Chapter 5

Generation of self-induced transparency gap solitary waves through modulational instability in uniformly doped fiber Bragg grating

5.1 Introduction

In chapters 2 and 3, we have discussed the interaction between two co-propagating modes of birefringent single mode fiber. In this chapter, turn our attention to investigate propagation of forward and backward direction modes in an uniformly doped fiber with two-level resonant atoms. By introducing a periodic modulation in an uniformly doped fiber, it is possible to create an interaction between forward and backward directions of an uniformly doped fiber.

The electromagnetic wave propagation in a periodic medium is a well studied phenomenon in the field of solid-state physics [187]. The electronic band gap which exists in semiconductors arises because the electrons move in a periodic medium defined by the crystal lattice. The diffraction of X-rays by a crystal lattice is another example of wave propagation in a periodic structure [187]. The problem of electromagnetic wave
propagation in a periodic waveguide is a similar phenomenon, with the exception that the light is confined in the transverse directions so that the interactions occur only in one dimension. The Bragg grating may be thought of as a one-dimensional diffraction grating which diffracts light from the forward direction into the backward direction. In order to efficiently diffract light in the opposite direction, the reflections from subsequent periods of the grating must interfere constructively. This means that the Bragg period $\Lambda$ (which is the distance between modulations of the refractive index in the grating) must be related to the free space wavelength $\lambda_0$ by:

$$\Lambda = \frac{m\lambda_0}{2n_{\text{eff}}},$$

(5.1)

where $n_{\text{eff}}$ is the effective index of refraction of the structure which depends upon the materials comprising the waveguide and $m$ is the order of Bragg grating. In this chapter, we consider only first-order ($m = 1$) Bragg gratings, because the efficiency of diffraction (cross coupling strength) is generally strongest for the first-order of diffraction [187].

The invention of the fiber Bragg grating (FBG) was considered as one of the prominent milestone events in the history of optical communication. FBG is just a piece of ordinary single-mode fiber a few centimetres long. The grating is constructed by varying the refractive index of the core lengthwise along the fiber periodically as sketched in figure 5.1. As light moves along the fiber and encounters the changes in

![Figure 5.1: The schematic illustration of fibre Bragg grating.](image-url)
refractive index, a small amount of light is reflected at each boundary. Light of the specified wavelength traveling along the fiber is reflected back from the grating in the direction from which it came. This is referred to as Bragg condition (which is given in equation (5.1)), and the wavelength at which this reflection occurs is called Bragg wavelength. The Bragg grating is essentially transparent for incident light at wavelengths other than the Bragg wavelength where the phase matching of the incident and reflected beams occurs. This is the most important characteristic of FBGs; selected wavelengths are reflected back toward the source and non-selected wavelengths are transmitted through the device without attenuation. When the period of the grating and the wavelength of the light are the same then there is constructive reinforcement and power is coupled from the forward direction to the backward direction. Light of other wavelengths encounters interference from out of phase reflections and therefore cannot propagate. If the non-selected wavelengths are reflected but they interfere destructively with themselves. In an electromagnetic resonant circuit, power from the forward direction is coupled into the resonant circuit and then reflected back. As the evanescent field of a wave propagating in a single-mode fiber extends into the cladding, it was affected and controlled by the periodic grating structure written there.

In a FBG, the periodic modulation is created by illuminating a photosensitive fiber with a periodic ultraviolet (UV) standing-wave. Typically, the fiber core is photorefractive, means that the index of refraction can be permanently changed by exposing the fiber to UV radiation. In 1978, Hill et al. [188] discovered experimentally that the refractive index changes in a germanium doped silica optical fiber by launching a beam of intense light into a fiber. They exposed the fiber core to intense contra-directionally propagating coherent beams with single-mode argon-ion laser. A standing wave pattern appeared in the fiber core at visible radiation. The periodic perturbations formed in the fiber core because of its photosensitivity. In 1989, a new
writing technology for FBGs, the UV light side-written technology at 0.244 μm, was
demonstrated by Meltz et al. [189]. FBG developed rapidly after UV light side-written
technology was developed. Since then, much research has been done to improve the
quality and durability of FBGs. This principle was extended to fabricate reflection
gratings at 1.53 μm, a wavelength of interest in telecommunications, also allowing
the demonstration of the first laser operating from the reflection of the photosensitive
fiber grating [190].

In recent years, considerable research has been focused to generate solitons in non-
linear optical periodic structures in fibers. In simple, one-dimensional geometries of
Bragg solitons are very similar to the solitons of conventional optical fibers. In both
of the structures the wave gains its stability through a counter-balancing of the GVD
and the Kerr-nonlinearity. The difference is that for the solitons of conventional opti-
cal fibers the GVD is primarily due to the underlying dispersion of the conventional
optical fibers, while for a grating soliton, it is due to the photonic band structure.
Optical dispersion for wavelengths near to PBG are nearly six-orders of magnitude
larger than the propagation in a uniform material. Large dispersion with nonlinear
changes in the refractive index results in soliton formation in length scales of only
in centimeters. Grating solitons are solitary waves that propagate through a grating
without changing their shapes. They arise from the balancing of the dispersion of the
grating and the self-phase modulation due to the Kerr nonlinearity, and are predicted
theoretically using nonlinear coupled-mode equations [155]. In general, grating soli-
tons are categorized into Bragg solitons and gap solitons with respect to the photonic
band structure. Gap soliton is an optical pulse that propagates at the wavelength
within the PBG. Bragg soliton is formed when the pulse wavelengths are outside the
gap (even at wavelengths nearly outside the gap). The first experimental observation
of Bragg solitons in a fiber with Bragg grating was performed in 1996 [191] under
laser pulse irradiation at a frequency near the PBG.
In the present chapter, we investigate the MI process without threshold conditions and generate SIT gap solitons (both bright and dark) at the PBG edges in the uniformly doped FBG with two-level resonant atoms. SIT solitons are coherent optical pulses propagating through a resonant medium without loss and distortion. Such coherent propagation is described by the Maxwell-Bloch equations [46–48, 59, 60]. Aközbek and John reported that both SIT Bragg and SIT gap solitons exists at the PBG edge and the near PBG edge respectively [62]. Because the dopant density and the atomic detuning frequency dramatically change the characteristics of a SIT-gap soliton, it has been suggested that such solitary propagation may be very useful in optical telecommunications and optical computing [61]. Authors showed that SIT gap soliton solution indicates the coexistence of a SIT soliton and a conventional grating solitons [61, 192].

This chapter is laid out as follows: In section 5.2, we derive NLCM-MB equations to describe the propagation of intensive electromagnetic waves in uniformly doped FBGs with two-level atoms and also present the steady state solutions of Bloch equations. In section 5.3, we apply LSA to identify MI conditions near the edges of the PBG in both anomalous and normal dispersion regimes. We also perform the dint numerical simulation of NLCM-MB equations and then compare the MI gain spectra produced by the LSA with that of MI gain spectra generated by numerical simulation in section 5.4. The next natural step is to discuss the generation of SIT Bragg solitons in terms of the resulting MI maximum gain. Here, we analytically discuss the generation of periodic (cnoidal) waves as well as bright and dark SIT Bragg solitons near the PBG edges in section 5.5. We conclude the result in section 5.6.

5.2 Theoretical Model

In this chapter, we consider a one-dimensional Bragg grating formed in an uniformly doped FBG with two-level resonant atoms. The intensive electromagnetic waves
propagation in such a medium can be described as [61,62,192]

\[
\frac{\partial^2 E}{\partial z^2} = \frac{n(z, \omega)^2}{c^2} \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} + \mu_0 N_D \frac{\partial^2 P_{res}}{\partial t^2}, \quad (5.2a)
\]

\[
\frac{\partial P_{res}}{\partial t} = i \Delta P_{res} - i R W E, \quad (5.2b)
\]

\[
\frac{\partial W}{\partial t} = i \frac{R}{2} (E P_{res}^* - E^* P_{res}), \quad (5.2c)
\]

where $E$ represents the slowly varying envelope of the electric field, $z$ and $t$ represent the longitudinal propagation direction and local time respectively, $P_{NL} = \chi^{(3)} |E|^2 E$ is the nonlinear polarization that results from the Kerr effect, $\chi^{(3)}$ is the Kerr constant, $P_{res}$ is the resonant polarization, $W$ is the population difference, $N_D$ is the density of the doped two-level resonant atoms, $n(z, \omega)$ represents both the nonlinear changes and the periodic variation of the refractive index, $c$ is the light speed in vacuum, $\mu_0$ is the vacuum permeability, $R = \mu/h$, $\mu$ is the dipole matrix element of the individual atom, $h$ is the Planck’s constant divided by $2\pi$, $\Delta (= \omega_r - \omega_B)$ is the frequency detuning from the transition frequency of the resonant atoms $\omega_r$ to the Bragg frequency $\omega_B (= 2\pi c/\lambda_B)$ and $\lambda_B$ is the Bragg wavelength of the grating. In the Fourier domain, equation (5.2a) becomes

\[
\nabla^2 \tilde{E} + \frac{\omega^2}{c^2} \tilde{n}(z, \omega)^2 \tilde{E} + \mu_0 \omega^2 N_D \tilde{P}_{res} = 0, \quad (5.3)
\]

where $\tilde{E}$ and $\tilde{P}_{res}$ are the Fourier transform of $E$ and $P_{res}$ respectively and $\tilde{n}(\omega)$ is defined by

\[
\tilde{n}(\omega) = n_0(\omega) + n_2 |E|^2 + \delta n_g \cos(2k_B z), \quad (5.4)
\]

where $n_0(\omega)$ is the frequency-dependent refractive index of the host medium, $n_2$ is the intensity-dependant refractive index of the host medium, $\delta n_g$ is the magnitude of the periodic-index variation and $k_B (= \pi/\Lambda)$ is the Bragg wave number for a first-order grating. Bragg wave number, $k_B$, is related to the Bragg wavelength through the Bragg condition $\lambda_B = 2\tilde{n}/\Lambda$ and can be used to define the Bragg frequency as
\( \omega_B = \pi c/(n\Lambda) \). To obtain the pulse propagation equations for this uniformly doped PBG structure, we use the perturbation theory of distributed feedback [155] to reduce equation (5.4). Using equation (5.4), the refractive index of the periodic structure, \( \tilde{n}(\omega)^2 \), in equation (5.3) is approximated by

\[
\tilde{n}(\omega)^2 \approx n_0(\omega)^2 + 2n_0(\omega) \left( n_2|\tilde{E}|^2 + \delta n_g \cos(2k_Bz) \right), \quad (5.5)
\]

On substituting equation (5.5) into equation (5.3), we obtain

\[
\nabla^2 \tilde{E} + \frac{n_0(\omega)^2}{c^2} \left( n_0(\omega) + 2(n_2|\tilde{E}|^2 + \delta n_g \cos(2k_Bz)) \right) \tilde{E} + \mu_0\omega^2 N_D \tilde{P}_{res} = 0. \quad (5.6)
\]

The electric field, \( \mathbf{E} \), the resonant polarization, \( \mathbf{P}_{res} \) and the population difference \( W \), in a uniformly doped FBG can be expressed as [61, 62, 192]

\[
\mathbf{E}(x, y, z, t) = \frac{1}{2} \tilde{x} F(x, y) (q_+(z, t)e^{i(k_B z - \omega_B t)} + q_-(z, t)e^{i(k_B z + \omega_B t)} + c.c), \quad (5.7a)
\]

\[
\mathbf{P}_{res}(x, y, z, t) = \frac{1}{2} \tilde{x} F(x, y) (P_+(z, t)e^{i(k_B z - \omega_B t)} + P_-(z, t)e^{i(k_B z + \omega_B t)} + c.c) \quad (5.7b)
\]

\[
W(z, t) = W_0 + W_1 e^{2ik_B z} + W_1^* e^{-2ik_B z}, \quad (5.7c)
\]

where \( F(x, y) \) is the transverse modal distribution, \( q_\pm \) and \( P_\pm \) are the slowly varying envelopes of electric field and resonant polarization respectively and the lower index + and – represent forward and backward direction respectively. For the simultaneous presence of forward and backward direction fields, the population difference oscillates harmonically and \( W \) represents the complex amplitude of the population oscillation [11]. It is noted that the population inversion, \( W \) to be a real quantity for \( W_1 \) must be equal to \( W_1^* \) [11]. By substituting equations (5.7a) and (5.7b) into equation (5.6), we obtain

\[
\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \left( k_0^2 n_0(\omega)^2 - \tilde{k}(\omega)^2 \right) F = 0, \quad (5.8a)
\]

\[
\frac{\partial^2 \tilde{q}_\pm}{\partial z^2} \pm 2ik_B \frac{\partial \tilde{q}_\pm}{\partial z} + \left( \tilde{k}(\omega)^2 - k_B^2 \right) \tilde{q}_\pm + \mu_0\omega^2 N_D \tilde{P}_\pm = 0, \quad (5.8b)
\]
where \( \tilde{k}(\omega) = k(\omega) + \Delta k_{\pm} \) and \( k_0(= \omega/c) \) are the wave numbers that are determined according to the eigenvalues of equation (5.8a). The transverse mode function \( F(x, y) \) can be averaged out by introducing the effective core area \( A_{\text{eff}} \) [155]. Likewise the averaged effects of the coupling strength \( (\kappa) \) and the Kerr nonlinearity \( (\Delta k_{\pm}) \) can be described by

\[
\kappa = \frac{k_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta n_g |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}, \quad (5.9a)
\]

\[
\Delta k_{\pm} = \frac{k_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta n_{\pm} |F(x, y)|^2 dx dy}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}, \quad (5.9b)
\]

\[
\Delta n_{\pm} = n_2 (|\tilde{g}_{\pm}|^2 + 2|\tilde{g}_e|^2). \quad (5.9c)
\]

In equation (5.8b), we have approximated \((\tilde{k}(\omega)^2 - k_B^2)\) by \(2k_B(\tilde{k}(\omega) - k_B)\) and \(k(\omega)/k_B \approx 1\), which indicate that the grating wave number is close to the mode-propagation constant. The effect of dispersion can be accounted for by expanding the mode-propagation constant \((k(\omega))\) in Taylor series about the Bragg frequency \(\omega_B\):

\[
k(\omega) = k_0 + (\omega - \omega_B) \beta_1 + ..., \quad (5.10)
\]

and higher-order terms are neglected where \(\beta_1(= d k(\omega)/d \omega = 1/v_g)\) is related inversely to the group velocity \((v_g)\). Taking the inverse Fourier transform of equation (5.8b) results in the time-domain propagation equations with equations (5.9c) and (5.10) and \((\omega - \omega_B)\) and \(\omega^2\) replaced by \(i \partial/\partial t\) and \(\omega_B^2\) respectively. Equation (5.8b) can be written in the time-domain as follows [61, 62, 192]

\[
\frac{\partial q_+}{\partial z} + \frac{\partial q_+}{\partial t} - i \delta q_+ - i k q_- - i \gamma (|q_+|^2 + 2|q_-|^2) q_+ - i \Gamma P_+ = 0, \quad (5.11a)
\]

\[
\frac{\partial q_-}{\partial z} - \frac{\partial q_-}{\partial t} + i \delta q_- + i k q_+ + i \gamma (2|q_+|^2 + |q_-|^2) q_- - i \Gamma P_- = 0, \quad (5.11b)
\]

where \(\Gamma = \mu \omega_B^2 N_D/(2k_B)\), \(q_+\) and \(P_+\) are the slowly varying envelopes of the electric field and the resonant polarization respectively, \(\delta\) is the pulse wave number that
detunes from the exact Bragg resonance, $\kappa$ is the linear coupling coefficient, $\gamma(= n_2 k_0 / A_{\text{eff}})$ is the self phase modulation (SPM) and $A_{\text{eff}}$ is the effective core area. Here, we considered that the group velocity ($v_g$) is maximum, is normalized to be unity [193,194]. Next, we turn to derive the atomic Bloch equations for the uniformly doped periodic medium. Substituting equations (5.7a), (5.7b) and (5.7c) into equations (5.2b) and (5.2c), we obtain [61,62,192]

$$\frac{\partial P^+}{\partial t} - i\Delta P^+ + iR(q_+ W_0 + q_- W_1) = 0,$$  
(5.12a)

$$\frac{\partial P^-}{\partial t} - i\Delta P^- + iR(q_- W_0 + q_+ W_1^*) = 0,$$  
(5.12b)

$$\frac{\partial W_0}{\partial t} - i\frac{R}{2}(q_+ P^*_0 + q_- P^*_1 - q_1^* P_+ - q_+^* P_-) = 0,$$  
(5.12c)

$$\frac{\partial W_1}{\partial t} - i\frac{R}{2}(q_+ P^*_0 - q_-^* P_+) = 0.$$  
(5.12d)

Equations (5.11)-(5.12) are the governing equations from which one can obtain the MI conditions and SIT gap solitons at PBG edges.

### 5.2.1 Steady state solution to the Bloch equations

The resonant polarization, $P_{\text{res}}$, and the population difference, $W$, satisfy the following normalization condition [48, 59, 60]:

$$W^2 + |P_{\text{res}}|^2 = 1,$$  
(5.13)

which reflects the conservation of probability in the sense that the total probability for an atom to be found either in the upper or lower levels is equal to unity. The steady-state response follows from setting the time derivative equal to zero in the Bloch equations (5.2b) and (5.2c). The quasi-static atomic polarization satisfies the equations as

$$P_{\text{res}} = \left(\frac{R}{\Delta}\right) WE,$$  
(5.14)
On substituting equation (5.14) into equation (5.13), we obtain the steady state solution of population difference as

$$W = \pm \left(1 - \frac{|E|^2}{2p_s}\right),$$  \hspace{1cm} (5.15)

where $p_s = (\Delta/R)^2$. Here, we have assumed that the field intensity ($|E|^2$) is sufficiently small compared with $p_s$ ($|E|^2 \ll p_s$). Without the field-induced polarization, the two-level system population is not inverted, hence the lower sign must be chosen in equation (5.15) [161]. By substituting equation (5.15) into equation (5.14), we obtain

$$P_{res} = -\left(\frac{R}{\Delta}\right)\left(1 - \frac{|E|^2}{2p_s}\right)E,$$  \hspace{1cm} (5.16)

The steady state solutions of the resonant polarization, $P_{res}$ and the population difference, $W$ are obtained so that equation (5.13) is satisfied automatically. On inserting equations (5.7) into equations (5.15) and (5.16), obtain the steady state solution of the Bloch equations for an uniformly doped FBG as

$$P_\pm = -\left(\frac{R}{\Delta}\right)\left(1 - \frac{|q_\pm|^2 + 2|q_\mp|^2}{2p_s}\right)q_\pm,$$  \hspace{1cm} (5.17a)

$$W_0 = -\left(1 - \frac{|q_\pm|^2 + |q_\mp|^2}{2p_s}\right),$$  \hspace{1cm} (5.17b)

$$W_1 = \left(\frac{q_+q_-^*}{2p_s}\right).$$  \hspace{1cm} (5.17c)

Before studying the properties of the full set of NLCM-MB equations we first consider the limiting case in which the amplitude of the forward and backward traveling waves is small enough to neglect the nonlinear terms. The properties of the thus obtained linearized set are quite important as they can help in the understanding of the full set of equations. Specifically, an eigen vector analysis of the linear system proves to be a very convenient starting point for describing the properties of the nonlinear system. Thus neglecting now the nonlinear terms ($\gamma = 0$) and resonant polarization ($\Gamma = 0$),
we rewrite equations (5.11a) and (5.11b) as

\[
\begin{pmatrix}
\frac{\partial}{\partial z} - i\delta & -i\kappa \\
 i\kappa & \frac{\partial}{\partial z} + i\delta
\end{pmatrix}
\begin{pmatrix}
q_+ \\
q_-
\end{pmatrix} = 0.
\]

(5.18)

We can find the dispersion relation associated with these equations by using the following ansatz

\[
\begin{pmatrix}
q_+ \\
q_-
\end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} e^{i\phi z},
\]

(5.19)

into equation (5.18), we obtain

\[
i \begin{pmatrix}
\phi - \delta & -\kappa \\
\kappa & \phi + \delta
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix} = 0.
\]

(5.20)

Setting the determinant equal to zero implies that \(\phi\) and \(\delta\) are related by [155, 194]

\[
\delta = \pm \sqrt{\kappa^2 + \phi^2}.
\]

(5.21)

This equation is of paramount importance for gratings. This relation is illustrated in figure 5.2; the dashed line gives the result in the absence of the grating, i.e., when \(\kappa \to 0\), while solid lines refers to a finite value for \(\kappa\). We see from these expressions that no solutions for the Bragg detuning can be found in the range \(-\kappa \leq \delta \leq \kappa\), so that no traveling-wave solutions in this range are allowed. The parameter \(\phi\) becomes purely imaginary when Bragg detuning, \(\delta\), of the incident light falls in this range. Most of the incident wave is reflected in that case since the grating does not support a propagation wave. The range \(|\delta| \leq \kappa\) is referred to as PBG [155,194,195], in analogy with electronic energy bands occurring in crystals. For frequencies within PBG the grating reflectivity is high, as the field envelopes are then evanescent, while outside PBG the reflectivity is much lower. Outside PBG, propagating wave solutions do exist, with Bloch functions which in general are superpositions of forward and backward propagating plane waves.
Since the Bloch functions are not pure forward and backward propagating waves, the associated group velocity is expected to be below that of the speed of light in bare fiber. For frequencies within PBG no propagating wave solutions for the electric field exist. Indeed, this is easily seen from figure 5.2 as the slope $d\delta/d\phi$ becomes progressively smaller toward the edges of PBG. At the edges, the slope vanishes, implying the absence of energy transport. The reduction in the group velocity is associated with the multiple Fresnel reflections at the grating rulings, which, in effect, slow down the light. In FBG, by using equation (5.21) the group velocity can be calculated as

$$V_g \approx \frac{d\omega_B}{dk_B} = \left( \frac{c}{\bar{n}} \right) \frac{d\delta}{d\phi},$$  \hspace{1cm} (5.22)
where

\[
\frac{d\delta}{d\phi} = \pm \frac{c}{n} \sqrt{1 - \frac{\kappa^2}{\delta^2}},
\]

the plus sign/minus sign applies to the forward/backward propagating mode respectively. Thus at PBG edges, where \( \delta = \pm \kappa \) we find \( V_g = 0 \) as required, while for \( |\delta| \to \infty \), \( V_g = c/\bar{n} \). The group velocity of the light can vary between zero and \( c/\bar{n} \) which shows that the optical signals can travel at any velocity from zero to the average speed of light in the medium. At frequencies close to PBG edges (\( |\delta| < \kappa \)), the grating exhibits the strong GVD, which strongly affects light propagation. The effect of the grating induced GVD can be calculated from equation (5.22). By differentiating equation (5.22), we calculate GVD at PBG as

\[
\beta_{g2} = \frac{d^2 \omega_B}{d k_B^2} = \pm \frac{c}{\bar{n} \kappa},
\]

(5.23)

here the plus sign is the upper branch (\( \delta > 0 \)) of the dispersion curve (figure 5.2) where grating induced GVD is anomalous dispersion and the minus sign represent the lower branch of the dispersion curve (\( \delta < 0 \)) of the dispersion curve (figure 5.2) where grating induced GVD is normal dispersion. For typical physical parameter values in the above equation, GVD in FBG is calculated as, \( \beta_{g2} = 6 \times 10^5 m^2/s \), while for a typical conventional optical fiber \( \beta_2 \approx 0.22 m^2/s \). From this fact, it is clear that the grating induced GVD in FBG is about six-order of the magnitude greater than GVD in conventional fiber [194].

### 5.3 Modulational instability

From a portable device point of view, a special type of fiber called FBG is preferred rather than conventional telecommunication fiber, as FBG offers a huge amount of dispersion [155, 181, 197]. The MI has also been studied in a FBG at low and high
power levels for anomalous (upper branches) and normal (lower branches) dispersion regimes respectively [155, 181, 201]. MI has been observed experimentally in an apodized grating structure wherein a single pulse has been converted into a train of ultra-short pulses [202, 203]. In addition to temporal instabilities, spatial temporal instabilities have also been studied in a nonlinear bulk medium with Bragg gratings in the presence of Kerr-type nonlinearity [204]. The impact of non-Kerr nonlinearity in terms of MI has also been studied in a FBG where the system possesses cubic-quintic nonlinearities [197]. At this juncture it should be pointed out that in the aforementioned conventional FBGs, MI in the normal dispersion regime has threshold condition [155, 181]. To overcome this problem, dynamic grating has been proposed wherein the occurrence of MI was demonstrated experimentally without any threshold condition in the normal dispersion regime [205, 206]. Very recently, the role of nonlinearity management in terms of MI has also been investigated in Refs. [198–200].

So far, for PBG structured material, dynamic grating has been the only choice to achieve MI in the normal dispersion regime with low power without any threshold condition [205, 206]. In general, from the implementation point of view, it may be difficult to achieve ultra-short pulses in the normal dispersion regime of dynamic FBGs. However, keeping this in mind, we propose to use FBGs doped uniformly with two-level resonant atoms, with which MI can be achieved in the normal dispersion regime at low power and without any threshold condition. In this work, we investigate the occurrence of MI for both anomalous and normal dispersion regimes. Especially, we show that there is no threshold condition for occurrence of MI in the normal dispersion regime. In the detailed analysis, we find that the SIT effect induces the non-conventional sidebands in doped FBGs. In addition, the formation of SIT solitons near the PBG edge is also discussed. Non-conventional sidebands induced by SIT effect in the normal dispersion regime and the generation of SIT solitons in terms of the resulting MI are considered to be the main theme of the present chapter.
In this section, we apply the LSA to investigate the occurrence of MI. In LSA, the governing equations ((5.11a) and (5.11b)) are linearized and find the appropriate steady state solutions. Thus, the prime aim of LSA is to perturb the CW solution. Then, we analyze whether this small perturbation grows or decays with propagation. It is obvious that LSA is valid as long as the perturbation amplitude remains low compared with the CW beam amplitude. By using equations (5.17a)-(5.17c), the steady state solutions of the equations (5.11) and (5.12) can be written as follows:

\[ q_\pm = U_\pm e^{i\phi z}, \]  
\[ P_\pm = \gamma \pm U_\pm e^{i\phi z}, \]  
\[ W_J = s_J, \]  

where \( J = 0, 1, U_+ = \sqrt{p_0/(1+f^2)}, U_- = \sqrt{p_0/(1+f^2)} f, \gamma_\pm = (R/\Delta) s_\pm, s_0 = -(1-p_0/2p_\mp), s_1 = (fU_+^2/(2p_\pm)), s_+ = -(1-(1+2f^2)s_1/f), s_- = -(1-(2+f^2)s_1/f) \) and \( p_0 \) is the total power. Here, \( f = q_-/q_+ \) can be positive or negative. For values of \( |f| < 1 \), the backward wave dominates. On substituting equations (5.24) into equations (5.11a) and (5.11b), we obtain the following nonlinear dispersion relation

\[ \phi = -\frac{\kappa}{2f} (1-f^2) - \frac{1}{2} (1-f^2) H + \frac{1}{2} (\gamma_+ - \gamma_-) \Gamma, \]  
\[ \delta = -\frac{\kappa}{2f} (1+f^2) - 3\gamma p_0 - \frac{1}{2} (\gamma_+ + \gamma_-) \Gamma, \]  

where \( H \equiv \gamma p_0/(1+f^2) \) is an effective nonlinear parameter. Here, the preceding nonlinear dispersion relation leads to the conventional FBG case if we switch off the SIT effect (\( \Gamma = 0 \)) [155]. In addition, the resulting nonlinear dispersion relation reduces to the linear dispersion relation when the nonlinear effect is turned off (\( p_0 = 0 \)). Now, through this linear dispersion relation, we try to explore the role of dopants, that is, SIT effect by means of dispersion curves. Figures 5.3 describe the linear (\( p_0 = 0 \)) dispersion curves for uniformly doped FBG with two-level resonant atoms.
and conventional FBG. In figures 5.3 (a) and 5.3(b), dotted, dashed, and dot-dashed lines illustrate the dispersion curves of uniformly doped FBG for $\Delta = \pm 10^{12} Hz$, $\Delta = \pm 10^{13} Hz$ and $\Delta = \pm 10^{14} Hz$, respectively and the solid lines represent the dispersion curves of the conventional FBG (for $\Gamma = 0$ in equation (5.25)). For comparatively lower values of the atomic resonant detuning parameter $\Delta$ the dispersion curves do undergo shift-down (figure 5.3(a)) or shift-up (figure 5.3(b)) depending upon the negative or positive sign of $\Delta$. For instance, for $\Delta = \pm 10^{14} Hz$, the linear dispersion curves (dot-dashed lines in figures 1(a) and 1(b)) nearly close to the conventional FBG. From figures 5.3(a) and 5.3(b), it is very clear that the linear dispersion curves

**Figure 5.3:** Linear dispersion relation curve for $\kappa = 20 cm^{-1}$, $\Gamma = 0$ (solid line), $\Delta = \pm 10^{12} Hz$ (dotted line), $\Delta = \pm 10^{13} Hz$ (dashed line), $\Delta = \pm 10^{14} Hz$ (dot-dashed line) and $p_0 = 0$. The solid curve represents the linear dispersion relation for conventional FBG ($\Gamma = 0$).
have been dramatically changed their characteristics by the atomic resonant detuning frequency $\Delta$. The detuning parameter $\delta$ of the CW beam from the Bragg frequency determines the values of $f$, which in turn fixes the values of $\phi$ in equations (5.25a) and (5.25b). The band edges occur when $\phi = 0$ at wave numbers $\delta_\pm = \pm \kappa$. For wave numbers $-\kappa < \delta > \kappa$, $\phi$ is purely imaginary and the field is exponentially attenuated within the medium. As a consequence, incident radiation of low intensity is completely reflected back. Outside the band gap, $\phi$ is real, facilitating linear wave propagation within the medium. The group velocity ($V_G$) inside the grating also depends on $f$ and is given by

$$V_G = \frac{d\delta}{d\phi} = \left(\frac{1-f^2}{1+f^2}\right).$$  \hspace{1cm} (5.26)

The upper ($f < 0$) and lower ($f > 0$) branches of the dispersion curve represent the anomalous and normal dispersion regimes, respectively. The PBG edges occur at $f = \pm 1$. The LSA of steady state solutions can be examined by introducing perturbed fields of the following form:

$$q_\pm = \left(U_\pm + a_\pm \cos(Kz + \Omega t) + ib_\pm \sin(Kz + \Omega t)\right)e^{i\phi z}, \hspace{1cm} (5.27a)$$

$$P_\pm = \Upsilon_\pm \left(U_\pm + c_\pm \cos(Kz + \Omega t) + id_\pm \sin(Kz + \Omega t)\right)e^{i\phi z}, \hspace{1cm} (5.27b)$$

$$W_0 = s_0 \left(1 + w_0 \cos(Kz + \Omega t)\right), \hspace{1cm} (5.27c)$$

$$W_1 = s_1 \left(1 + w_+ \cos(Kz + \Omega t) + iw_- \sin(Kz + \Omega t)\right), \hspace{1cm} (5.27d)$$

where $a_\pm, b_\pm, c_\pm, d_\pm, w_0,$ and $w_\pm$ are real amplitudes of infinitesimal perturbations, $K$ is the perturbed wave number, and $\Omega$ is the respective eigenvalue. Substituting ansatz (5.27a)-(5.27d) into equations (5.11)-(5.12) and performing the linearization, we obtain the eleven nontrivial equations for the perturbed fields $a_\pm, b_\pm, c_\pm, d_\pm, w_0,$ and $w_\pm$, which can be written in an $11 \times 11$ matrix form. This set has a nontrivial solution only when the $11 \times 11$ determinant formed by the coefficients matrix vanishes
as follows:

$$
\begin{array}{cccccccc}
  m_{11} & 0 & 0 & 0 & 0 & 0 & m_{17} & -\kappa & m_{19} & 0 & 0 \\
  0 & m_{22} & 0 & 0 & 0 & 0 & \kappa & m_{28} & 0 & m_{210} & 0 \\
  0 & 0 & m_{33} & 0 & 0 & 0 & s_0 & -s_1 & s_+ & 0 & m_{311} \\
  0 & 0 & 0 & m_{44} & 0 & 0 & -s_1 & s_0 & 0 & s_- & 0 \\
  0 & 0 & 0 & 0 & m_{55} & 0 & m_{57} & m_{58} & m_{59} & m_{510} & 0 \\
  0 & 0 & 0 & 0 & 0 & m_{66} & m_{67} & m_{68} & m_{69} & m_{610} & 0 \\
  m_{71} & m_{72} & m_{73} & 0 & 0 & 0 & m_{77} & 0 & 0 & 0 & 0 \\
  m_{81} & m_{82} & 0 & m_{84} & 0 & 0 & 0 & m_{88} & 0 & 0 & 0 \\
  -s_0 & s_1 & -s_+ & 0 & 0 & m_{96} & 0 & 0 & m_{99} & 0 & 0 \\
  s_1 & -s_0 & 0 & -s_- & 0 & 0 & 0 & 0 & 0 & m_{1010} & 0 \\
  m_{111} & m_{112} & m_{113} & m_{114} & 0 & 0 & 0 & 0 & 0 & 0 & m_{1111} \\
\end{array}
\right) = 0, \quad (5.28)
$$

where

$$
\begin{align*}
  m_{11} & = -m_{77} \equiv K - \Omega, \quad m_{17} = f \kappa + \frac{\Gamma R s_+}{\Delta}, \quad m_{19} = -m_{73} = -\frac{\Gamma R s_+}{\Delta}, \\
  m_{22} & = m_{88} \equiv K + \Omega, \quad m_{28} = -\frac{\kappa}{f} - \frac{\Gamma R s_-}{\Delta}, \quad m_{210} = m_{84} \equiv \frac{\Gamma R s_-}{\Delta}, \\
  m_{33} & = -m_{99} \equiv \frac{\Omega s_+}{\Delta}, \quad m_{311} = -m_{96} \equiv -s_1 U_-, \quad m_{44} = m_{1010} \equiv \frac{\Omega s_-}{\Delta}, \\
  m_{55} & = \Omega s_0, \quad m_{57} = -m_{59} = -\frac{R^2 U_+ s_1}{\Delta}, \quad m_{58} = m_{510} = -\frac{R^2 U_- s_+}{\Delta}, \\
  m_{66} & = m_{1111} \equiv \Omega s_1, \quad m_{67} = -\frac{m_{610}}{f} = -m_{111} = -\frac{m_{114}}{f} = -\frac{R^2 U_- s_-}{2\Delta}, \\
  m_{68} & = -f m_{69} = m_{112} = f m_{113} \equiv \frac{R^2 U_+ s_+}{2\Delta}, \quad m_{71} = 2H - f \kappa - \frac{\Gamma R s_+}{\Delta}, \\
  m_{72} & = m_{81} \equiv \kappa + 4f H, \quad m_{82} \equiv 2f H f^2 - \frac{\kappa}{f} - \frac{\Gamma R s_-}{\Delta}.
\end{align*}
$$

It is well established that MI occurs when there is an exponential growth rate (gain) in the amplitude of the perturbed wave, which in turn implies the existence of a non-vanishing imaginary part in the complex parameter $\Omega$. For the case of
FBG uniformly doped with two-level resonant atoms, MI occurs when there is an exponential growth in the amplitude of the perturbed wave which implies the existence of a non-vanishing largest imaginary part in the complex parameter $G(K) \equiv \Omega$. For $\Upsilon_{\pm} = 0$, the eigenvalue in equation (5.28) is tantamount to that found in [155,181].

![Figure 5.4: MI gain spectrum obtained from the LSA for anomalous dispersion regime (upper branch) for $\kappa = 20 \, cm^{-1}$, $H = 0.5 \, \kappa$, $\Upsilon_{\pm} = 0$ (solid line), $\Delta = \pm 10^{12} \, Hz$ (dotted line), $\Delta = \pm 10^{13} \, Hz$ (dashed line) and $\Delta = \pm 10^{14} \, Hz$ (dot-dashed line). The solid line represents the MI gain spectrum of conventional FBG ($\Upsilon_{\pm} = 0$).](image)

Here, we aim to display the MI gain spectra, as functions of $K$, $\Delta$, $f$, and $H$, for both the anomalous (upper branch for $f < 0$) and the normal (lower branch for $f > 0$) dispersion regimes. We examine below various kinds of behaviors that arise when the sign and the magnitude of the atomic resonant detuning parameter $\Delta$ are varied. For demonstration purposes, we consider the following physical parameters:
\[ \gamma = 0.002 \text{W}^{-1}\text{m}^{-1}, \quad \kappa = 20 \text{cm}^{-1}, \quad \lambda_B = 1.55 \times 10^{-6} \text{m}, \quad \epsilon_0 = 8.854 \times 10^{-12} \text{F}\text{m}^{-1}, \]
\[ \mu = 1.4 \times 10^{-32} \text{C}\text{m}, \quad \hbar = 1.0545 \times 10^{-34} \text{Js} \text{ and } N_D = 8 \times 10^{24} \text{m}^{-3}. \]
Here, we carry out the MI analysis for different values of the atomic resonant detuning parameter, \[ \Delta = \pm 10^{12} \text{Hz}, \pm 10^{13} \text{Hz} \text{ and } \pm 10^{14} \text{Hz}, \]
as we are interested in investigating the influence of the atomic resonant detuning parameter. Figures 5.4 (a)-(d) represent the MI gain spectra for both the PBG edge and near the PBG edge for various values \( \Delta \) in the range \( \pm 10^{12} \text{Hz} \) to \( \pm 10^{14} \text{Hz} \). Figures 5.4 (a) and 5.4 (c) illustrate the MI gain spectra at the PBG edge and near the PBG edge, respectively for a negative sign of the atomic resonant detuning parameter \( \Delta \) with various values \( (-10^{12} \text{Hz} \text{ to } -10^{14} \text{Hz}) \), whereas figures 5.4 (b) and 5.4 (d) show the MI gain spectra for a positive sign of \( \Delta \) \( (10^{12} \text{Hz} \text{ to } 10^{14} \text{Hz}) \). The solid lines in figures 5.4 represent the MI gain spectra for the conventional FBG \( (\Gamma = 0) \). When \( \Delta = \pm 10^{14} \text{Hz} \) (dot-dashed line), the MI gain spectra coincide with conventional FBG. From this physical process, we infer that the system ceases to hold the effect of SIT for relatively higher values of atomic resonant detuning parameter, \( \Delta = \pm 10^{14} \text{Hz} \). Thus, the impact can be realized only for relatively lower values of atomic resonant detuning parameter \( (|\Delta| < 10^{14}) \). In figures 5.4 (a) and 5.4 (c), dotted lines illustrate that the corresponding optimum modulation wave number (It is the wave number at which maximum gain occurs) and peak gain (maximum gain) are relatively large for \( \Delta = -10^{12} \text{Hz} \) (dotted lines). However, increasing the magnitude of the negative values of \( \Delta \) leads to a shrinkage of the MI gain spectra, whereas both the peak gain and the bandwidth decrease rapidly for \( \Delta < 0 \). From figures 5.4 (b) and 5.4 (d), we observe the following general features as: Both the peak gain and the bandwidth of the MI gain spectra increase with the atomic resonant detuning parameter, \( \Delta (> 10^{12} \text{Hz}) \). For \( \Delta = 10^{12} \text{Hz} \) (dotted lines), all the wave numbers with a magnitude below a certain values are stable, while all wave numbers with a larger magnitude are unstable, which is in great contrast with the conventional FBG [155,181]. Another important point is that the MI bandwidth
becomes infinite for $\Delta = 10^{12}\,Hz$. Here, we infer that the bandwidth is infinite for $\Delta \leq 10^{12}\,Hz$ whereas the bandwidth is finite when $\Delta > 10^{12}\,Hz$. The dashed lines depict the MI gain spectra for $\Delta = \pm 10^{13}\,Hz$. In addition to the two-dimensional plots in (figures 5.4), the three-dimensional plots shown in figures 5.5 (a) and 5.5 (b), represent the MI sidebands for a low power ($H < \kappa$). The peak gain and the bandwidth of the MI sidebands increase as the effective nonlinear parameter, $H$, increases. Figures 5.4 and 5.5 clearly show that the peak gain is relatively higher at the top of the PBG ($f = -1$) than near the PBG edge ($f < 0$).

Now we turn to discuss the MI in the normal dispersion (lower branch) regime. Our main aim is to overcome the finite threshold conditions wherein the conventional FBG MI process in the normal dispersion regime. The conventional FBG MI process has threshold conditions in the normal dispersion regime as follows [181,201]. (i) The instability required a finite threshold condition, $H > \kappa/2$, where $H$ is the effective nonlinear parameter. Thus, if the input power is sufficiently low, then the continuous wave signal is stable against small perturbations: only when the power exceeds threshold condition ($H > \kappa/2$) is the signal unstable. (ii) For $f > 0.447$, all the wave numbers with a magnitude below a certain value are stable, while all wave numbers with a larger magnitude are unstable. In this case, wave numbers are infinite. Most of

![Figure 5.5: MI gain spectrum obtained from the LSA for anomalous dispersion regime [(a) f=-1 and (b) f=-0.5] for $\kappa = 20\,cm^{-1}$ and $\Delta = -10^{12}\,Hz$.](image-url)
the unstable wave numbers are finite at $f > 0.447$. Figures 5.6 and 5.7 represent the occurrence of MI in the normal dispersion regime. Also, the resulting MI gain spectra strongly depends upon the atomic resonant detuning parameter $\Delta$ in the normal dispersion regime (figures 5.6). Figures 5.6 (a)-5.6(d) illustrate the following important features: One can easily see that the MI gain spectra are close to the conventional FBG for the larger magnitude values of the atomic resonant detuning parameter, $|\Delta| > 10^{12} \text{Hz}$. From this physical process, we conclude that the doped FBG system ceases to hold the effect of SIT for $\Delta = \pm 10^{13} \text{Hz}$ (dashed lines) and $\Delta = \pm 10^{14} \text{Hz}$ (dot-dashed lines). In this case, we find that all wave numbers with a magnitude...
below a certain values are stable, while all wave numbers of a larger magnitude are unstable. This behaviour confirms the similarity to the conventional FBG where the MI process has a finite threshold condition [155,181,201]. The role of the SIT effect can be realized only close to resonance for $\Delta = \pm 10^{12} \text{Hz}$ (dotted lines). In the normal dispersion regime, the SIT effect can be nullified by large magnitude values of the atomic resonant detuning parameter ($|\Delta| > 10^{12} \text{Hz}$), which means doped FBG acts as a conventional FBG. In the normal dispersion regime, the atomic resonant detuning parameter $\Delta$ can induce the MI with non-conventional gain spectra where there is no any threshold condition (dotted lines in figures 5.6 (a) and 5.6 (c)). Here, we observe that all wave numbers are unstable with finite wave number and the instability exists even at an offset wave number ($K=0$). In general, the non-convention MI processes have a substantial advantage over the ordinary conventional MI process, that is, they offer more possibilities to obtain a large MI bandwidth.

Figure 5.7: MI gain spectrum obtained from the LSA for normal dispersion regime [(a) $f=1$ and (b) $f=0.5$]. The physical parameters are $\kappa = 20 \text{cm}^{-1}$ and $\Delta = -10^{12} \text{Hz}$.

Figures 5.7(a) and 5.7(b) show that the MI gain spectra can occur without any threshold conditions such as low power ($H < \kappa$) and all wave numbers are unstable at the normal dispersion regime. The peak value of the gain spectrum grows and the sidebands broaden as the effective nonlinear parameter $H$ increases, which is represented in figure 5.7. From this physical process, the MI gain spectra depend
strongly on the input power. Figures 5.7 (a) and (b) represent the MI spectra at the bottom (normal dispersion regime) of the PBG edge ($f = 1$) and near the PBG edge ($f = 0.5$), respectively.

Near the resonance, $\Delta = -10^{12} \text{Hz}$ (doted lines in figures 5.6(a), 5.6(c), and 5.7), we find the non-conventional MI processes in the lower branch (normal dispersion regime) of uniformly doped FBG, which are induced by the atomic resonant detuning parameter $\Delta$. In this case, the MI process differs qualitatively from the conventional FBG in two respects. The first of these is that this MI process has no threshold, which in turn implies that the continuous wave is unstable. The second major difference is the shape of the MI gain spectrum. In the lower branch, all the wave numbers with all values are unstable, and instability can occur at a finite wave number. Thus, such kind of abnormal behavior should be contrasted with the conventional FBG where the MI process has a threshold.

5.4 Direct simulations

In the previous section, the MI was analyzed by means of LSA. The LSA is primarily of approximate nature, and therefore, leads to results that should be confirmed by the direct simulation of the NLCM-MB equations (equations (5.11) and (5.12)). Here, we aim to compare the predictions with direct simulation of the NLCM-MB equations, adding small initial perturbations to CW states, for the cases of both the anomalous and the normal dispersion regimes. It is shown by the typical examples displayed in figures 5.8 and 5.9 that the gain spectra, especially the optimum modulation wave number and the peak gain, as predicted by the LSA, agree with their counterparts that can be extracted from direct numerical results (Table 5.1). The stability of the steady-state solution given by equation (5.24) was tested by adding small initial perturbations to it. Simulations of NLCM-MB equations with such initial conditions were performed (using Matlab), by means of the SSFM. The MI gain was extracted
from results of the simulations which show the growth of the intensity fluctuations seeded by the small initial perturbations.

Figure 5.8: MI gain spectrum [(a1) and (b1)] obtained from the LSA for anomalous dispersion regime, and restored from direct simulations the NLCM-MB equations [(a2) and (b2)]. The physical parameters are \( \kappa = 20 \, \text{cm}^{-1} \), \( H = 0.5 \kappa \) and \( \Delta = -10^{12} \, \text{Hz} \).
Table 5.1: The optimum modulation wave number \( K_{\text{opt}} \) and peak gain \( G_{\text{peak}} \) are obtained from Figs. 5.8 and 5.9.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>( f )</th>
<th>( K_{\text{opt}} , \text{cm}^{-1} )</th>
<th>( G_{\text{peak}} , \text{dB/cm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>LSA</td>
<td>NLCM-MB</td>
</tr>
<tr>
<td>1</td>
<td>-1.0</td>
<td>48.00</td>
<td>48.00</td>
</tr>
<tr>
<td>2</td>
<td>-0.5</td>
<td>47.00</td>
<td>47.00</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>15.00</td>
<td>15.00</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>06.75</td>
<td>06.75</td>
</tr>
</tbody>
</table>

Figure 5.9: MI gain spectrum [(a1) and (b1)] obtained from the LSA for normal dispersion regime, and restored from direct simulations the NLCM-MB equations [(a2) and (b2)]. The physical parameters are \( \kappa = 20 \, \text{cm}^{-1}, H = 0.5 \kappa \) and \( \Delta = -10^{12} \, \text{Hz} \).
5.5 Generation of self-induced transparency solitons at photonic band gap edges

Having discussed the MI conditions, we proceed to find the peak gain with a minimum power, which will be utilized to discuss the generation of SIT gap solitons near the PBG of uniformly doped FBG. Bragg and gap solitons have been extensively investigated by many research groups in FBG [207–210] and investigations of these exciting entities are still alive. Gap solitons have pulse spectra lying entirely within a PBG [210, 211]. The term ‘gap soliton’ was first introduced in 1987 by Chen and Mills [212]; then Mills and Trullinger [213] proved the existence of gap solitons by analytic methods. Later, Sipe and Winful [214] and de Sterke and Sipe [215] showed that the electric field satisfies a NLS equation, which allows soliton solutions with carrier frequencies close to the edge of the stopband. Christodoulides and Joseph [209], and Aceves and Wabnitz [207] obtained soliton solutions with carrier frequencies close to the Bragg resonance. In 1996, Eggleton et al. [211] reported a direct observation of soliton propagation and pulse compression in uniform fiber gratings, verifying experimentally for the first time the theories developed by Christodoulides et al. [209] and Aceves et al. [207]. This was followed by a further report [203], which both refined the experimental technique and broadened the experimental understanding of the dynamics of pulse propagation in these structures. In doped FBGs, the grating induced dispersion balances with both the material Kerr nonlinearity and the resonant effects determined by the Bloch equations. The resulting solitons are known as SIT Bragg solitons which are essentially distortionless optical pulses. Because of the balance based on the pure grating dispersion, the doping concentration and the atomic resonant detuning frequency can dramatically change the characteristics of a SIT Bragg soliton when the carrier frequency is close to the original edges of the band gap. The distortionless SIT Bragg soliton pulses have been investigated in a
uniformly doped nonlinear PBG structure [61, 62, 192]. The authors have found both the gap SIT soliton in a periodic array of thin layers of resonant two-level atoms separated by the half-wavelength non-absorbing dielectric layers, that is, a resonantly absorbing Bragg reflector [161, 185, 216–218].

Mantsyzov and Kuzmin [219] studied nonlinear pulse propagation in a discrete one-dimensional medium made of two-level atoms. Kozhekin and Kurizkai [217] extended the model first discussed to a continuous medium in which thin layers of resonant atoms were placed at regular intervals inside the periodic dielectric medium. Aközbek and John [62] investigated the properties of SIT solitary waves in a one-dimensional nonlinear periodic structure doped uniformly with resonant two-level atoms. In addition, they have also reported the SIT gap solitons whose central frequency was detuned near the PBG edge. It has been mentioned that these SIT solitons could be useful in optical communications and optical computing since the dopant density and atomic resonant detuning frequency dramatically change the characteristics of SIT gap solitons. Recently, Tseng and Chi [61] investigated the existence of moving SIT pulse train in a uniformly doped PBG structure. They reduced the NLCM-MB equations into equivalent NLCM equations called effective NLCM equations. They have solved the effective NLCM equations and investigated the aforementioned pulse-train soliton solutions in a uniformly doped nonlinear periodic structure. Following the similar work, they also discussed the co-existence of a SIT soliton and a Bragg soliton in a nonlinear PBG medium doped uniformly with inhomogeneously broadening two-level atoms. For this purpose, they derived the effective NLS equation from the effective NLCM equations to discuss the SIT-Bragg soliton near the PBG structure.

In this section, we will discuss how to generate SIT gap solitons at PBG edges. If the grating parameters are constant over the spectral bandwidth of the pulse and the central frequency of the incident pulse is near to the PBG edges, NLCM-MB equations
can be reduced to a nonlinear Schrödinger-Maxwell-Bloch (NLS-MB) equations [61, 62]. By using the 2 × 2 Pauli matrices, NLCM-MB equations can be written in a compact form as [62]

\[
\sigma_x \frac{\partial Q}{\partial z} + \frac{\partial Q}{\partial t} - i\kappa \sigma_z Q - \frac{i\gamma}{2} \left(3(Q^\dagger Q) - (Q^\dagger \sigma_x Q)\sigma_x\right)Q - i\Gamma P = 0, \quad (5.29a)
\]

\[
\frac{\partial P}{\partial t} - i\Delta P + iRW_0Q + i\frac{R}{\sqrt{2}} \left((W^\dagger Q)\varphi_1 - (W^\dagger \sigma_z Q)\varphi_2\right) = 0, \quad (5.29b)
\]

\[
\frac{\partial W_1}{\partial t} - i\frac{R}{\sqrt{8}} \left(((P^\dagger \sigma_x Q) - (Q^\dagger \sigma_x P))\varphi_1 - ((P^\dagger i\sigma_y Q) - (Q^\dagger i\sigma_y P))\varphi_2\right) = 0, \quad (5.29d)
\]

where

\[
Q = \sqrt{\frac{1}{2}} \begin{pmatrix} q_+ + q_- \\ q_+ - q_- \end{pmatrix} e^{i\delta z}; \quad Q^\dagger = \sqrt{\frac{1}{2}} \begin{pmatrix} (q_+^* + q_-^*) \\ (q_+^* - q_-^*) \end{pmatrix} e^{-i\delta z},
\]

\[
P = \sqrt{\frac{1}{2}} \begin{pmatrix} P_+ + P_- \\ P_+ - P_- \end{pmatrix} e^{i\delta z}; \quad P^\dagger = \sqrt{\frac{1}{2}} \begin{pmatrix} (P_+^* + P_-^*) \\ (P_+^* - P_-^*) \end{pmatrix} e^{-i\delta z},
\]

\[
W = \sqrt{2} \begin{pmatrix} W_1 + W_1^* \\ W_1 - W_1^* \end{pmatrix}; \quad W^\dagger = \sqrt{2} \begin{pmatrix} (W_1^* + W_1) \\ (W_1^* - W_1) \end{pmatrix},
\]

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \varphi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

\[Q, P \text{ and } W \text{ are the two component "spinor" fields, } \sigma_x, \sigma_y \text{ and } \sigma_z \text{ are the Pauli spin matrices and } ^\dagger \text{ represents the complex conjugate. In Fourier space, equation (5.29a) becomes}
\]

\[-i\phi \sigma_z \tilde{Q} + \frac{\partial \tilde{Q}}{\partial t} - i\kappa \sigma_x \tilde{Q} - iF_{nl}(\tilde{Q}) - i\Gamma \tilde{P} = 0, \quad (5.30)\]
where $\tilde{Q}$ and $\tilde{P}$ are the Fourier transform of $Q$ and $P$ respectively and $F_{nl}$ is the nonlinear function given by

$$F_{nl}(\tilde{Q}) = \frac{\gamma}{2} \int d\phi_1 \int \left( 3\tilde{Q}(\phi_1, t)^\dagger \tilde{Q}(\phi_2, t) - (\tilde{Q}(\phi_1, t)^\dagger \sigma_z \tilde{Q}(\phi_2, t)\sigma_z \right) \times \tilde{Q}(k + \phi_1 - \phi_2, t)d\phi_2.$$ 

The linear part of equation (5.30) can be diagonalized using the $\phi$ dependant unitary operator

$$S = \begin{pmatrix} \sin(\theta/2) & \cos(\theta/2) \\ \cos(\theta/2) & -\sin(\theta/2) \end{pmatrix},$$

(5.31)

where $\tan\theta = \kappa/\phi$. Introducing the following new spinor fields as

$$\tilde{\Psi}(\phi, t) = S^\dagger(\phi)\tilde{Q}(\phi, t),$$

(5.32a)

$$\tilde{p}_1(\phi, t) = S^\dagger(\phi)\tilde{P}(\phi, t),$$

(5.32b)

$$w_1 = S^\dagger(\phi)W_1,$$

(5.32c)

into equation (5.30) in the absence of SPM ($\gamma = 0$) and resonant polarization ($\Gamma = 0$) becomes

$$\frac{\partial \tilde{\Psi}}{\partial t} + i\sqrt{\kappa^2 + \phi^2} \sigma_z \tilde{\Psi} = 0,$$

(5.33)

The PBG edge behaviour described by expanding the dispersion relation (5.21) for small $\phi$ ($\phi/\kappa \ll 1$):

$$\delta(q) = \left( \delta_0 + \frac{\partial \delta}{\partial \phi} \phi + \frac{1}{2} \frac{\partial^2 \delta}{\partial \phi^2} \phi^2 + \ldots \right),$$

(5.34)

where $\delta_0|_{\phi=0} = \kappa$, $(\partial \delta/\partial \phi)|_{\phi=0} = 0$ (from equation (5.26)) and $(\partial^2 \delta/\partial \phi^2)|_{\phi=0} = 1/\kappa$ at $f = \pm 1 \ [155, 181]$. Inserting equation (5.34) into equation (5.30), we obtain

$$\frac{\partial \tilde{\Psi}}{\partial t} - i \left( \kappa + \frac{\phi^2}{2\kappa} \right) \sigma_z \tilde{\Psi} - i \frac{\gamma}{2} \left( 3(\tilde{\Psi}^\dagger \tilde{\Psi}) - (\tilde{\Psi}^\dagger \sigma_x \tilde{\Psi})\sigma_x \right) \tilde{\Psi} - i\Gamma \tilde{p}_1 = 0.$$ 

(5.35)

The above equation can be converted into the spatial domain by replacing $\phi^2$ with

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the differential operator \(-\partial^2/\partial z^2\). The resulting effective nonlinear wave equation as

\[
\frac{\partial \Psi}{\partial t} - i\left(\kappa - \left(\frac{\text{sgn}(f)}{2\kappa}\right)\frac{\partial^2}{\partial z^2}\right) \sigma_z \Psi - \frac{i\gamma}{2} \left(3(\Psi^\dagger \Psi) - (\Psi^\dagger \sigma_z \Psi)\sigma_x\right) \Psi - i\Gamma p_1 = 0. \tag{5.36}
\]

By inserting equations (5.32) into equations (5.29b)-(5.29d), we get

\[
\frac{\partial p_1}{\partial t} - i\Delta p_1 + iRW_0 \Psi + i\frac{R}{\sqrt{2}} \left((w^\dagger_1 \Psi)\varphi_1 - (w^\dagger_1 \sigma_z \Psi)\varphi_2\right) = 0, \tag{5.37a}
\]

\[
\frac{\partial \Psi}{\partial z} - \frac{R}{\sqrt{2}} \left(p^\dagger_1 \Psi - \Psi^\dagger p_1\right) = 0, \tag{5.37b}
\]

\[
\frac{\partial \varphi_1}{\partial t} - \frac{R}{\sqrt{8}} \left((p^\dagger_1 \sigma_x \Psi) - (\Psi^\dagger \sigma_x p_1)\right)\varphi_1 - \left((p^\dagger_1 i \sigma_y \Psi) - (\Psi^\dagger i \sigma_y p_1)\right)\varphi_2 = 0. \tag{5.37c}
\]

Here \(\varphi^\dagger_1 = (1,0)\) and \(\varphi^\dagger_2 = (0,1)\). We consider two specific solutions \(\Psi^\dagger = (Q_1,0)\) and \(\Psi^\dagger = (0,Q_2)\), where \(Q_1 = (q_+ + q_-)/\sqrt{2}\) and \(Q_2 = (q_+ - q_-)/\sqrt{2}\), corresponds to the lower and the upper PBG edges, respectively. At the upper PBG edge \((f = -1)\), inserting the following ansatzs \(\Psi^\dagger = (0,Q_2)\), \(p^\dagger_1 = (0,P_2)\) and \(w^\dagger = \sqrt{2}(w_1,0)\) into equations (5.36)-(5.37c), we obtain

\[
\frac{\partial q_+}{\partial t} - i\kappa q_+ - \frac{\beta_2^g q_+}{2} \frac{\partial^2}{\partial z^2} - i\gamma_g|q_+|^2 q_+ - i\Gamma P_+ = 0, \tag{5.38a}
\]

\[
\frac{\partial P_+}{\partial t} - i\Delta P_+ + iR(W_0 - W_1)q_+ = 0, \tag{5.38b}
\]

\[
\frac{\partial W_0}{\partial t} - \frac{R}{2} \left(P^*_+ q_+ - q^*_+ P_+\right) = 0, \tag{5.38c}
\]

\[
\frac{\partial W_1}{\partial t} - \frac{R}{4} \left(P^*_+ q_+ - q^*_+ P_+\right) = 0, \tag{5.38d}
\]

where \(Q_1 = \sqrt{2}q_+\), \(P_2 = (P_+ - P_-)/\sqrt{2} = \sqrt{2}P_+\), \(\beta_2^g = 1/\kappa\), \(\gamma_g = 3\gamma\). Using the following transformations \(q_+ = q e^{i\kappa z}\), \(P_+ = p e^{i\kappa z}\) and \(w = W_0 - W_1\) into equations (5.38b)-(5.38d), we obtained the NLS-MB equations as

\[
\frac{\partial q}{\partial t} - i\frac{\beta_2^g}{2} \frac{\partial^2 q}{\partial z^2} - i\gamma_g|q|^2 q - i\Gamma p = 0, \tag{5.39a}
\]

\[
\frac{\partial p}{\partial t} - i\Delta p + iRwq = 0, \tag{5.39b}
\]

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To start with, we consider the general solution of the form $q(z,t) = A(\varsigma)e^{i(kt-\omega t)}$, $p(z,t) = [u(\varsigma) + iv(\varsigma)]e^{i(kt-\omega t)}$ and $w(z,t) = \eta(\varsigma)$, where $\varsigma \equiv t - (z/V)$, $V$ is the soliton velocity, $k$ the wave number, and $\omega$ the frequency of the solution sought for.

Upon substituting this ansatz into equations (5.39a)-(5.39c) and separating the real and imaginary parts, we obtain

$$- \left( \frac{\beta_2^2}{2V^2} \right) \frac{d^2 A}{d\varsigma^2} + \gamma_9 A^3 + \left( \frac{\beta_2^2 k^2}{2} + \omega \right) A + \Gamma u = 0, \quad (5.40a)$$

$$\left( 1 + \frac{k\beta_2}{V} \right) \frac{dA}{d\varsigma} + \Gamma v = 0, \quad (5.40b)$$

$$\frac{du}{d\varsigma} + (\Delta + \omega)v = 0, \quad (5.40c)$$

$$\frac{dv}{d\varsigma} - (\Delta + \omega)u + RA\eta = 0, \quad (5.40d)$$

$$\frac{d\eta}{d\varsigma} - \left( \frac{3}{2} \right) RA v = 0. \quad (5.40e)$$

Further, substituting $v$ from equation (5.40b) into equation (5.40e), we can eliminate the population variable,

$$\eta = -1 - \left( \frac{3R(1 + k\beta_2/V)}{4\Gamma} \right) A^2. \quad (5.41)$$

and inserting $u$, $v$, and $\eta$ from equations (5.40a), (5.40b), and (5.41) into equations (5.40c) and (5.40d), we arrive at a second-order equation:

$$\frac{d^2 A}{d\varsigma^2} = \rho_{11} A + \rho_{12} A^3, \quad (5.42a)$$

$$\frac{d^2 A}{d\varsigma^2} = \rho_{21} A - \rho_{22} A^3, \quad (5.42b)$$

where

$$\rho_{11} \equiv \left( \frac{2V^2(\Delta + 2\omega) + Vk(Vk + 2(\Delta + \omega))\beta_2^2}{\beta_2^2} \right).$$
\[
\begin{align*}
\rho_{12} & \equiv \left( \frac{2\gamma g V^2}{\beta_2^g} \right), \\
\rho_{21} & \equiv \left( \frac{V^2((\Delta + \omega)(\beta_2^g k^2 + 2\omega) - 2R\Gamma)}{2V^2 + (2kV + \Delta + \omega)\beta_2^g} \right), \\
\rho_{22} & \equiv \left( \frac{V (3R^2 (V + k\beta_2^g) - 4V(\Delta + \omega)\gamma g)}{2(2V^2 + (2kV + \Delta + \omega)\beta_2^g)} \right).
\end{align*}
\]

We can derive the solitary wave solutions from equations (5.42a) and (5.42b). It is clear that equations (5.42a) and (5.42b) can be equivalent only under the following conditions: \(\rho_{11} = \rho_{21}\) and \(\rho_{21} = -\rho_{22}\). The integration of equation (5.42b) produces an equation for the traveling wave,

\[
\left( \frac{dA}{d\zeta} \right)^2 = \rho_{21} A^2 - \frac{1}{2} \rho_{22} A^4 + 2C, \quad (5.43)
\]

where \(C\) is an arbitrary integration constant. Equation (5.43) can be solved in terms of the Jacobi elliptic functions. In particular, the choice of \(C = (m^2(1-m^2)\rho_{21}^2)/((2m^2 - 1)^2\rho_{22})\) yields a solution for the cnoidal waves (the one written in terms of elliptic function \(cn, sn,\) and \(dn\), which depend on modulus \(m\)):

\[
\begin{align*}
q(z, t) &= \sqrt{p_0} \ cn \left[ \frac{t - (z/V)}{T_0}, m \right] e^{i(kz-\omega t)}, \\
P(z, t) &= \frac{\sqrt{p_0}}{\Gamma} \left( p_1 \ cn \left[ \frac{t - (z/V)}{T_0}, m \right] - p_2 \ cn^3 \left[ \frac{t - (z/V)}{T_0}, m \right] \\
&\quad - i \left( \frac{V + k\beta_2^g}{VT_0} \right) \ sn \left[ \frac{t - (z/V)}{T_0}, m \right] \ dn \left[ \frac{t - (z/V)}{T_0}, m \right] \right) e^{i(kz-\omega t)}, \\
W(z, t) &= -1 - \left( \frac{3Rp_0 (V + k\beta_2)}{2VT_0} \right) \ cn^2 \left[ \frac{t - (z/V)}{T_0}, m \right], \quad (5.45)
\end{align*}
\]

where we have defined

\[
T_0 \equiv \sqrt{\frac{2m^2 - 1}{\rho_{21}}},
\]
\[ \varphi_1 \equiv \left( \frac{\beta_2^g}{2V^2 T_0^2} \right) \left( m^2 + m - 1 \right) - \left( \frac{\beta_2^g}{2} \right) k^2 + \omega, \]

\[ \varphi_2 \equiv \left( \frac{\beta_2^g}{2T_0^2 \rho_2} \right) m(m+1) + \gamma g p_0, \]

and the elliptic modulus is implicitly determined by power \( p_0 \) through the following relation:

\[ p_0 \equiv \frac{2m^2 \rho_2 - 1}{(2m^2 - 1) \rho_2}. \]

Parameter \( T_0 \), which determines the period of the cnoidal-wave solution, can be expressed in terms of \( p_0 \) as

\[ T_0 = \sqrt{\frac{2m^2}{p_0 \rho_2}}. \]  

(5.46)

Figure 5.10: (a) The intensity profile of the \( cn \) \( (m = 0.4) \) solution and (b) the intensity profile of the bright \( (m = 1) \) SIT soliton solution, for \( f = -1, p_0 = 2W, k = 12 m^{-1}, \omega = 0.7 Hz, \kappa = 20 cm^{-1} \) and \( V = 0.4 m s^{-1} \).

Figure 5.10 (a) displays a typical example of the local power distribution corresponding to the cn solution in the anomalous \( (top \ PBG \ edge) \) dispersion regime. Further, figure 5.10 (b) shows the profile of the bright SIT gap soliton, which is obtained from solution (5.45) in the limit of \( m = 1 \). Actually, these soliton solutions were reported in Refs. [62, 196].
Similarly, if the integration constant is chosen to be \( C = -(m\rho_{21})^2/((m^2 + 1)^2 \rho_{22}) \), equation (5.43) generates another exact periodic solution in terms of function \( \text{sn} \):

\[
q(z, t) = \sqrt{p_0} \text{sn} \left[ \frac{t - (z/V)}{T_0}, m \right] e^{i(kz-\omega t)},
\]

\[
P(z, t) = -\frac{\sqrt{p_0}}{\Gamma} \left( \sigma_1 \text{sn} \left[ \frac{t - (z/V)}{T_0}, m \right] + \sigma_2 \text{sn}^3 \left[ \frac{t - (z/V)}{T_0}, m \right] \right. \]
\[
+ i \left( V + k\beta_2^g \right) \text{cn} \left[ \frac{t - (z/V)}{T_0}, m \right] \text{dn} \left[ \frac{t - (z/V)}{T_0}, m \right] \left. \right) e^{i(kz-\omega t)},
\]

\[
W(z, t) = -1 - \frac{3Rp_0 (V + k\beta_2^g)}{2VT_0} \text{sn}^2 \left[ \frac{t - (z/V)}{T_0}, m \right],
\]

where the following relations should be imposed on the parameters:

\[
p_0 \equiv \frac{2m^2 \rho_{21}}{(m^2 + 1) \rho_{22}},
\]

\[
T_0 \equiv \sqrt{\frac{m^2 + 1}{\rho_{21}}},
\]

\[
\sigma_1 \equiv \left( \frac{\beta_2^g}{2V^2T_0^2} \right) (m + 1) + \left( \frac{\beta_2^g}{2} \right) k^2 + 2\omega,
\]

\[
\sigma_2 \equiv \gamma g p_0 - \left( \frac{\beta_2^g}{2V^2T_0^2} \right) m(m + 1).
\]

Figure 5.11: (a) The intensity profile of the \( \text{sn} \) \((m = 0.4)\) solution and (b) the intensity profile of the dark \((m = 1)\) SIT soliton solution, for \( f = -1 \), \( p_0 = 2W \), \( k = 12 \text{m}^{-1} \), \( \omega = 0.7 \text{Hz} \) and \( V = 0.4 \text{m s}^{-1} \).
The intensity distribution in the sn solution, in the anomalous dispersion regime, is displayed in figure 5.11 (a). In addition, in figure 5.11 (b) we show that the profile of the limit solution with $m = 1$, which corresponds to a dark SIT gap soliton in the anomalous dispersion regime. Similarly, we can generate the cnoidal solutions, as well as their limit forms for $m = 1$, which corresponds to the bright and dark SIT gap solitons in the case of the normal (bottom PBG edge for $f=1$) dispersion regime. Our results describe that both bright and dark SIT gap solitons can be generated in the cases of the anomalous and normal dispersion regimes [196]. This should be contrasted to the well known situations in Ref. [198], where bright and dark SIT gap solitons exist, solely with the anomalous and normal dispersion regimes. On the contrary, it is known that bright and dark solitons coexist in the model based on a periodic array of narrow layers of two-level atoms, which simultaneously plays the role of the Bragg reflector [185,218]. It is noteworthy that the Painlevé analysis could not reveal the existence of dark solitons in the NLS-MB system [186], while the present results make it possible to find them in both cases, normal and anomalous dispersion.

It is clear that equations (5.42) are similar to equations (4.47). The soliton solutions of equations (5.42) indicate the coexistence of a SIT soliton and a SIT Bragg soliton. The physical mechanism for this coexistence is attributed to the fact that a uniformly doped PBG structure. Because the resonant atoms dominate the quadratic grating dispersion and the third-order nonlinearity, the original forbidden band has been shifted by the dopants (figure 5.3). Furthermore, such a soliton solution can also satisfy the atomic Bloch equations for the choice of parameters $\kappa = \gamma = 0$.

From an experimental viewpoint, we have calculated the important and interesting physical parameters such as pulse power ($p_0$) and pulse width ($T_0$). Based on the arguments, we believe that SIT gap solitons could be generated experimentally in uniformly doped FBG. Experimental observation of SIT gap solitons in the uniformly doped FBG is an attractive subject, which could lead to practical expansions of
gap solitons in the vast area of light wave systems. In contrast to the fiber SIT soliton [196], the SIT gap solitons can be realized experimentally since uniformly doped FBGs have a length of only a few centimeters, owing to the large dispersion, this is long enough for generating SIT gap solitons.

5.6 Conclusion

By means of MI analysis, the MI conditions are identified to generate the ultra-short pulses in uniformly doped FBG systems. It should be pointed out that there is a threshold condition for the occurrence of MI in the normal dispersion regime of a conventional FBG whereas in the case of dynamic grating the same can be achieved without threshold condition. However, this is difficult to realize practically. Keeping this in mind, the MI conditions are identified and proposed, in the normal dispersion regime, which do not require any power threshold condition. In this case, the MI process differs qualitatively from the conventional FBG in the following two respects. The first of these is that this MI process has no threshold, which in turn implies that the CW is unstable. The second major difference is the shape of the MI gain spectrum. In the lower branch, all the wave numbers with all values are unstable, and instability can occur at a finite wave number. We have also performed the numerical analysis to solve the governing NLCM-MB equations. The numerical results of the prediction of the optimum modulation wave number and the peak gain agree well with those of the LSA. Finally, our results described that both bright and dark SIT gap solitons can be generated in the cases of the anomalous as well as normal dispersion regimes. This should be contrasted to the well established results, where bright and dark SIT gap solitons exist, solely with the anomalous and normal dispersion regimes. Another important point to be noted is that the non-conventional MI gain spectra in the normal dispersion regime and generation of both bright and dark SIT gap solitons in anomalous and normal dispersion regimes have not been observed so far.