Chapter 5

AN $M/M/1$ QUEUE WITH INTERRUPTION AND PROTECTION

5.1 Introduction

In chapter 4 we described a model in which the final few phases of the service is protected from interruptions so as to improve the system performance. This runs parallel to the N-policy. However, if the chance of interruption is considerably large, this method may not be fruitful enough. In such cases it may take much time to finish the unprotected phases. Here protecting more phases may not be a good idea. Moreover, in many real life situations, the instant at which protection is needed may vary from customer to customer. For example, in the treatment of chronic diseases, patients undergo a series of phases and during some of these phases the risk is higher. So patients in the
risky phases should be given additional attention. The time to reach a serious phase depends on the physical condition of the patients. So one cannot insist that special attention (Protection) will begin only at a particular phase. Some patients may require protection soon after they develop the disease while some others can withstand for a long duration. This procedure of protecting a service is similar to the T-policy in Queueing theory.

Motivated by this chronic care model, we introduce a system in which the server is protected from interruptions after a random time from the start of each service.

\section{Model Description}

We consider a single server queueing model in which customers arrive according to a Poisson process with parameter $\lambda$. The service time follows exponential distribution with mean $\frac{1}{\mu}$. While rendering service the server may face some interruptions. The interruptions occur according to a Poisson process with parameter $\theta$. The interrupted service restarts after a repair and the repair time follows exponential distribution with mean $\frac{1}{\delta}$. To diminish the effect of interruptions a protection mechanism is arranged. Once the protection mechanism is on, the service will continue without any further interruptions. This mechanism is provided after getting an uninterrupted service for a random time. This is done with the help of a random clock whose realization time is exponentially distributed with mean
The clock is started simultaneously with the service process. If there were no interruptions until the realization of the clock, the protection for the service is provided at the epoch of the realization of the clock. If there is any interruption before the realization of the clock, it is reset and started again with the restart of the service. This model is described in Figure 5.1.

![Figure 5.1: An M/M/1 Queue with Interruption and Protection](image)

### 5.3 Mathematical Model

The system described in the previous section can be mathematically modelled as a Markov process

\[
X = \{X(t)/t \geq 0\} = \{(N(t), C(t), J(t))/t \geq 0\}
\]

where \(N(t)\) is the number of customers in the system, \(C(t)\) is the status of the clock; it is 0 if clock is running and 1 if the clock is realized and \(J(t)\) is the state of the server which is 0 if server is not interrupted and 1 if it is interrupted. The state space of
the process is given by \{0\} \cup (\{1, 2, 3, \ldots\} \times \{0, 1\} \times \{0, 1\})$. The infinitesimal generator matrix of the process is given by

\[
Q = \begin{bmatrix}
A_{10} & A_{00} \\
A_{21} & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
... & ... & ... \\
... & ... & ... \\
\end{bmatrix}
\]

where

\[
A_{10} = \begin{bmatrix} -\lambda \end{bmatrix}, 
A_{00} = \begin{bmatrix} \lambda & 0 & 0 \end{bmatrix}, 
A_{21} = \begin{bmatrix} \mu \\
0 \\
0 \\
\mu \end{bmatrix}, 
A_1 = \begin{bmatrix} -\left(\mu + \theta + \varphi + \lambda\right) & \theta & \varphi \\
\delta & -\left(\delta + \lambda\right) & 0 \\
0 & 0 & -\left(\mu + \lambda\right) \end{bmatrix}, 
A_0 = \lambda I.
\]

5.4 Stability Analysis

Let \( A = A_0 + A_1 + A_2 = \begin{bmatrix} -\left(\theta + \varphi\right) & \theta & \varphi \\
\delta & -\delta & 0 \\
\mu & 0 & -\mu \end{bmatrix} \) and \( \pi = (\pi_1, \pi_2, \pi_3) \) be the invariant vector of \( A \) such that \( \pi e = 1 \). Then the system is stable if and only if \( \pi A_0 e < \pi A_2 e \). Hence we have
the following theorem.

**Theorem 5.4.1.** The system $X$ is stable if and only if

$$\frac{\lambda}{\mu} < \left(1 + \frac{\varphi}{\mu}\right) \left(1 + \frac{\varphi}{\mu} + \frac{\theta}{\delta}\right)^{-1}$$

5.5 Steady State Analysis

The steady state probability vector is obtained from the equation $xQ = 0$. Writing $x = (x_0, x_1, x_2, ...)$, a matrix geometric solution of the above equation is given by

$$x_0 = 1 - \rho$$

$$x_i = (1 - \rho)\alpha R^i, i = 1, 2, 3, ...$$

where

$$\rho = \frac{\lambda}{\mu + \varphi} \left(1 + \frac{\varphi}{\mu} + \frac{\theta}{\delta}\right)$$

$$R = \frac{\lambda}{\mu (\mu + \lambda + \varphi)} \begin{bmatrix}
\frac{\mu + \lambda}{\delta + \lambda} & \frac{\theta (\mu + \lambda)}{\delta + \lambda} & \varphi \\
\frac{\mu + \lambda}{\delta + \lambda} & \frac{\theta (\mu + \lambda)}{\delta + \lambda} + \frac{\mu (\mu + \lambda + \varphi)}{\delta + \lambda} & \varphi \\
\frac{\mu + \lambda}{\delta + \lambda} & \frac{\theta \lambda}{\delta + \lambda} + \frac{\theta \lambda}{\delta + \lambda} & \mu + \lambda
\end{bmatrix}$$

and $\alpha = (1, 0, 0)$.
5.6 Analysis of the Service Process

Obviously we can see that the service time follows phase type distribution with representation \((\alpha, S)\) where \(\alpha = (1, 0, 0)\) and

\[
S = \begin{bmatrix}
-(\mu + \theta + \varphi) & \theta & \varphi \\
\delta & -\delta & 0 \\
0 & 0 & -\mu
\end{bmatrix}
\]

The following results can be easily derived.

- The expected service time, \(E(\tau) = \frac{1}{\mu} + \frac{\theta}{\mu + \varphi} \cdot \frac{1}{\delta}\)
- Increase in service time due to interruption = \(\frac{\theta}{\mu + \varphi} \cdot \frac{1}{\delta}\)
- During a service, the time spent in the unprotected uninterrupted state = \(\frac{1}{\mu + \varphi}\)
- Time spent in the protected state = \(\frac{\varphi}{\mu + \varphi} \cdot \frac{1}{\delta}\)
- Probability that the service is completed before protection starts = \(\frac{\mu}{\mu + \varphi}\)

5.6.1 Expected number of interruptions during any particular service

Let \(N(t)\) be the number of interruptions during a particular service at time \(t\). Let \(J(t)\) be the status of the server at time
5.6. Analysis of the Service Process

Then \( Y = \{ (N(t), J(t)) / t \geq 0 \} \) is a Markov Process whose infinitesimal generator matrix is given by

\[
\tilde{Q} = \begin{bmatrix}
0 & 0 & 0 \\
S_2 & S_1 & S_0 \\
S_2 & 0 & S_1 & S_0 \\
S_2 & 0 & 0 & S_1 & S_0 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

where

\[
S_0 = \begin{bmatrix}
\mu + \varphi \\
0
\end{bmatrix},
S_1 = \begin{bmatrix}
-(\mu + \theta + \varphi) & 0 \\
0 & -\delta
\end{bmatrix}
\text{ and } S_2 = \begin{bmatrix}
0 & \theta \\
\delta & 0
\end{bmatrix}.
\]

Let \( y_k \) be the probability that there are exactly \( k \) interruptions during a service. Then

\[
y_0 = \alpha (-S_1)^{-1} S_2
\]

and

\[
y_k = \alpha \left[ (-S_1)^{-1} S_0 \right]^k (-S_1)^{-1} S_2, \quad k = 1, 2, 3, \ldots
\]

Simplifying we get,

\[
y_k = \frac{\mu + \varphi}{\mu + \varphi + \theta} \left( \frac{\theta}{\mu + \varphi + \theta} \right)^k, \quad k = 0, 1, 2, \ldots
\]

**Theorem 5.6.1.** The mean number of interruptions occur dur-
ing a single service is

\[ E(I) = \frac{\theta}{\mu + \varphi}. \]

**Proof.** The mean number of interruptions,

\[ E(I) = \sum_{k=0}^{\infty} ky_k = \frac{\theta}{\mu + \varphi}. \]

\[ \square \]

### 5.7 Expected Waiting Time

Consider a customer who joins as the \( r^{th} \) customer in the queue, \( r > 0 \). The waiting time of this customer may be described as the time until absorption of a Markov Chain \( W = \{W(t)/t \geq 0\} = \{(N(t), C(t), J(t))/t \geq 0\} \) where \( N(t) \) is the rank of the tagged customer, \( C(t) = 0 \) if the service is unprotected and 1 otherwise and \( J(t) = 1 \) if the service is interrupted and 0 otherwise at time \( t \). Thus the waiting time of the tagged customer has a phase type distribution with representation \((\beta, B)\) where

\[
B = \begin{bmatrix}
S & S^0 \alpha \\
S & S^0 \alpha \\
\cdots & \cdots \\
S & S^0 \alpha \\
\end{bmatrix}
\]
and $\beta$ is the initial probability vector which ensures that the chain always starts from level $r$.

Therefore the expected waiting time of the tagged customer according to the state of the server at the time of joining the queue,

$$E^r_W = -B^{-1}e$$

$$= -S^{-1} \left[ I - S^0 \alpha S^{-1} + \ldots + (-1)^{r-1} (S^0 \alpha S^{-1})^{r-1} \right] e$$

$$= -S^{-1}e + \frac{r - 1}{\mu + \varphi} \left( 1 + \frac{\theta \varphi}{\delta \mu} \right) e$$

$$= -S^{-1}e + (r - 1)E(\tau)e.$$

Hence the expected waiting time of a customer who has to wait is

$$E(W) = \sum_{r=1}^{\infty} x_r E^r_W.$$ 

### 5.8 Some Performance Measures

- Probability that the system is idle,

$$P_s(\text{idle}) = 1 - \rho$$

- Probability that the system is busy and uninterrupted without protection on

$$\frac{\lambda}{\mu + \varphi}$$

- Probability that the system is in interruption,
• Probability that the system is under protection,
\[
\frac{\varphi \lambda}{\mu + \varphi \mu}
\]

• Expected number of customers in the system
\[
\rho + \frac{\lambda}{1 - \rho} \left[ \frac{\rho}{\mu} + \frac{\lambda \theta (\mu + \varphi + \theta + \delta)}{(\mu + \varphi)^2 \delta^2} \right]
\]

5.9 Numerical Illustration

A numerical study of the effect of various parameters involved on various performance measures is carried out in this section.

Table 5.1 shows the effect of \(\varphi\), the rate of realization of the protection clock. As \(\varphi\) increases the service is protected from interruption that much quicker and hence reduces the chance of being interrupted. This improves the performance of the system. As the expected number of interruptions is decreased, the service time is reduced. This results in a reduction of expected number of customers in the system and expected waiting time. When \(\varphi \to \infty\), the system reduces to an ordinary \(M/M/1\) queue.

In Table 5.2 the effect of interruption rate \(\theta\) on various performance measures is illustrated. As expected, an increase in \(\theta\) increases the probability that the system is interrupted and hence the service time increases. As a consequence, the ex-
pected number of customers in the system and expected waiting
time are also increased.

Table 5.3 explains the effect of the repair rate \( \delta \) on the 
performance of the system. As the repair rate increases, the time 
spent in the interrupted state decreases and results in a higher 
service rate. This explains the changes in different measures. If 
\( \delta \) is very large, the repair is done instantaneously and the time 
spent in the interrupted state becomes zero.

Table 5.1: Variation in different performance measures 
with the rate of realization of protection clock

Arrival Rate \( \lambda = 3 \), Service Rate \( \mu = 10 \), Interruption Rate \( \theta = 3 \), repair rate \( \delta = 5 \).

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( P_s(Idle) )</th>
<th>( P_s(Prot) )</th>
<th>( P_s(I) )</th>
<th>( E_s(C) )</th>
<th>( E(W) )</th>
<th>( E(Ser) )</th>
<th>( E(I) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4818</td>
<td>0.0273</td>
<td>0.2182</td>
<td>0.8166</td>
<td>0.1827</td>
<td>0.1727</td>
<td>0.3636</td>
</tr>
<tr>
<td>2</td>
<td>0.5000</td>
<td>0.0500</td>
<td>0.2000</td>
<td>0.7200</td>
<td>0.1567</td>
<td>0.1667</td>
<td>0.3333</td>
</tr>
<tr>
<td>3</td>
<td>0.5154</td>
<td>0.0692</td>
<td>0.1846</td>
<td>0.6458</td>
<td>0.1370</td>
<td>0.1615</td>
<td>0.3077</td>
</tr>
<tr>
<td>4</td>
<td>0.5286</td>
<td>0.0857</td>
<td>0.1714</td>
<td>0.5873</td>
<td>0.1217</td>
<td>0.1571</td>
<td>0.2857</td>
</tr>
<tr>
<td>5</td>
<td>0.5400</td>
<td>0.1000</td>
<td>0.1600</td>
<td>0.5400</td>
<td>0.1095</td>
<td>0.1533</td>
<td>0.2667</td>
</tr>
<tr>
<td>6</td>
<td>0.5500</td>
<td>0.1125</td>
<td>0.1500</td>
<td>0.5011</td>
<td>0.0995</td>
<td>0.1500</td>
<td>0.2500</td>
</tr>
<tr>
<td>7</td>
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<td>0.1235</td>
<td>0.1412</td>
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<td>0.0913</td>
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<td>0.2353</td>
</tr>
<tr>
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<td>0.1333</td>
<td>0.4412</td>
<td>0.0845</td>
<td>0.1444</td>
<td>0.2222</td>
</tr>
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<td>9</td>
<td>0.5737</td>
<td>0.1421</td>
<td>0.1263</td>
<td>0.4176</td>
<td>0.0786</td>
<td>0.1421</td>
<td>0.2105</td>
</tr>
<tr>
<td>10</td>
<td>0.5800</td>
<td>0.1500</td>
<td>0.1200</td>
<td>0.3972</td>
<td>0.0736</td>
<td>0.1400</td>
<td>0.2000</td>
</tr>
<tr>
<td>large</td>
<td>0.7000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1286</td>
<td>0.0129</td>
<td>0.1000</td>
<td>0.4000</td>
</tr>
</tbody>
</table>
Table 5.2: **Variation in different performance measures with the interruption rate**

Arrival Rate $\lambda = 3$, Service Rate $\mu = 10$, Protection clock realization rate $\varphi = 4$ Rate $\theta = 3$, repair rate $\delta = 5$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$P_s(Idle)$</th>
<th>$P_s(Prot)$</th>
<th>$P_s(I)$</th>
<th>$E_s(C)$</th>
<th>$E(W)$</th>
<th>$E(Ser)$</th>
<th>$E(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6571</td>
<td>0.0857</td>
<td>0.0429</td>
<td>0.2124</td>
<td>0.0316</td>
<td>0.1143</td>
<td>0.0714</td>
</tr>
<tr>
<td>2</td>
<td>0.6143</td>
<td>0.0857</td>
<td>0.0857</td>
<td>0.3410</td>
<td>0.0551</td>
<td>0.1286</td>
<td>0.1429</td>
</tr>
<tr>
<td>3</td>
<td>0.5714</td>
<td>0.0857</td>
<td>0.1286</td>
<td>0.4371</td>
<td>0.0845</td>
<td>0.1429</td>
<td>0.2143</td>
</tr>
<tr>
<td>4</td>
<td>0.5286</td>
<td>0.0857</td>
<td>0.1714</td>
<td>0.5873</td>
<td>0.1217</td>
<td>0.1571</td>
<td>0.2571</td>
</tr>
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<td>5</td>
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</tr>
<tr>
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<td>0.2000</td>
<td>0.5000</td>
</tr>
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<td>0.0857</td>
<td>0.3429</td>
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<td>0.4125</td>
<td>0.2143</td>
<td>0.5714</td>
</tr>
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<td>0.0857</td>
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<td>0.5524</td>
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<td>0.6429</td>
</tr>
<tr>
<td>10</td>
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<td>0.4286</td>
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<td>0.7456</td>
<td>0.2429</td>
<td>0.7143</td>
</tr>
</tbody>
</table>

### 5.10 Analysis of Cost Function

The results in the previous sections show the effect of interruptions on the system performance and the extend to which it can be minimized through giving protection. The cost of service will increase due to the introduction of protection. Longer the time the service is protected the more will be the cost. So a question naturally arises about the epoch at which the protection starts so that the service cost is optimum. For checking this optimality we consider the following factors:- The expected number of interruptions $E(I)$, The fraction of time in which the system is protected $T(Pr)$, The fraction of time in which the system is unprotected $T(unpr)$, The number of customers in
Table 5.3: **Variation in different performance measures with the repair rate**

Arrival Rate $\lambda = 3$, Service Rate $\mu = 10$, Protection clock realization rate $\varphi = 5$ Interruption rate $\theta = 3$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$P_s(Idle)$</th>
<th>$P_s(Prot)$</th>
<th>$P_s(I)$</th>
<th>$E_s(C)$</th>
<th>$E(W)$</th>
<th>$E(Ser)$</th>
<th>$E(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1000</td>
<td>0.1</td>
<td>0.6000</td>
<td>25.5000</td>
<td>8.2300</td>
<td>0.3000</td>
<td><strong>0.2</strong></td>
</tr>
<tr>
<td>2</td>
<td>0.4000</td>
<td>0.1</td>
<td>0.3000</td>
<td>1.9500</td>
<td>0.5300</td>
<td>0.2000</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.5000</td>
<td>0.1</td>
<td>0.2000</td>
<td>0.8600</td>
<td>0.2033</td>
<td>0.1667</td>
<td>0.2</td>
</tr>
<tr>
<td>4</td>
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<td>0.1</td>
<td>0.1500</td>
<td>0.5455</td>
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</tr>
<tr>
<td>5</td>
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<td>0.1</td>
<td>0.1200</td>
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<td>0.2</td>
</tr>
<tr>
<td>6</td>
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<td>0.1000</td>
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<td>0.0578</td>
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<td>0.2</td>
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<tr>
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<td>0.0857</td>
<td>0.2880</td>
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<td>0.2</td>
</tr>
<tr>
<td>8</td>
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<td>0.1</td>
<td>0.0750</td>
<td>0.2580</td>
<td>0.0391</td>
<td>0.1250</td>
<td>0.2</td>
</tr>
<tr>
<td>9</td>
<td>0.6333</td>
<td>0.1</td>
<td>0.0667</td>
<td>0.2368</td>
<td>0.0341</td>
<td>0.1222</td>
<td>0.2</td>
</tr>
<tr>
<td>10</td>
<td>0.6400</td>
<td>0.1</td>
<td>0.0600</td>
<td>0.2213</td>
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<td>0.2</td>
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<tr>
<td>large</td>
<td>0.7000</td>
<td>0.1</td>
<td>0.0000</td>
<td>0.1286</td>
<td>0.0129</td>
<td>0.1000</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Each time an interruption occurs, the repair has to be done and let $RC$ be the repair cost. Let $PC$ be the unit time cost of running the server with protection and $UPC$ be that without protection. Let $HC$ be the holding cost for retaining a customer for unit time. Then the service cost per unit time is given by

$$C = E(I) \times RC + E_s(C) \times HC + T(pr) \times PC + T(upr) \times UPC.$$ 

The variation in cost with the time to start protection when $\mu = 10, \lambda = 3, \theta = 4$ and $\delta = 5$ with costs $RC = 4, HC = 0.5, PC = 20$ and $UPC = 2.5$ is given in Table 5.4. In this case the optimum value for $\varphi$ is 6.
Table 5.4: The variation in cost with the time to start protection

Arrival Rate $\lambda = 3$, Service Rate $\mu = 10$, Interruption rate $\theta = 4$, repair rate $\delta = 5$.

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>5.985</td>
<td>5.9768</td>
<td>5.9722</td>
<td>5.97</td>
<td>5.9693</td>
<td>5.9696</td>
<td>5.9706</td>
<td>5.972</td>
</tr>
</tbody>
</table>

Figure 5.2: Variation in Cost with $\varphi$