Chapter 4

Two commodity Inventory System with Compliment for Bulk Demand
4.1 Introduction

The challenge of managing a production/inventory system in the presence of random demand and lead time has inspired a considerable amount of research effort in recent years. Marketing of consumable goods which has high obsolescence rate have to be dealt carefully. Incentive for bulk purchase is one strategy to boost sale volume at the same time the risk of getting obsolescence. This motivated the researchers to build an inventory models having two items in stock, and of which one is regular with stochastic demand. Other one is a gift item supplied to a customer for the bulk purchase of the regular item. The bulk order scheme is actually equivalent to price break for bulk purchase, but it differs because the sales promotion for gift article. The significance of modeling such stochastic systems can be directly attributed to the severity of their potential negative effort on operating costs and customer service measures in modern manufacturing and business environments.

In this chapter we investigate the issue of random demand and lead time together with the supply of gift for a bulk demand greater than a specific number \( r > 0 \). The problem considered here touches the two boundaries of classified models (i) the two commodity inventory system and (ii) inventory system with price break. One for one gift policy already exist in the retail business, but proper modeling for the inventory control system for random demand and lead time is not studied earlier.

In multi-item continuous review inventory systems, Ballintify [17] and Silver [217] have considered a can order policy which is represented by the triplet \((S_i, c_i, s_i)\), where the three parameters \( S_i, c_i \) and \( s_i \) are specified for each item \( i \) with \( s_i \leq c_i \leq S_i \), under the unit sized Poisson demand and constant
lead time. Subsequently, many articles have appeared with models involving the above policy and another article of interest is Federgruen, Groenevelt and Tijms [71], which deals with the general case of compound Poisson demands and non-zero lead times. Gök et al. [88] studied the impacts of various factors on cost savings achieved by coordinating replenishments. By a heuristic approach, Atkins and Iyogun [16] proposed a lower bound for the optimal cost of the coordinated multi-item inventory systems. A review of various inventory models under joint replenishment is provided by Goyal and Satir [90].

Two commodity inventory control system with (i) correlated (ii) uncorrelated demand for items with (i) joint and (ii) independent ordering policies are studied elaborately ([3], [4], [5], [81], [89], [115], [126], [133], [134], [167], [168], [170], [218], [223], [241], [242] and [252]). Continuous review inventory system with lost sales has been studied in the last three decades. Even though their models are particular cases of multi commodity inventory systems, the differ in the same of joint ordering policy and correlated demand.

The system considered in this chapter is logically equivalent to price break inventory control system, which was studied earlier by many researchers [100], but have different marketing policy. At the outset the problem we considered here is a realistic retail marketing oriented one which involves high complexity in terms of tractability. The authors has been motivated by the real time application of the system and simplified the same of the complex constraints and studied the problem with mathematical frame.

In section 1, a brief introduction to the problem proposed for study and the motivation behind the model is given. In Section 2, we present the model
formulation and the notations. Analysis of the model and the steady state solution of the model are given in Section 3. In Section 4, we derive various measures of system performance in the steady state. In the last section a numerical example is provided to illustrate our model in a lucid manner.

4.2 Model Formulation

We consider a two commodity inventory system with one major item having stochastic demand and one unit of second item is supplied to the customer for each bulk ordering \( r(>0) \) or more than \( r \) units of first item. Here the number of units demanded is also random, with known probability distribution \((p_i)_{i=0}^\infty\). The time points of demand occurrences form Poisson process with parameter \( \lambda (>0) \). The compliment item is issued to the customer who demands exactly \( r \) or more than \( r \) items. The replenishment policy for major item is \((s,S)\) type, with order quantity \( Q(=S-s) \). The replenishment of gift item is instantaneous that is the reorder rate is infinite. The demand during the stock out period for the major item is assumed to be lost sales.

**Notations:**

\[ 0 \] : zero matrix
\[ 1_N' \] : \((1,1,\cdots,1)_{1\times N}\)
\[ I_N \] : identity matrix of order \( N \)
\[ E_1 \] : \(\{f+1, f+2, \cdots, F\}\)
\[ E_2 \] : \(\{0,1,\cdots,S\}\)
\[ \delta_{ij} \] : Kronecker delta
\[ < m - k > = \begin{cases} m - k & \text{if } m - k > 0 \\ 0 & \text{if } m - k \leq 0 \end{cases} \]
\[ \delta_{ij} : (1 - \delta_{ij}) \]
\[ [A]_{ij} : (i, j)\text{-th element of the matrix } A \]
\[ \sum_{i=j}^{k} c_i = \begin{cases} c_j c_{j-1} \cdots c_k & \text{if } j \geq k \\ 1 & \text{if } j < k \end{cases} \]

### 4.3 Analysis

Let \( L_1(t) \) denote the inventory level of compliment item and \( L_2(t) \) denote the inventory level of main commodity in the system at time \( t \). Clearly \( \{(L_1(t), L_2(t)), t \geq 0\} \) is a vector valued stochastic process with continuous time parameter and state space \( E = E_1 \times E_2 \). From the assumptions made on the input and output processes, it can be shown that \( \{(L_1(t), L_2(t)), t \geq 0\} \), on the state space \( E \), is a Markov process. Let \( \{p_k\} \) be the probability distribution of demand size \( k \). The transition probabilities \( p_{ij}(t) \) of the Markov process has derivatives \( a(i, j) \) at \( t = 0 \), which given the intensity of the transition from state \( i \) to \( j \). The infinitesimal generator \( A \) of the Markov process with entries are of the form,

\[
A = (\{ a((i, m), (j, n)) \}), \quad (i, m), (j, n) \in E.
\]

Our policy on customer service with supply of complimentary item for bulk demand size \( r \) or greater than \( r \) (fixed) gives raise to the following argument on state transitions in the vector process \( \{(L_1(t), L_2(t)), t \geq 0\} \).

* A bulk demand of \( k \) items for main commodity takes the state from
  
  \( \bullet (i, m) \) to \( (i, m - k), i = f + 1, f + 2, \ldots, F, m = r, r + 1, \ldots, S, \) 
  
  \( k = 1, 2, \ldots, r - 1 \) and the intensity of transition is given by \( \lambda p_k \).
• \((i, m)\) to \((i-1, < m-k >), i = f+2, f+3, \ldots, F, m = r, r+1, \ldots, S, k = r, r+1, \ldots, \) and the respective intensity of transitions are given by \(\lambda p_k\) and \(\lambda \sum_{u=k}^{\infty} p_u\).

• \((i, m)\) to \((i, < m-k >), i = f+1, f+2, \ldots, F, m = 0, 1, \ldots, r-1, k = 1, 2, \ldots, \) and the respective intensity of transitions are given by \(\lambda p_k\) and \(\lambda \sum_{u=k}^{\infty} p_u\).

• \((f+1, m)\) to \((F, < m-k >), m = r, r+1, \ldots, S, k = r, r+1, \ldots, \) due to the instantaneous replenishment of gift item and the respective intensity of transitions are given by \(\lambda p_k\) and \(\lambda \sum_{u=k}^{\infty} p_u\).

* From the state \((i, m)\), a replenishment takes it to \((i, m+Q), i = f+1, f+2, \ldots, F, m = 0, 1, \ldots, s\) and the intensity of transition is given by \(\mu\). In this case

\[
a((i, m), (i, m+Q)) = \mu.
\]

* No other transitions from \((i, m)\) to \((j, n)\), except \((j, n) \neq (i, m)\) are possible. Hence their rates of transitions are zero.

* To obtain the intensity of passage, \(a((i, m), (i, m))\) of state \((i, m)\), we note that the entries in any row of this matrix add to zero. Hence the diagonal entry is equal to the negative of the sum of the other entries in that row. More explicitly

\[
a((i, m), (i, m)) = - \sum\sum_{(j,n)\neq(i,m)} a((i, m), (j, n)).
\]

Hence we get,
\[
a((i, m), (j, n)) =
\begin{cases}
\lambda p_k, & j = i, \quad n = m - k, \\
i = f + 1, f + 2, \cdots, F, \quad m = r, r + 1, \cdots, S \\
k = 1, 2, \cdots, r - 1
\end{cases}
\begin{cases}
\lambda p_k, & j = i - 1, \quad n = m - k, \\
i = f + 2, f + 3, \cdots, F, \quad m = r + 1, r + 2, \cdots, S \\
k = r, r + 1, \cdots, m - 1
\end{cases}
\begin{cases}
\lambda p_m', & j = i - 1, \quad n = 0, \\
i = f + 2, f + 3, \cdots, F, \quad m = r, r + 1, \cdots, S
\end{cases}
\begin{cases}
\lambda p_k, & j = i, \quad n = m - k, \\
i = f + 1, f + 2, \cdots, F, \quad m = 2, 3, \cdots, r - 1 \\
k = 1, 2, \cdots, m - 1
\end{cases}
\begin{cases}
\lambda p_m', & j = i, \quad n = 0, \\
i = f + 1, f + 2, \cdots, F, \quad m = 1, 2, \cdots, r - 1
\end{cases}
\begin{cases}
\lambda p_k, & j = F, \quad n = m - k, \\
i = f + 1, \quad m = r + 1, \cdots, S \\
k = r, r + 1, \cdots, m - 1
\end{cases}
\begin{cases}
\lambda p_m', & j = F, \quad n = 0, \\
i = f + 1, \quad m = r, r + 1, \cdots, S
\end{cases}
\begin{cases}
\mu, & j = i, \quad n = m + Q, \\
i = f + 1, f + 2, \cdots, F, \quad m = 0, 1, \cdots, s
\end{cases}
\begin{cases}
-\lambda, & j = i, \quad n = m, \\
i = f + 1, f + 2, \cdots, F, \quad m = s + 1, s + 2, \cdots, S
\end{cases}
\begin{cases}
-(\lambda + \mu), & j = i, \quad n = m, \\
i = f + 1, f + 2, \cdots, F, \quad m = 0, 1, \cdots, s
\end{cases}
\begin{cases}
-\mu, & j = i, \quad n = m, \\
i = f + 1, f + 2, \cdots, F, \quad m = 0
\end{cases}
\begin{cases}
0, & \text{otherwise.}
\end{cases}
\]

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where $p_m = \sum_{k=m}^{\infty} p_k$.

The infinitesimal generator $A$ can be conveniently expressed as a block partitioned matrix:

$$A = ( (A_{ij}) ),$$

where

$$A_{ij} = \begin{cases} 
D & j = i, \ i = f + 1, f + 2, \ldots, F \\
B & j = i - 1, \ i = f + 2, f + 3, \ldots, F \\
B & j = F, \ i = f + 1 \\
0 & \text{Otherwise.}
\end{cases}$$

More explicitly,

$$A = \begin{pmatrix} 
D & B & B \\
D & B & B \\
\vdots & \ddots & \ddots & B & B \\
D & B & B \\
B & B . & D \\
\end{pmatrix}$$

where,

$$[D]_{ij} = \begin{cases} 
\lambda_{p_{i-j}}, & j = i - 1, i - 2, \ldots, i - (r - 1) \ i = r, r + 1, \ldots, S, \\
\lambda_{p_{i-j}}, & j = 1, 2, \ldots, i - 1, \ i = 2, 3, \ldots, r - 1, \\
\lambda \sum_{k=i}^{\infty} p_k, & j = 0, \ i = 1, 2, \ldots, r - 1, \\
\mu, & j = i + Q, \ i = 0, 1, \ldots, s, \\
-\lambda, & j = i, \ i = s + 1, s + 2, \ldots, S, \\
-(\lambda + \mu), & j = i, \ i = 1, 2, \ldots, s, \\
-\mu, & j = i, \ i = 0, \\
0, & \text{otherwise}
\end{cases}$$

$$[B]_{ij} = \begin{cases} 
\lambda_{p_{i-j}}, & j = i - r, i - r - 1, \ldots, 1 \ i = r + 1, r + 2, \ldots, S, \\
\lambda \sum_{k=i}^{\infty} p_k, & j = 0, \ i = r, r + 1, \ldots, S, \\
0, & \text{otherwise}
\end{cases}$$
It can be seen from the structure of $A$ that the homogeneous Markov process $\{(L_1(t), L_2(t)), t \geq 0\}$ on the state space $E$ is irreducible. Hence the limiting distribution

$$\Phi = (\phi(S), \phi(S-1), \ldots, \phi(0))$$

with $\phi(m) = (\phi(m,F), \phi(m,F-1), \ldots, \phi(m,f+2), \phi(m,f+1))$, $m = 0, 1, \ldots, S$, where $\phi^{(i,j)}$ denotes the steady state probability for the state $(i, j)$ of the inventory level process, exists and is given by

$$\Phi A = 0 \quad \text{and} \quad \sum_{(i,j) \in E} \phi^{(i,j)} = 1.$$  

(4.1)

The first equation of the above yields the following set of equations:

$$\phi^{(i)} D + \phi^{(i+1)} B = 0, \quad i = f + 1, f + 2, \ldots, F - 1,$$

and

$$\phi^{(F)} D + \phi^{(F+1)} B = 0. \quad (*)$$

After simplifications, the above equations, except (*), yield

$$\phi^{(i)} = \phi^{(f+1)} (-1)^{i-f-1} (DB^{-1})^{i-f-1}, \quad i = f + 1, f + 2, \ldots, F$$

where $\phi^{(f+1)}$ can be obtained by solving,
\[ \phi^F D + \phi^{f+1} B = 0 \] and \[ \Phi \mathbf{e} = 1, \]

that is

\[ \phi^{f+1} \left\{ B + (-1)^{f-1}(DB^{-1})^{f-1} \right\} = 0 \]

and

\[ \phi^{f+1} \left[ \sum_{i=f+1}^{F} (-1)^{i-f-1}(DB^{-1})^{i-f-1} \right] \mathbf{e} = 1. \]

### 4.4 System Performance Measures

In this section, we derive some performance measures of the system under consideration.

#### 4.4.1 Mean Inventory Level

Let \( I_1 \) denote the expected inventory level of the main commodity in the steady state which is given by

\[ I_1 = \sum_{i=f+1}^{F} i \left( \sum_{j=0}^{S} \phi^{(i,j)} \right) \]

Let \( I_2 \) denote the expected inventory level of the complimentary commodity in the steady state which is given by
\[ I_2 = \sum_{j=1}^{S} j \left( \sum_{i=f+1}^{F} \phi^{(i,j)} \right) \]

### 4.4.2 Mean Reorder Rate

Let \( R_1 \) denote the mean reorder rate of the main commodity in the steady state. Then,

\[
R_1 = \sum_{i=f+1}^{F} \sum_{j=1}^{Q} \lambda \phi^{(i,s+j)} \left( \sum_{k=j}^{\infty} p_k \right)
\]

Let \( R_2 \) denote the mean reorder rate of the complimentary commodity in the steady state. Then,

\[
R_2 = \sum_{j=r}^{S} \lambda \phi^{(f+1,j)} \left( \sum_{k=r}^{\infty} p_k \right)
\]

### 4.4.3 Mean Shortage Rate

Let \( MS_1 \) denote the mean shortage rate of the main commodity in the steady state then we have

\[
MS_1 = \sum_{i=f+1}^{F} \lambda \phi^{(i,0)} \left( \sum_{k=1}^{\infty} p_k \right).
\]

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4.4.4 Total Expected Cost Rate

To compute the total expected cost per unit time, the following cost structure is imposed.

- $h_1$: the inventory carrying cost per item per unit time of main commodity.
- $h_2$: the inventory carrying cost per item per unit time of compliment commodity.
- $K_1$: setup cost per order of main commodity.
- $K_2$: setup cost per order of compliment commodity.
- $c_{sh_1}$: shortage cost per unit item of main commodity.

The long run expected cost rate is given by

$$TC(S, s, F, f) = h_1 I_1 + h_2 I_2 + K_1 R_1 + K_2 R_2 + c_{sh_1} M S_1.$$ 

Since the total expected cost rate is known only implicitly, the analytical properties such as convexity of the total expected cost rate cannot be proved in the present form. However we present the following numerical examples to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima, when the total cost function is treated as a function of only two variables.

4.5 Numerical Study

We have used simple numerical search procedures to find the “local” optimal values for the decision variables $S, s, F$ and $f$ by considering a small set of integer values for these variables. With a large number of numerical examples, we have found that the total cost rate per unit time in the long run is either convex function or an increasing function.
Table 4.1 gives the total expected cost rate as a function of $S$ and $F$ by fixing other variables as constant. After obtaining the local optima, $S^*$ and $F^*$, the sensitivity analysis is carried out to see how the changes in $S$ and $F$ affect the total expected cost rate (refer figure 4.1). We have computed the values of

$$\frac{TC(S, 8, F, 5)}{TC(S^*, 8, F^*, 5)}$$

Figure 4.1: Effect of $S$ and $F$ on Total Expected Cost Rate by fixing the parameters and costs as: $\lambda = 20.8; \mu = 11.9; r = 6; p_i = 0.2 \times 0.8^i, i = 1, 2, \ldots; h_1 = 4.7; h_2 = 2.9; K_1 = 180; K_2 = 50; c_{sh1} = 6$. Varying $S$ from 129 to 149 in steps 5 and $F$ from 16 to 20, we get the optimal values for $S$ and $F$ are $S^* = 139$ and $F^* = 18$ and $TC(S^*, 8, F^*, 5) = 453.868595$. It appears that the total expected cost rate is more sensitive to the changes in $F$ than that of in $S$.

Holding costs for both main and compliment items are contributing more to the total expected cost rate $TC(S, s, F, f)$. But due to the impact of other parameters on total expected cost rate, the optimal values $S^*$ and $F^*$ are
Table 4.1: Sensitivity of $S$ and $F$ on total expected cost rate

<table>
<thead>
<tr>
<th>$S$</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
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<td>1.000000</td>
<td>1.000840</td>
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<td>1.002193</td>
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Table 4.2: Sensitivity analysis of $h_1$ and $h_2$ on total expected cost rate

<table>
<thead>
<tr>
<th>$h_2$</th>
<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
</tr>
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<tbody>
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<td>323.000078</td>
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<td>429.579453</td>
<td>482.869140</td>
<td>536.158828</td>
</tr>
<tr>
<td>4</td>
<td>331.132793</td>
<td>384.422480</td>
<td>437.712168</td>
<td>491.001855</td>
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</tr>
<tr>
<td>5</td>
<td>339.265508</td>
<td>392.555195</td>
<td>445.844882</td>
<td>499.134570</td>
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</tr>
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<td>6</td>
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<tr>
<td>7</td>
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<td>408.820625</td>
<td>462.110312</td>
<td>515.399999</td>
<td>568.689687</td>
</tr>
</tbody>
</table>

$s = 8; f = 5; \lambda = 20.8; \mu = 11.9; r = 6; p_i = 0.2 \times 0.8^i, i = 1, 2, \ldots; c_{s1} = 180; c_{s2} = 50; c_{sh1} = 6.$

Table 4.3: Sensitivity analysis of $h_1$ and $h_2$ on total expected cost rate

A subtle change in holding and shortage costs gives rise to same effect on total expected cost rate but not had considerable impact on local optimal values $S^*$ and $F^*$ (see table 4.3).

Change in ordering costs $K_1$ and $K_2$ has some incremental effect on total expected cost rate but it does not change the optimal values $S^*$ and $F^*$ (see table 4.4).


<table>
<thead>
<tr>
<th>$h_1$</th>
<th>$c_{sh1}$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>688.816805</td>
<td>689.703612</td>
<td>690.590419</td>
<td>691.477225</td>
</tr>
</tbody>
</table>

$s = 8; f = 5; \lambda = 20.8; \mu = 11.9; r = 6; p_i = 0.2 \times 0.8^i, i = 1, 2, \ldots; c_{s1} = 170; c_{s2} = 150; h_2 = 3.9.$

Table 4.3: Sensitivity analysis of $h_1$ and $c_{sh1}$ on total expected cost rate

### 4.6 Conclusion

This chapter has presented a inventory control model for two commodities whose demands are inter dependent. The system performance measures obtained are used to get optimal policy parameters. This model has several important applications in real business world to boost the sale volume. The authors are working in other kind of models such as price break and gift item for bulk order. In our future work, we wish to explore to issue a certain quality of the main item for each fixed bulk quantity demanded instead of giving a complimentary item which the customer may not like.
\( c_{s2} = 8; f = 5; \lambda = 20.8; \mu = 11.9; r = 6; p_i = 0.2 \times 0.8^i, i = 1, 2, \ldots; h_1 = 5.4; h_2 = 3.9; c_{sh1} = 6. \)

Table 4.4: Sensitivity analysis of \( c_{s1} \) and \( c_{s2} \) on total expected cost rate