Chapter 6

AN $M/E_m/1$ QUEUE WITH PROTECTION BASED ON T-POLICY

6.1 Introduction

In the last chapter we analysed a queueing model having a service process with interruption and a time based method of protection, some thing like a T-policy. But the service time, if there were no interruption was assumed to be exponential. In many practical situations as pointed out in chapter 4, the service process may consist of more than one phase. In such systems, the repeat and resumption of service have distinct roles to play. The chance of repeating an interrupted service makes this T-policy protection very interesting.

Moreover, we have already introduced a method of protecting some phases of the service process from interruption, say an N-
policy. It would be meaningful to compare the two types of protection. This is one of the motive of this chapter

### 6.2 Model Description

Consider a single server queuing system in which the arrival process is Poisson with rate $\lambda$. The service time distribution is Erlang of order $m$ with mean $\frac{1}{\mu}$. Interruptions occur to the service process at an exponentially distributed time durations with mean $\frac{1}{\theta}$. As in the previous models, the interrupted server is taken for repair immediately with repair duration follows an exponential distribution with mean $\frac{1}{\delta}$. A random clock is started at the beginning of each repair to decide whether to restart or resume the service after repair. If the random clock realizes before a repair, the service needs to be repeated, otherwise the service is resumed in the phase at which the interruption occurred. The realization time of the random clock also follows an exponential distribution, with mean $\frac{1}{\gamma}$. To avoid the situation where the system is interrupted while the service nearing completion, and had to repeat the service from the beginning, a protective mechanism is provided. To minimize the cost of running the system, this mechanism will be used only while the service is continued for sufficiently long time. Thus a clock is started simultaneously with the beginning of a new or an interrupted service. The protection is provided for the part of the service that remains after the realization of this random clock. The realization time of this
6.2. Model Description

clock is assumed to be an exponential variable with mean $\frac{1}{\varphi}$. In effect we extend the service protection mechanism introduced in Chapter 5 to Erlang distributed service time.

This queueing model can be defined by the Markov process $X = \{X(t)/t \geq 0\} = \{(N(t), P(t), S(t), J(t)), t \geq 0\}$ where $N(t)$ is the number of customers in the system, $P(t)$ is the status of the protective mechanism which is 0 or 1 according as the mechanism is off or on, $S(t)$ is the status of the server which is 0, 1 or 2 according as the service is uninterrupted, interrupted with a running clock or interrupted with a realized clock and $J(t)$ is the phase of the service process at time $t$. The state space is given by $\{0, 1, 2, 3, \ldots\} \times \{0, 1\} \times \{0\} \times \{0\} \times \{1, 2, \ldots, m\}$ and the infinitesimal generator matrix given by

$$Q = \begin{bmatrix} A_{10} & A_{00} \\ A_{21} & A_{1} & A_{0} \\ & A_{2} & A_{1} & A_{0} \\ & & & \ddots & \ddots \end{bmatrix}$$

where

$$A_{10} = [-\lambda] \quad A_{00} = \left[\begin{array}{ccc} \lambda & 0 & 0 \\ 1 & 0 & \ldots & 0 \end{array}\right] \quad \alpha = \left[1 \quad 0 \quad \ldots \quad 0\right]$$
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\[ A_0 = \lambda I_{4m} \]

\[
A_{21} = \begin{bmatrix} S^0 \\ 0 \\ 0 \\ S^0 \end{bmatrix} \quad A_2 = \begin{bmatrix} S^0 \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ S^0 \alpha & 0 & 0 & 0 \end{bmatrix}
\]

\[
A_1 = \begin{bmatrix} S - (\theta + \varphi + \lambda)I & \theta I & 0 & \varphi I \\ \delta I & -(\gamma + \delta + \lambda)I & \gamma I & 0 \\ \delta e\alpha & 0 & -(\delta + \lambda)I & 0 \\ 0 & 0 & 0 & S - \lambda I \end{bmatrix}
\]

\[
S^0 = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ m\mu \end{bmatrix} \quad S = m\mu 
\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}
\]

and $I$ is the identity matrix of order $m$.

\section*{6.3 Analysis of the service process}

\subsection*{6.3.1 Expected service time}

The service process with the introduction of interruption, becomes a Markov Process $\Psi$ with $4m$ transient states given by

\[ \{0, 1, 2, 3\} \times \{1, 2, \ldots, m\} \]
6.3. Analysis of the service process

and one absorbing state \( \tilde{0} \). The absorbing state denotes the service completion. The process \( \Psi \) can be represented by \( \psi(t) = (i, j) \) where \( i = 0 \) if the service is uninterrupted, \( i = 1 \) if the service is interrupted with restart clock running \( i = 2 \) if the service is interrupted with a finished restart clock and \( i = 3 \) if the service is protected and \( j \) is the phase of service. The initial probability vector is \( \beta = (\alpha, 0, 0, 0) \) Let \( \tau \) be the time until absorption of the process \( \Psi \). The infinitesimal generator of this process is given by \( \tilde{Q} = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix} \), where

\[
T = \begin{bmatrix}
S - (\theta + \varphi) I & \theta I & 0 & \varphi I \\
\delta I & -(\gamma + \delta) I & \gamma I & 0 \\
\delta e\alpha & 0 & -\delta I & 0 \\
0 & 0 & 0 & S
\end{bmatrix}
\]

\[
T^0 = \begin{bmatrix} S^0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\beta(-T)^{-1} = (y_1, y_2, ..., y_{4m}) \text{ gives us the time spend in each state in a single service and can be calculated as}
\]

\[
y_i = \begin{cases} 
\frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right)^{m-i} \omega, & i = 1, 2, ..., m \\
\frac{\theta}{\gamma + \delta} \left( \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right) \right)^{m-i} \omega, & i = m + 1, ..., 2m \\
\frac{\gamma\theta}{\delta(\gamma + \delta)} \left( \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right) \right)^{m-i} \omega, & i = 2m, ..., 3m \\
\frac{\varphi}{m\mu} \left( \sum_{j=4}^{m-1} \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right)^j \right) \omega, & i = 3m + 1, ..., 4m.
\end{cases}
\]
where

\[ \frac{1}{\omega} = m\mu + \varphi \sum_{j=0}^{m-1} \left[ \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right) \right]^j \]

Therefore, during a single service

- The expected time spent in the protected phases is

\[ \frac{\varphi}{m\mu} \left( \sum_{j=1}^{m} j \left[ \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right) \right]^{j-1} \right) \omega. \]

- The expected time spent in the uninterrupted unprotected phases is

\[ \left( \sum_{j=1}^{m} \left[ \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right) \right]^{j-1} \right) \omega. \]

- Expected interruption duration is

\[ \frac{\theta}{\delta} \left( \sum_{j=1}^{m} \left[ \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right) \right]^{j-1} \right) \omega. \]

The expected service time is the expect time until absorption of the above process which is given by,

\[ E(\tau) = \beta(\sim T)^{-1} e \]

\[ = \left( 1 + \frac{\theta}{\delta} \right) \left( \sum_{j=1}^{m} \left[ \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\gamma\theta}{\gamma + \delta} \right) \right]^{j-1} \right) \omega \]
6.3. Analysis of the service process

\[ + \frac{\varphi}{m \mu} \left( \sum_{j=1}^{m} j \left[ \frac{1}{m \mu} \left( m \mu + \varphi + \frac{\gamma \theta}{\gamma + \delta} \right) \right]^{j-1} \right) \omega. \]

6.3.2 Expected number of interruptions during a single service

To compute the expected number of interruptions faced by a customer during his service, we consider the Markov process \( \chi = \{(N(t), J(t)) / t \geq 0\} \) where \( N(t) \) is the number of interruptions that occurred up to time \( t \) and \( J(t) \) is the phase of the service process at time \( t \). \( \chi \) has the state space \( \{0, 1, 2, 3, \ldots\} \times \{1, 2, 3, \ldots, m\} \cup \{\Delta\} \) where \( \Delta \) is the absorbing state which denotes that the service process reached the first protected phase and there will be no more interruptions. The infinitesimal generator \( \tilde{Q} \) of this process is

\[
Q = 
\begin{bmatrix}
D_2 & D_1 & D_0 & \cdot & \cdot & \cdot \\
D_2 & 0 & D_1 & D_0 & \cdot & \cdot \\
D_2 & 0 & 0 & D_1 & D_0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]
where

\[
D_1 = -m\mu \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & \vdots \\ & \vdots & \ddots \end{bmatrix} - (\theta + \varphi) I_m
\]

\[
D_0 = \begin{bmatrix} \frac{\theta}{\gamma + \delta} & 0 & 0 & \cdots \\ \frac{\delta \theta}{\gamma + \delta} & 0 & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\gamma \theta}{\gamma + \delta} & 0 & \cdots & \frac{\delta \theta}{\gamma + \delta} \end{bmatrix}_{m} \quad \text{and} \quad D_2 = \begin{bmatrix} \varphi \\ \varphi \\ \vdots \\ m\mu + \varphi \end{bmatrix}_{(m-n) \times 1}
\]

If \(z_j\) is the probability that there are exactly \(j\) interruptions during a single service, then

\[
z_j = \zeta (-D_1^{-1}D_0)^j (-D_1^{-1}D_2), \quad j = 0, 1, 2, ...
\]

Expected number of interruptions during a single service,

\[
E(I) = \sum_{j=0}^{\infty} jz_j
\]

\[
= \zeta (-D_1^{-1}D_0) [I - (-D_1^{-1}) D_0]^{-1} e
\]

\[
= \theta (\kappa^m - 1) \left( \varphi \kappa^m + \frac{\gamma \theta}{\gamma + \delta} \right)^{-1}
\]
6.4. Steady state Analysis

\[ \kappa = 1 + \frac{\varphi}{m\mu} + \frac{\gamma\theta}{m\mu(\gamma + \delta)} \]

6.4 Steady state Analysis

6.4.1 Stability analysis

Let \( A = A_0 + A_1 + A_2 = \begin{bmatrix} S - (\theta + \varphi)I + S^0\alpha & \theta I & 0 & \varphi I \\ \delta I & -(\gamma + \delta)I & \gamma I & 0 \\ \delta e\alpha & 0 & -\delta I & 0 \\ S^0\alpha & 0 & 0 & S \end{bmatrix} \)

and \( \pi = (\pi_0, \pi_1, \pi_2, \pi_3) \) be the vector such that \( \pi A = 0 \) and \( \pi e = 1 \). Then

\[ \pi_0 \left( S - (\theta + \varphi)I + S^0\alpha \right) + \delta \pi_1 + \pi_2 \delta e\alpha + \pi_3 S^0\alpha = 0 \quad (6.1) \]

\[ \theta \pi_0 - (\gamma + \delta) \pi_1 = 0 \quad (6.2) \]

\[ \gamma \pi_1 - \delta \pi_2 = 0 \quad (6.3) \]

\[ \varphi \pi_0 - \pi_3 S = 0 \quad (6.4) \]

From the above equations, we get

\[ \pi_0 \left[ (S + S^0\alpha) - \left( \frac{\gamma\theta}{\gamma + \delta} + \varphi \right) (I - e\alpha) \right] = 0 \]
Solving, we get
\[
\pi_0 = \sigma(1, k, k^2, ..., k^{m-1})
\]
where \( \sigma \) is a non-zero constant and
\[
k = \frac{1}{m\mu} \left( m\mu + \varphi + \frac{\theta}{\gamma + \delta} \right).
\]
Therefore using equations (6.1), (6.2) and (6.3) we have,
\[
\begin{align*}
\pi_0 e &= (1 + k + k^2 + ... + k^{m-1})\sigma \\
\pi_1 e &= \frac{\theta}{\gamma + \delta} (1 + k + k^2 + ... + k^{m-1})\sigma \\
\pi_2 e &= \frac{\gamma \theta}{\delta (\gamma + \delta)} (1 + k + k^2 + ... + k^{m-1})\sigma \\
\pi_3 e &= \frac{\varphi}{m\mu} (1 + 2k + 3k^2 + ... + mk^{m-1})\sigma
\end{align*}
\]
and
\[
\pi e = \left[ \left( 1 + \frac{\theta}{\gamma + \delta} + \frac{\gamma \theta}{\delta (\gamma + \delta)} \right) (1 + k + k^2 + ... + k^{m-1}) + \frac{\varphi}{m\mu} (1 + 2k + 3k^2 + ... + mk^{m-1}) \right] \sigma
\]
Therefore
\[
\sigma = \left[ \left( 1 + \frac{\theta}{\gamma + \delta} + \frac{\gamma \theta}{\delta (\gamma + \delta)} \right) (1 + k + k^2 + ... + k^{m-1}) + \frac{\varphi}{m\mu} (1 + 2k + 3k^2 + ... + mk^{m-1}) \right]^{-1}
\]
Also it follows that \( \pi A_2 e = [\varphi (1 + k + k^2 + ... + k^{m-1}) + m\mu] \sigma \) and \( \pi A_0 e = \lambda \) Hence the condition for stability is
\[
\lambda \left[ \left( 1 + \frac{\theta}{\gamma + \delta} + \frac{\gamma \theta}{\delta (\gamma + \delta)} \right) (1 + k + k^2 + ... + k^{m-1}) + \frac{\varphi}{m\mu} (1 + 2k + 3k^2 + ... + mk^{m-1}) \right] < \varphi (1 + k + k^2 + ... + k^{m-1}) + m\mu \quad (6.5)
\]
6.4.2 Stationary probabilities

The stationary probability vector $x$ is given by $xQ = 0$, together with the normalizing condition $xe = 1$. On partitioning the steady state vector as $x = (x_0, x_1, x_2, ...)$, the equation $xQ = 0$ reduces to the following equations.

$$
\begin{align*}
    x_0A_{10} + x_1A_{21} &= 0 \\
    x_0A_{00} + x_1A_1 + x_2A_2 &= 0 \\
    x_iA_0 + x_{i+1}A_1 + x_{i+2}A_2 &= 0, \quad i = 1, 2, 3, ... .
\end{align*}
$$

Also $\lambda x_i e = x_{i+1}A_{21}, i = 0, 1, 2, ...$ and $A_2 = A_{21} \begin{bmatrix} \alpha & 0 & 0 & 0 \end{bmatrix}$

Solving we get $x_0 = 1 - \rho, x_i = (1 - \rho)\beta R^i$ for $i = 1, 2, 3, ...$ where $\beta = (\alpha, 0, 0, 0)$, $R = -\lambda[A_1 + \lambda e\beta]^{-1}$, and $\rho = -\lambda\alpha T^{-1}e$.

6.5 Performance Analysis

6.5.1 Expected waiting time

The expected waiting time $W_L$ of a customer in the queue is found by the tagged customer method as in section (4.5.1). Thus $W_L$ is given by

$$W_L = \lambda\beta(-T)^{-2}e + E(\tau)E_Q(C)$$
where $E_Q(C) = \sum_{r=1}^{\infty} (r - 1)x(r)e$ is the expected number of customers in the queue and $E(\tau)$ is the expected service time.

### 6.5.2 Other performance measures

Each probability vector $x_i$ except $x_0$ in the steady state probability vector
\[
\mathbf{x} = (x_0, x_1, x_2, \ldots)
\]
can be partitioned as
\[
x_i = \left( x_i^{(0)}, x_i^{(1)}, x_i^{(2)}, x_i^{(3)} \right)
\]
where
\[
x_i^{(j)} = (x_i^{(j)}(1), x_i^{(j)}(2), \ldots, x_i^{(j)}(m))
\]
for $j = 0, 1, 2, 3$ and $i = 1, 2, 3, \ldots$.

With this notation, we have the following results.

- Probability of no customer in the system,
  \[
P_C(0) = x_0.
  \]
- Probability of $i$ customers in the system,
  \[
P_C(i) = x_i e.
  \]
6.5. Performance Analysis

- Expected number of customers in the system,

\[ E_S(C) = \sum_{i=0}^{\infty} iP_C(i). \]

- Expected queue length,

\[ E_Q(C) = \sum_{i=0}^{\infty} (i - 1)P_C(i). \]

- Probability that there are \( i \) customers in the system when the server is interrupted,

\[ P^I_C(i) = \left( x^{(1)}_i + x^{(2)}_i \right) e. \]

- Expected number of customers in the system when the system is under interruption,

\[ E^I(C) = \sum_{i=0}^{\infty} iP^I_C(i). \]

- Probability that there are \( i \) customers in the system when the server is uninterrupted,

\[ P^I_C(i) = \left( x^{(0)}_i + x^{(3)}_i \right) e. \]

- Expected number of customers in the system when the
server is uninterrupted,

\[ E^B(C) = \sum_{i=0}^{\infty} iP_C^B(i). \]

- Variance of the number of customers in the system,

\[ V_S(C) = \sum_{i=0}^{\infty} i^2 P_C(i) - \left( \sum_{i=0}^{\infty} iP_C(i) \right)^2. \]

- Probability that the system is under interruption,

\[ P_S(I) = \sum_{i=0}^{\infty} P_C^I(i). \]

- Probability that the system is protected state,

\[ P_{pr} = \sum_{i=1}^{\infty} x_i^{(3)}e. \]

- Effective interruption rate,

\[ EI = \theta \sum_{i=1}^{\infty} x_i^{(0)}e. \]

- Effective rate of service resumption,

\[ ERSM = \delta \sum_{i=1}^{\infty} x_i^{(1)}e. \]
6.6. Numerical Study of the Model

This section a numerical analysis of the model is done. This will be helpful in understanding the effect of various parameters in the performance of the system. In this examination, we consider a service process in which the service time distribution is Erlang-5. The arrival rate is assumed to be 2. For the easiness of comparison, we use the same notations used in section 4.6.

The results of the study with different values of \( \theta \) is given in Table 6.1. The value of \( \theta \) is increased from 0 to 6. As \( \theta \) increases, the interruption occurs more frequently. Thus the number of interruptions per service as well as the duration of interruption during a service are increased. This results in an increase in the service time. This explains the changes in the other measures like expected number of customers, the waiting time, probability that the system is interrupted and the probability that the system is idle.

One would be interested to see what happens to the system when rate of realization of protection clock \( \varphi \) is increased. Table 6.2 gives us an idea about this. As \( \varphi \) increases, the system gets protected sooner. So the probability of being interrupted

\[
ERSM = \delta \sum_{i=1}^{\infty} x_i^{(2)} e.
\]
decreases, resulting in a faster service. Hence the service time is lowered and so is the number of customers in the system and the waiting time. Also note the increase in idle probability with increase in $\varphi$.

Next we will see how the rate of realization of repeat clock, $\gamma$ acts on the system performance measures. Table 6.3 comprises the numerical results obtained. As the possibility of repeating the service increases with increase in $\gamma$, the service rate is slightly increased. So there is more chance that the system will be protected. This is the reason why the change in service rate is small. Because of the same reason the number of interruption during a service remains almost unchanged as the interruption duration. Increase in service time results in increased number of customers waiting and longer waiting times.

Our Numerical experiment shows heavy dependency of the system performance measures on the rate of repair $\gamma$ as shown in Table 6.4. As $\gamma$ increases, the interruptions are repaired quickly so the duration of each interruption decreases. Also the chance that an interrupted service is resumed from where it was interrupted increases. Due to these reasons the service time decreases. The service may be completed even before it is protected. So the probability that the system is protected is decreased slightly. Due to the increased service rate, the number of customers in the system, average queue length and the waiting time are decreased. Changes in the number of interruptions during a service and probability that system is idle are to be noted.
Variations in the service rate $\mu$ have a direct impact on the performance of the system. As the expected service time decreases with increase in $\mu$, the system behaves in a nice way with high values for $\mu$ as illustrated in Table 6.5

6.7 Comparison of the models described in chapter 4 and chapter 6

The difference in the models described in chapter 4 and chapter 6 is in where and when the protection is applied. In the first model the final $n$ phases are protected in a deterministic way, irrespective of the circumstances. But in the latter, protection is given only if the service could not be completed in a reasonable time. Thus in this case, we wait for a while after the start of a new service and if it seems that the service may go long due to interruption, the protection is given to the rest of the service. Hence any number of phases may be protected. The protection can start even from the middle of a phase. In the first model we have no control over the time to complete the unprotected phases whereas the second model has a handicap of protecting unnecessary phases. The two types of protection methods have some similarity to the N-policy and T-Policy in queueing systems.

Table 4.1 and Table 6.1 can be compared to get a glimpse of the behaviour of the two systems with respect to the interruption rate $\theta$. It may be noted that, for the 'N-Policy' model the
probability that the system is protected remains constant with respect to $\theta$. This is because we protect only the last $n$ phases of the service process, whatever may be the value of $\theta$. Hence the protection cost remains the same. But in the 'T-Policy' model as $\theta$ increases the chance that the system is protected increases. This means that the protection is started at an earlier phase for high values of $\theta$ compared to the N-Policy model. This increases the cost of protection. But the values of the expected service times from these tables shows that even though more protection is is provided in the T-Policy model, the N-Policy model has the lowest service time for large values of $\theta$. In the T-Policy model to start the protection, we wait until we get an uninterrupted service for sufficiently long duration. For large values of $\theta$ it takes long to fulfil this condition as follows from the high values of $E(I)$ and $E^I(D)$. So much time is elapsed before switching the protection on. This results in a high service time. This is reflected in the values of the number of customers in the system and their waiting time. Thus the comparison of these tables reveals that when the chance of interruption is small, the T-Policy would be beneficial, but for large values of $\theta$, the N-Policy model dominates the other.

A comparison of Table 4.2 and Table 6.2 gives us a relation between the number of protected phases in the N-Policy model and the rate of realization of the protection clock in the corresponding T-Policy model. For the choice of other parameters, it can be seen that an N-Policy model with 1 protected phase performs almost similarly to a T-Policy model with $\varphi = 2$. 
The role of the service rate $\mu$ in the performance of the two systems is revealed in the comparison of Table 4.3 and Table 6.5. It can be seen that the probability $P_{pr}$ decreases considerably as $\mu$ increases for both the models. For the N-Policy model this decrement is due to the increase in the speed of service. But for the T-Policy model there is one more reason other than this. As the service is rendered quickly, most part of service might have completed before giving protection. So for high values of $\mu$ we are not getting the benefit of protection. This is why the expected number and duration of interruptions is more for the T-Policy model. This is reflected in the number of customers and their waiting time also. So we conclude that for large values of $\mu$, the N-Policy is better.

The role of the repeat rate $\gamma$ on the two models can be unveiled on the comparison of Table 4.4 and Table 6.3. With high repeat rate, most of the interrupted services have to be repeated. So the time to reach the protected phases is more for the N-Policy model. Therefore the service time increases with increase in repeat rate. The effect of $\gamma$ on the T-Policy model is the same. With a proper choice of $\varphi$ both the models give almost the same performance.

On the other hand, a high repair rate $\delta$ forces most of the interrupted services to be resumed. Hence in an N-Policy model, the service process reaches the protected phases quickly lowering the service time. So the model performs better with high repair rate. See Table 4.5 for numerical illustration.
But for the T-Policy model, with a high repair rate, though the repair is done very fast, the probability further interruptions denies a customer uninterrupted service for sufficient time to switch on the protection. Hence it may take long to get the service protected. As a result the service time will not be decreased in spite of high repair rate. Thus we conclude that with a high repair rate, N-Policy model is superior to the N-Policy model.

6.8 Cost analysis

In systems with interruptions, the interruptions is a cause of loss. We introduce protection to minimize the loss due to interruption. To protect the server from interruptions, additional resources might be used. This adds to the cost of service. So one must be interested to know the time to start the protection so that the cost of running the system is minimum.

we conducted a numerical experiment with the same cost function used in chapter 4, given by

\[
CF = CRPT \times ERPT + CRSM \times ERSM \\
+ CHOLD \times E_S(C) + CINT \times EI \\
+ CP \times P_{pr}.
\]

where CRPT and CRSM are the unit time costs for repeating or resuming an interrupted service respectively. CHOLD is the holding cost per unit time per customer and CINT is the cost
6.8. Cost analysis

per unit time per interruption. Finally CP is the unit time cost for giving service in the protected phases.

The results obtained for two different costs of protection are given in Table 6.6 and Table 6.7 with the values of the parameters involved. For this analysis we took

$$\lambda = 1.5, m = 10, \mu = 5.0, \theta = 1.0, \gamma = 1.0, \delta = 10.0$$

The results shows that for $CP = 600$, the optimum duration until protection is $42.5$ and for $CP = 650$, the optimal value is $6.505$. This assures that we can control the value of $\varphi$ in favour of the system.
Table 6.1: **Effect of interruption rate.**

Service rate, $\mu = 8$, Repair rate, $\delta = 2$ Rate of realization of the repeat clock, $\gamma = 3$, Rate of realization of the protection clock, $\varphi = 15$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_S(C)$</td>
<td>0.3</td>
<td>0.4669</td>
<td>0.6648</td>
<td>0.9030</td>
<td>1.1953</td>
<td>1.5625</td>
<td>2.0377</td>
</tr>
<tr>
<td>$V_S(C)$</td>
<td>0.3267</td>
<td>0.8717</td>
<td>1.5890</td>
<td>2.5557</td>
<td>3.8962</td>
<td>5.8216</td>
<td>8.7121</td>
</tr>
<tr>
<td>$E_Q(C)$</td>
<td>0.05</td>
<td>0.1613</td>
<td>0.3030</td>
<td>0.4845</td>
<td>0.7196</td>
<td>1.0291</td>
<td>1.4460</td>
</tr>
<tr>
<td>$E_I(C)$</td>
<td>0</td>
<td>0.1229</td>
<td>0.2682</td>
<td>0.4426</td>
<td>0.6563</td>
<td>0.9241</td>
<td>1.2701</td>
</tr>
<tr>
<td>$E_B(C)$</td>
<td>0.3</td>
<td>0.3440</td>
<td>0.3966</td>
<td>0.4603</td>
<td>0.5390</td>
<td>0.6384</td>
<td>0.7675</td>
</tr>
<tr>
<td>$P_S(I)$</td>
<td>0</td>
<td>0.0534</td>
<td>0.1074</td>
<td>0.1620</td>
<td>0.2171</td>
<td>0.2728</td>
<td>0.3290</td>
</tr>
<tr>
<td>$P_S(idle)$</td>
<td>0.75</td>
<td>0.6944</td>
<td>0.6382</td>
<td>0.5815</td>
<td>0.5243</td>
<td>0.4666</td>
<td>0.4083</td>
</tr>
<tr>
<td>$E(I)$</td>
<td>0</td>
<td>0.0534</td>
<td>0.1074</td>
<td>0.1620</td>
<td>0.2171</td>
<td>0.2728</td>
<td>0.3290</td>
</tr>
<tr>
<td>$E_I(D)$</td>
<td>0</td>
<td>0.0267</td>
<td>0.0537</td>
<td>0.0810</td>
<td>0.1086</td>
<td>0.1364</td>
<td>0.1645</td>
</tr>
<tr>
<td>$E(\tau)$</td>
<td>0.1250</td>
<td>0.1528</td>
<td>0.1809</td>
<td>0.2092</td>
<td>0.2379</td>
<td>0.2677</td>
<td>0.2958</td>
</tr>
<tr>
<td>$P_x$</td>
<td>0.1438</td>
<td>0.1454</td>
<td>0.1469</td>
<td>0.1485</td>
<td>0.1500</td>
<td>0.1515</td>
<td>0.1530</td>
</tr>
<tr>
<td>$W_L$</td>
<td>0.0250</td>
<td>0.0807</td>
<td>0.1515</td>
<td>0.2422</td>
<td>0.3598</td>
<td>0.5145</td>
<td>0.7230</td>
</tr>
</tbody>
</table>
6.8. Cost analysis

Table 6.2: **Effect of Realization rate of protection clock.**

Service rate, $\mu = 5$, Repair rate, $\delta = 2$ Rate of realization of the repeat clock, $\gamma = 3$, Interruption rate, $\theta = 1$.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_S(C)$</td>
<td>1.9719</td>
<td>1.5227</td>
<td>1.1869</td>
<td>0.9372</td>
<td>0.8218</td>
<td>0.7575</td>
<td>0.5604</td>
</tr>
<tr>
<td>$V_S(C)$</td>
<td>7.3757</td>
<td>4.7936</td>
<td>3.1326</td>
<td>2.0508</td>
<td>1.5960</td>
<td>1.3557</td>
<td>0.6785</td>
</tr>
<tr>
<td>$E_Q(C)$</td>
<td>1.3424</td>
<td>0.9435</td>
<td>0.6556</td>
<td>0.4499</td>
<td>0.3580</td>
<td>0.3079</td>
<td>0.1603</td>
</tr>
<tr>
<td>$E^I(C)$</td>
<td>0.7972</td>
<td>0.5482</td>
<td>0.3602</td>
<td>0.2189</td>
<td>0.1529</td>
<td>0.1159</td>
<td>0.0002</td>
</tr>
<tr>
<td>$E^B(C)$</td>
<td>1.1747</td>
<td>0.9746</td>
<td>0.8267</td>
<td>0.7183</td>
<td>0.6688</td>
<td>0.6416</td>
<td>0.5601</td>
</tr>
<tr>
<td>$P_S(I)$</td>
<td>0.2098</td>
<td>0.1649</td>
<td>0.1219</td>
<td>0.0821</td>
<td>0.0606</td>
<td>0.0475</td>
<td>0.0001</td>
</tr>
<tr>
<td>$P_S(idle)$</td>
<td>0.3705</td>
<td>0.4208</td>
<td>0.4687</td>
<td>0.5127</td>
<td>0.5362</td>
<td>0.5504</td>
<td>0.5999</td>
</tr>
<tr>
<td>$E(I)$</td>
<td>0.2098</td>
<td>0.1649</td>
<td>0.1219</td>
<td>0.0821</td>
<td>0.0606</td>
<td>0.0475</td>
<td>0</td>
</tr>
<tr>
<td>$E^I(D)$</td>
<td>0.1049</td>
<td>0.0825</td>
<td>0.0609</td>
<td>0.0410</td>
<td>0.0303</td>
<td>0.0237</td>
<td>0</td>
</tr>
<tr>
<td>$E(\tau)$</td>
<td>0.3147</td>
<td>0.2896</td>
<td>0.2656</td>
<td>0.2437</td>
<td>0.2319</td>
<td>0.2248</td>
<td>0.2001</td>
</tr>
<tr>
<td>$P_{pr}$</td>
<td>0</td>
<td>0.0844</td>
<td>0.1656</td>
<td>0.2411</td>
<td>0.2821</td>
<td>0.3073</td>
<td>0.3998</td>
</tr>
<tr>
<td>$W_L$</td>
<td>0.6712</td>
<td>0.4718</td>
<td>0.3278</td>
<td>0.2249</td>
<td>0.1790</td>
<td>0.1539</td>
<td>0.0801</td>
</tr>
</tbody>
</table>
Table 6.3: **Effect of rate of realization of repeat clock.**

Service rate, $\mu = 5$, Repair rate, $\delta = 2$ Rate of realization of the protection clock, $\varphi = 15$, Interruption rate, $\theta = 1$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_s(C)$</td>
<td>0.8142</td>
<td>0.8218</td>
<td>0.8234</td>
<td>0.8253</td>
<td>0.827</td>
<td>0.828</td>
<td>0.8285</td>
</tr>
<tr>
<td>$V_s(C)$</td>
<td>1.5672</td>
<td>1.596</td>
<td>1.6014</td>
<td>1.6074</td>
<td>1.6126</td>
<td>1.6152</td>
<td>1.6166</td>
</tr>
<tr>
<td>$E_Q(C)$</td>
<td>0.3519</td>
<td>0.358</td>
<td>0.3592</td>
<td>0.3606</td>
<td>0.3619</td>
<td>0.3626</td>
<td>0.363</td>
</tr>
<tr>
<td>$E_I(C)$</td>
<td>0.1521</td>
<td>0.1529</td>
<td>0.1531</td>
<td>0.1533</td>
<td>0.1535</td>
<td>0.1536</td>
<td>0.1536</td>
</tr>
<tr>
<td>$E_B(C)$</td>
<td>0.6621</td>
<td>0.6688</td>
<td>0.6703</td>
<td>0.672</td>
<td>0.6736</td>
<td>0.6744</td>
<td>0.6749</td>
</tr>
<tr>
<td>$P_s(I)$</td>
<td>0.0604</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0607</td>
<td>0.0607</td>
</tr>
<tr>
<td>$P_s(idle)$</td>
<td>0.5377</td>
<td>0.5362</td>
<td>0.5358</td>
<td>0.5353</td>
<td>0.5349</td>
<td>0.5346</td>
<td>0.5345</td>
</tr>
<tr>
<td>$E(I)$</td>
<td>0.0604</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0606</td>
<td>0.0607</td>
<td>0.0607</td>
</tr>
<tr>
<td>$E_I(D)$</td>
<td>0.0302</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
<td>0.0303</td>
</tr>
<tr>
<td>$E(\tau)$</td>
<td>0.2311</td>
<td>0.2319</td>
<td>0.2321</td>
<td>0.2323</td>
<td>0.2326</td>
<td>0.2327</td>
<td>0.2328</td>
</tr>
<tr>
<td>$P_{pr}$</td>
<td>0.2809</td>
<td>0.2821</td>
<td>0.2824</td>
<td>0.2828</td>
<td>0.2832</td>
<td>0.2834</td>
<td>0.2835</td>
</tr>
<tr>
<td>$W_L$</td>
<td>0.176</td>
<td>0.179</td>
<td>0.1796</td>
<td>0.1803</td>
<td>0.181</td>
<td>0.1813</td>
<td>0.1815</td>
</tr>
</tbody>
</table>
Table 6.4: **Effect of rate of repair.**

Service rate, $\mu = 5$, Rate of realization of the repeat clock, $\gamma = 3$, Rate of realization of the protection clock, $\varphi = 15$, Interruption rate, $\theta = 1$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_S(C)$</td>
<td>1.4027</td>
<td>0.7044</td>
<td>0.6577</td>
<td>0.6183</td>
<td>0.5918</td>
<td>0.5802</td>
<td>0.5747</td>
</tr>
<tr>
<td>$V_S(C)$</td>
<td>5.7409</td>
<td>1.0776</td>
<td>0.9137</td>
<td>0.7985</td>
<td>0.735</td>
<td>0.7112</td>
<td>0.7011</td>
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<tr>
<td>$E_Q(C)$</td>
<td>0.8775</td>
<td>0.2614</td>
<td>0.2251</td>
<td>0.1964</td>
<td>0.1785</td>
<td>0.1712</td>
<td>0.168</td>
</tr>
<tr>
<td>$E^I(C)$</td>
<td>0.4978</td>
<td>0.0837</td>
<td>0.0563</td>
<td>0.0334</td>
<td>0.0181</td>
<td>0.0114</td>
<td>0.0083</td>
</tr>
<tr>
<td>$E^B(C)$</td>
<td>0.905</td>
<td>0.6207</td>
<td>0.6014</td>
<td>0.5849</td>
<td>0.5737</td>
<td>0.5687</td>
<td>0.5664</td>
</tr>
<tr>
<td>$P_S(I)$</td>
<td>0.1212</td>
<td>0.0403</td>
<td>0.0302</td>
<td>0.0201</td>
<td>0.0121</td>
<td>0.0081</td>
<td>0.006</td>
</tr>
<tr>
<td>$P_S(idle)$</td>
<td>0.4747</td>
<td>0.5569</td>
<td>0.5674</td>
<td>0.578</td>
<td>0.5867</td>
<td>0.591</td>
<td>0.5932</td>
</tr>
<tr>
<td>$E(I)$</td>
<td>0.0606</td>
<td>0.0605</td>
<td>0.0605</td>
<td>0.0604</td>
<td>0.0604</td>
<td>0.0604</td>
<td>0.0604</td>
</tr>
<tr>
<td>$E^I(D)$</td>
<td>0.0606</td>
<td>0.0202</td>
<td>0.0151</td>
<td>0.0101</td>
<td>0.006</td>
<td>0.004</td>
<td>0.003</td>
</tr>
<tr>
<td>$E(\tau)$</td>
<td>0.2626</td>
<td>0.2215</td>
<td>0.2163</td>
<td>0.211</td>
<td>0.2067</td>
<td>0.2045</td>
<td>0.2034</td>
</tr>
<tr>
<td>$P_x$</td>
<td>0.2828</td>
<td>0.2817</td>
<td>0.2814</td>
<td>0.2809</td>
<td>0.2805</td>
<td>0.2802</td>
<td>0.28</td>
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<tr>
<td>$W_L$</td>
<td>0.4387</td>
<td>0.1307</td>
<td>0.1126</td>
<td>0.0982</td>
<td>0.0892</td>
<td>0.0856</td>
<td>0.084</td>
</tr>
</tbody>
</table>
Chapter 6. An $M/E_m/1$ Queue with Protection based on T-Policy

Table 6.5: **Effect of service rate.**

Repair rate, $\delta = 2$, Rate of realization of the repeat clock, $\gamma = 3$, Rate of realization of the protection clock, $\varphi = 15$, Interruption rate, $\theta = 1$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_S(C)$</td>
<td>2.2076</td>
<td>0.8218</td>
<td>0.4669</td>
<td>0.3696</td>
<td>0.3077</td>
<td>0.2472</td>
<td>0.1872</td>
</tr>
<tr>
<td>$V_S(C)$</td>
<td>6.3922</td>
<td>1.596</td>
<td>0.8717</td>
<td>0.6926</td>
<td>0.5802</td>
<td>0.4702</td>
<td>0.36</td>
</tr>
<tr>
<td>$E_Q(C)$</td>
<td>1.472</td>
<td>0.358</td>
<td>0.1613</td>
<td>0.1189</td>
<td>0.0946</td>
<td>0.0727</td>
<td>0.0528</td>
</tr>
<tr>
<td>$E^I(C)$</td>
<td>0.2368</td>
<td>0.1529</td>
<td>0.1229</td>
<td>0.11</td>
<td>0.0996</td>
<td>0.0871</td>
<td>0.0719</td>
</tr>
<tr>
<td>$E^B(C)$</td>
<td>1.9708</td>
<td>0.6688</td>
<td>0.344</td>
<td>0.2596</td>
<td>0.2082</td>
<td>0.1601</td>
<td>0.1153</td>
</tr>
<tr>
<td>$P_S(I)$</td>
<td>0.0647</td>
<td>0.0606</td>
<td>0.0534</td>
<td>0.049</td>
<td>0.0451</td>
<td>0.0401</td>
<td>0.0337</td>
</tr>
<tr>
<td>$P_S(idle)$</td>
<td>0.2644</td>
<td>0.5362</td>
<td>0.6944</td>
<td>0.7493</td>
<td>0.7868</td>
<td>0.8255</td>
<td>0.8656</td>
</tr>
<tr>
<td>$E(I)$</td>
<td>0.0647</td>
<td>0.0606</td>
<td>0.0534</td>
<td>0.049</td>
<td>0.0451</td>
<td>0.0401</td>
<td>0.0337</td>
</tr>
<tr>
<td>$E^I(D)$</td>
<td>0.0324</td>
<td>0.0303</td>
<td>0.0267</td>
<td>0.0245</td>
<td>0.0226</td>
<td>0.0201</td>
<td>0.0169</td>
</tr>
<tr>
<td>$E(\tau)$</td>
<td>0.3678</td>
<td>0.2319</td>
<td>0.1528</td>
<td>0.1254</td>
<td>0.1066</td>
<td>0.0873</td>
<td>0.0672</td>
</tr>
<tr>
<td>$P_{pr}$</td>
<td>0.5415</td>
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<td>0.1454</td>
<td>0.1037</td>
<td>0.0779</td>
<td>0.0541</td>
<td>0.0333</td>
</tr>
<tr>
<td>$W_L$</td>
<td>0.736</td>
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<td>0.0594</td>
<td>0.0473</td>
<td>0.0364</td>
<td>0.0264</td>
</tr>
</tbody>
</table>

Table 6.6: **Time to Protection versus Cost**

Cost of protection per unit time $= 600$

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$ERPT$</th>
<th>$ERSM$</th>
<th>$ES(C)$</th>
<th>$EI$</th>
<th>$P_{pr}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>35.0</td>
<td>0.0004</td>
<td>0.0390</td>
<td>2.0303</td>
<td>0.0429</td>
<td>0.7073</td>
<td>648.5794</td>
</tr>
<tr>
<td>40.0</td>
<td>0.0003</td>
<td>0.0341</td>
<td>2.0248</td>
<td>0.0375</td>
<td>0.7126</td>
<td>648.5721</td>
</tr>
<tr>
<td>42.0</td>
<td>0.0003</td>
<td>0.0325</td>
<td>2.0229</td>
<td>0.0357</td>
<td>0.7144</td>
<td>648.5716</td>
</tr>
<tr>
<td>42.5</td>
<td>0.0003</td>
<td>0.0321</td>
<td>2.0225</td>
<td>0.0353</td>
<td>0.7148</td>
<td><strong>648.5715</strong></td>
</tr>
<tr>
<td>43.0</td>
<td>0.0003</td>
<td>0.0317</td>
<td>2.0221</td>
<td>0.0349</td>
<td>0.7152</td>
<td>648.5716</td>
</tr>
<tr>
<td>45.0</td>
<td>0.0003</td>
<td>0.0303</td>
<td>2.0205</td>
<td>0.0333</td>
<td>0.7167</td>
<td>648.5722</td>
</tr>
<tr>
<td>50.0</td>
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<td>0.0273</td>
<td>2.0171</td>
<td>0.0300</td>
<td>0.7201</td>
<td>648.5758</td>
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</table>
### Table 6.7: Time to Protection versus Cost

Cost of protection per unit time = 600

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$ERPT$</th>
<th>$ERSM$</th>
<th>$E_S(C)$</th>
<th>$EI$</th>
<th>$P_{pr}$</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.000</td>
<td>0.0021</td>
<td>0.2111</td>
<td>2.2686</td>
<td>0.2322</td>
<td>0.5204</td>
<td>679.9899</td>
</tr>
<tr>
<td>6.300</td>
<td>0.0020</td>
<td>0.2027</td>
<td>2.2551</td>
<td>0.2230</td>
<td>0.5295</td>
<td>679.9660</td>
</tr>
<tr>
<td>6.490</td>
<td>0.0020</td>
<td>0.1977</td>
<td>2.2472</td>
<td>0.2175</td>
<td>0.5349</td>
<td>679.9617</td>
</tr>
<tr>
<td>6.500</td>
<td>0.0020</td>
<td>0.1975</td>
<td>2.2468</td>
<td>0.2172</td>
<td>0.5352</td>
<td>679.9617</td>
</tr>
<tr>
<td>6.505</td>
<td>0.0020</td>
<td>0.1973</td>
<td>2.2466</td>
<td>0.2171</td>
<td>0.5353</td>
<td><strong>679.9616</strong></td>
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CONCLUSION

In this thesis we discussed a few queueing models involving interruption of service and protection against interruption. Second and third chapters where on interruption without protection of service. As a consequence service of a customer has to be resumed or repeated depending on factors deciding which one to opt. Chapters 4, 5 and 6 introduced protection mechanism against interruption. The protection mechanism of chapter 4 has the flavour of N-policy where as those in chapters 5 and 6 have the flavour of T-policy.

The applications of the models discussed in this thesis are numerous, some of which are indicated in the introduction and in the relevant chapters. The results of chapters 4 and 6 are compared for efficiency.

The models discussed in this thesis can be extended to Markovian arrival process and arbitrarily distributed service time with rational Laplace Stieltjes transform. Several other variations and generalizations are on the anvil.
Chapter 6. An $M/E_m/1$ Queue with Protection based on T-Policy