted that the matrices $A$, $A_1$, $B$ and $C$ are of size $(S_2 + 1) \times (S_2 + 1)$, $A_3$ and $B_3$ are of size $N_2 \times N_2$, $C_1$ is of size $(S_2 + N_2) \times (S_2 + 1)$, $C_2$ is of size $N_2 \times (S_2 + 1)$, $B_4$ is of size $(S_2 + N_2) \times (S_2 + 1)$, $B_5$ and $B_6$ are of size $N_2 \times (S_2 + 1)$, $B_2$ is of size $(S_2 + N_2) \times N_2$, $B_1$ is of size $(S_2 + 1) \times (S_2 + N_2)$, $A_2$ is of size $(S_2 + N_2) \times (S_2 + N_2)$. 

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Appendix B: Derivation for steady state probability vector.

After great deal of simplification, the limiting distribution $\Phi$ in terms of $\phi(s_1)$ is given below:

$$
\phi(i) = \phi(s_1)(-1)^{s_1-i}(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})^{-1-i},
$$

$$
i = -(N_1 - 1), \ldots, -2, -1
$$

$$
= \phi(s_1)(-1)^{s_1-i}(BA_1^{-1})^{s_1-1}(B_1A_2^{-1}),
$$

$$
i = 0
$$

$$
= \phi(s_1)(-1)^{s_1-i}(BA_1^{-1})^{s_1-i},
$$

$$
i = 1, 2, \ldots, s_1 - 1
$$

$$
= \phi(s_1)(-1)^{S_1+1-i} \left\{ (CA^{-1})(BA^{-1})^{S_1-i} + \sum_{j=1}^{s_1-1} (BA_1^{-1})^j(CA^{-1})(BA^{-1})^{s_1-i-j} + 
(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{-1+j}(C_2A^{-1})(BA^{-1})^{Q_1-i-j} \right\} 
$$

$$
i = s_1 + 1, \ldots, Q_1 - N_1 - 1, Q_1 - N_1
$$

$$
= \phi(s_1)(-1)^{S_1+1-i} \left\{ (CA^{-1})(BA^{-1})^{S_1-i} + \sum_{j=1}^{s_1-1} (BA_1^{-1})^j(CA^{-1})(BA^{-1})^{s_1-i-j} + 
(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{-1+j}(C_2A^{-1})(BA^{-1})^{Q_1-i-j} \right\} 
$$

$$
i = Q_1 - N_1 + 1, \ldots, Q_1 - 1, Q_1
$$

$$
= \phi(s_1)(-1)^{S_1+1-i} \left\{ (CA^{-1})(BA^{-1})^{S_1-i} + \sum_{j=1}^{s_1-1} (BA_1^{-1})^j(CA^{-1})(BA^{-1})^{S_1-i-j} \right\} 
$$

$$
i = Q_1 + 1, \ldots, S_1 - 1, S_1
$$

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where $\phi^{(s_1)}$ can be obtained by solving,

$$
\phi^{(s_1)} A_1 + \phi^{(s_1+1)} B + \phi^{(0)} B_4 + \sum_{j=1}^{N_1-2} \phi(-j) B_5 + \phi(-(N_1-1)) B_6 = 0 \text{ and }
$$

$$
\Phi e = 1,
$$

that is

$$
\phi^{(s_1)} \left( A_1 + (-1)^{Q_1} \left\{ (CA^{-1})(BA^{-1})^{Q_1-1} + \sum_{j=1}^{s_1-1} (BA_1^{-1})^j (CA^{-1})(BA^{-1})^{Q_1-1-j} + \\
(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(C_1A^{-1})(BA^{-1})^{Q_1-s_1-1} + \\
\sum_{j=1}^{N_1-1} (BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{-1+j}(C_2A^{-1})(BA^{-1})^{Q_1-s_1-1-j} \right\} B + \\
\{ (-1)^{s_1}(BA_1^{-1})^{s_1-1}(B_1A_2^{-1}) \} B_4 + \\
\left\{ \sum_{j=1}^{N_1-2} (-1)^{s_1+j}(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{-1+j} \right\} B_5 + \\
\{ (-1)^{s_1+N_1-1}(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{N_1-2} \} B_6 \right) = 0.
$$

and

$$
\phi^{(s_1)} \left[ \sum_{i=-(N_1-1)}^{s_1-1} (-1)^{s_1-i}(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{-1-i} + \\
(-1)^{s_1}(BA_1^{-1})^{s_1-1}(B_1A_2^{-1}) + \sum_{i=1}^{s_1-1} (-1)^{s_1-i}(BA_1^{-1})^{s_1-i} + I \right]
$$

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\[
\sum_{i=q_1+1}^{Q_1-N_1} (-1)^{s_1+1-i} \left\{ (CA^{-1})(BA^{-1})^{s_1-i} + \sum_{j=1}^{s_1-1} (BA_1^{-1})^j (CA^{-1})(BA^{-1})^{s_1-i-j} \right\} + \\
(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(C_1A^{-1})(BA^{-1})Q_1^{-i} + \\
\sum_{j=1}^{N_1-1} (BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{-1+j}(C_2A^{-1})(BA^{-1})Q_1^{-i-j} + \\
\sum_{i=q_1-N_1+1}^{Q_1} (-1)^{s_1+1-i} \left\{ (CA^{-1})(BA^{-1})^{s_1-i} + \sum_{j=1}^{s_1-1} (BA_1^{-1})^j (CA^{-1})(BA^{-1})^{s_1-i-j} \right\} + \\
(BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(C_1A^{-1})(BA^{-1})Q_1^{-i} + (1 - \delta_iQ_1) + \\
\sum_{j=1}^{Q_1-1} (BA_1^{-1})^{s_1-1}(B_1A_2^{-1})(B_2A_3^{-1})(B_3A_3^{-1})^{-1+j}(C_2A^{-1})(BA^{-1})Q_1^{-i-j} + \\
\sum_{i=Q_1+1}^{S_1} (-1)^{s_1+1-i} \left\{ (CA^{-1})(BA^{-1})^{s_1-i} + \sum_{j=1}^{S_1-i} (BA_1^{-1})^j (CA^{-1})(BA^{-1})^{s_1-i-j} \right\} e = 1.
\]