CHAPTER - II

Structural Properties of Generalized Negative Binomial Distribution
2.1. Introduction

The usefulness of a distribution to a greater extent rests on its structural properties and the basic assumptions inherent in the very derivation of the distribution. These properties considerably help in recognizing the empirical situations where the distributions may be applied successfully. The more clearly determined the structural properties, the more clear is the scope for distribution.

The various structural properties of generalized negative binomial distribution have already been studied by many researchers in past. But it is scattered in literature. First of all, an attempt has been made to collect all such properties and to make a critical study of these. Besides, the structural properties, there are some other interesting properties that the distribution may posses, such as properties regarding the relationship of the distribution with other distributions etc, a critical study of all such properties has been done in this chapter.

In this chapter the general expression for correlation co-efficient between boys and girls in a family has been obtained, considering that the family size has inflated modified power series distribution, and then specialized to generalized negative binomial distribution and generalized Poisson distribution and their particular cases.
Size-biased generalized negative binomial distribution (SBGNBD), a particular case of the weighted generalized negative binomial distribution, taking the weights as the variate value has been defined. The first four moments about origin and about mean of SBGNBD has been obtained.

Also a general class of GNBD’s has been defined. This class of GNBD form a wider class of distributions having any particular GNBD as its sub class. Some of the characterization properties have been discussed and a general expression for moments of this class of NBD’s has been obtained through which the moments of various particular GNBD’s can easily be obtained.

2.2: Generation of the GNBD Model

The generalized negative binomial distribution (GNBD) was first defined by Jain and Consul (1971), it was subsequently obtained by Consul and Shenton (1972a) as a particular family of Lagrangian probability distributions. The GNBD has been generated by many researchers, Consul and Famoye (1995) consolidated the four different methods of generating GNBD.

i. Jain and Consul (1971): The Lagrangian expansion of $(1-\theta)^{-m}$ in powers of $u = 0(1-\theta)^{0-1}$ by using the Lagrangian transformation $t = u(1-\theta)^{1-\theta}$ gives

$$
(1-\theta)^{-m} = 1 + \sum_{x=1}^{\infty} \frac{u^x}{x!} d^{x-1} \left[ \frac{m(1-\theta)^{-m-1-\beta xx}}{0} \right]_{0-0}
$$

$$
(1-\theta)^{-m} = \sum_{x=0}^{\infty} \frac{m}{m+\beta x} \left( \frac{m+\beta x}{x} \right)^x (1-\theta)^{\beta x-x}
$$

(2.2.1)

Dividing by $(1-\theta)^{-m}$, the above provides $P(X = x)$ as in (1.6.1) for $x = 0, 1, 2...$
II. **Consul and Shenton (1972a):** When \( g(t) = (1 - \theta + \theta t)^{\beta} \) for \( 1 - \theta + \theta = 1 \) such that \( \theta \beta < 1 \) and \( f(t) = (1 - \theta + \theta t)^{m} \) under transformation \( t = u g(t) \). (1.4.1) takes the form

\[
(1 - \theta + \theta t)^{m} = (1 - \theta)^{m} + \sum_{x=0}^{\infty} \frac{u^x}{x!} \left[ \frac{d}{dt}^{x-1} m \theta (1 - \theta + \theta t)^{m-\beta} \right]_{t=0}
\]

\[
= (1 - \theta)^{m} \sum_{x=0}^{\infty} \frac{m}{m + \beta x} \left( \frac{m + \beta x}{x} \right)^x [\theta (1 - \theta)^{\beta-1} u]^x \quad (2.2.2)
\]

Evidently for \( u = 1 \), the unique root of equation \( t = u (1 - \theta + \theta t)^{\beta} \) is \( t = 1 \), setting \( u = 1 \), in (2.2.2) reveals that

\[
P(X = x) = \frac{m}{m + \beta x} \left( \frac{m + \beta x}{x} \right)^x \theta^x (1 - \theta)^{m-\beta-x} \quad x = 0, 1, 2, \ldots
\]

III. **Consul and Famoye (1995):** The Lagrangian expansion of function \( f(t) = (1 - \theta)^{m} (1 - \theta t)^{-m} \) under transformation \( t = u g(t) \) where

\( g(t) = (1 - \theta)^{\beta-1} (1 - \theta t)^{1-\beta} \), \( m > 0 \), \( \beta \geq 1 \), \( 0 < \theta < 1 \) in powers of \( u \) also provide the value of \( P(X = x) \) given in (1.6.1), as the coefficient of \( u^x \) in the expansion.

IV. **Consul (1993):** defined it as a particular case of Lagrangian Katz family with probability function given as

\[
P(X = x) = \frac{a}{p \left( \frac{a}{p} + x \right)} \left( \frac{a}{p} + x \right)^{a-1} \left( \frac{b}{p} + x \right)^{b-1} p^x (1 - p)^p \quad (2.2.3)
\]

\[
x = 0, 1, 2, \ldots
\]

Where \( a > 0 \), \( b > 0 \) and \( p < 1 \). When \( 0 < p < 1 \) the GNBD is obtained as a particular case of (2.2.3) by putting \( a = mp \), \( b = (\beta - 1)p \) and \( 0 < p = \theta < 1 \).

2.3: **Moments Of Generalized Negative Binomial Distribution**

The \( r^{th} \) moment about origin \( (\mu'_r) \) of generalized negative binomial distribution is the sum of an infinite series given by
\[
\mu_r' = \sum_{x=0}^{\infty} x' \frac{m}{m + \beta x} \left( \frac{m + \beta x}{x} \right)^x \theta^x (1 - \theta)^{m + \beta x - x} \quad x = 0, 1, 2, \ldots
\]

Jain and Consul (1971) obtained the recurrence relation of moments by converting the series into \( \mu'_r \) functions given as

\[
\mu'_r = m \theta \sum_{j=0}^{r-1} \binom{r-1}{j} \left[ \mu'_j(m + \beta - 1) + \frac{\beta}{m + \beta - 1} \mu'_{j+1}(m + \beta - 1) \right] \quad r = 1, 2, \ldots
\]

(2.3.1)

The repeated use of (2.3.1), gives mean of GNBD as

\[
\mu'_1 = \frac{m \theta}{(1 - \theta \beta)}
\]

(2.3.2)

using successive reductions on functions \( \mu'_1 \) and substituting values of \( \mu'_1 \), \( \mu'_2 \) is obtained and is

\[
\mu'_2 = \frac{(m \theta)^2}{(1 - \theta \beta)^2} + \frac{m \theta (1 - \theta)}{(1 - \theta \beta)^3}
\]

(2.3.3)

thus the variance of the generalized negative binomial distribution is

\[
\mu_2 = \frac{m \theta (1 - \theta)}{(1 - \theta \beta)^3}
\]

(2.3.4)

The third and fourth moments about origin are given as

\[
\mu'_3 = \frac{(m \theta)^3}{(1 - \theta \beta)^3} + \frac{3(m \theta)^2 (1 - \theta)}{(1 - \theta \beta)^4} + \frac{m \theta (1 - \theta)}{(1 - \theta \beta)^5} \left[ 1 - 2 \theta + \theta \beta (2 - \theta) \right]
\]

(2.3.5)

\[
\mu'_4 = \frac{(m \theta)^4}{(1 - \theta \beta)^4} + \frac{6(m \theta)^3 (1 - \theta)}{(1 - \theta \beta)^5} + \frac{(m \theta)^2 (1 - \theta) [7 - 11 \theta + 4 \theta \beta (2 - \theta)]}{(1 - \theta \beta)^6}
\]

\[
+ \frac{m \theta (1 - \theta) [1 - 6 \theta + 6 \theta^2 + 2 \theta \beta (4 - 9 \theta + 4 \theta^2) + \theta^2 \beta^2 (6 - 6 \theta + \theta^2)]}{(1 - \theta \beta)^7}
\]

(2.3.6)

using relation between moments about mean \( (\mu_r) \) in terms of moments about origin \( (\mu'_r) \) the third and fourth central moments are obtained as
\[ \mu_3 = \frac{m\theta(1-\theta)}{(1-\theta\beta)^3} \left[ 1 - 2\theta + \theta\beta(2 - \theta) \right] \]  
\[ \mu_4 = \frac{3(m\theta)^2(1-\theta)^2}{(1-\theta\beta)^4} + \frac{m\theta(1-\theta)}{(1-\theta\beta)^5} \left[ 1 - 6\theta + 6\theta^2 + 2\theta\beta(4 - 9\theta + 4\theta^2) + \theta^2\beta^2(6 - 6\theta + \theta^2) \right] \]  
\[ \frac{1}{(1-\theta\beta)^7} \]  
\[ (2.3.7) \]  
\[ (2.3.8) \]

In particular taking \( \beta = 1 \) and \( \beta = 0 \), in the expressions for moments of generalized negative binomial distribution, the moments for negative binomial and binomial distributions are obtained respectively.

The mean and variance of GNBD are approximately equal if \( \beta = \frac{1}{2} \) and exactly equal if \( \beta = \left[ 1 - (1-\theta)^{1/2} \right]/\theta \).

### 2.4: Moments of GNBD Under the Framework of MPSD

Gupta (1974a) defined the modified power series distribution (1.3.1) of which generalized negative binomial distribution (1.6.1) is a member with

\[ f(\theta) = (1-\theta)^{-m} \text{ and } g(\theta) = \theta(1-\theta)^{q-1} \]

The recurrence relation between moments about origin of modified power series distribution is given as

\[ \mu'_{r+1} = \frac{g(\theta)}{g'(\theta)} \frac{d\mu'_{r}}{d\theta} + \mu'_r \mu'_1 \]  
\[ (2.4.1) \]

Using (2.4.1) the recurrence relation between the moments about zero of GNBD takes the form

\[ \mu'_{r+1} = \frac{\theta(1-\theta)}{1-\theta\beta} \frac{d\mu'_{r}}{d\theta} + \mu'_r \mu'_1 \]  
\[ (2.4.2) \]

The recurrence relation between central moments can be obtained by using the relation (1.3.4), which gives

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Using the relation (1.3.6), the recurrence relation between the factorial moments of GNBD is given as

$$
\mu_{r+1} = \frac{\theta(1-\theta)}{1-\theta^r} \frac{d\mu_r}{d\theta} + r\mu_r \mu_{r-1}
$$

(2.4.3)

Using the relation (1.3.6), the recurrence relation between the factorial moments of GNBD is given as

$$
\mu_{[r+1]} = \frac{\theta(1-\theta)}{1-\theta^r} \frac{d\mu_r}{d\theta} + \mu_{[r]} \mu_{[r]} - r\mu_r
$$

(2.4.4)

2.5: Negative Integer Moments of GNBD

Stephan (1945) demonstrated the importance of study of negative or reciprocal moments of discrete random variables taking positive values. The reciprocal moments are used in different context, notably in life testing and in the ratio estimations. Kumar and Consul (1979) obtained the negative moments of GNBD. For positive integer $r$, the $r^{th}$ negative moment about arbitrary constant $k$ of MPSD is given by

$$
M(r,k) = E[(X+k)^r] = \sum_{x=1}^\infty \frac{a(x)[g(\theta)]^k}{(x+k)^r f(\theta)}
$$

(2.5.1)

on differentiating (2.5.1) and multiplying the resultant expression by $f(\theta) \cdot [g(\theta)]^k$ and integrating from 0 to $\theta$ one gets the relation

$$
M(r,k) = \left[f(\theta)[g(\theta)]^k\right]^{-1} \int_0^\theta M(r-1,k)I(k)d\theta
$$

(2.5.2)

where $I(k) = f(\theta) \cdot g'(\theta) \cdot \{g(\theta)\}^{k-1}$

and $M(0,k) = 1$

GNBD being the member of MPSD with $g(\theta) = \theta(1-\theta)^{\theta-1}$ and $f(\theta) = (1-\theta)^{-m}$. The first negative integer moment (NIM) about the $-k$ ($k > 0$) for GNBD is obtained by using recursive formula (2.5.2) as
\[ M(1, k) = E \left[ \frac{1}{X + k} \right] = \frac{(1 - \theta)^{m+k-\beta k}}{\theta^k} \int_0^\infty (1 - \theta) \theta^{k-1} (1 - \theta)^{\beta k - k - m - 1} d\theta \] (2.5.3)

i) For \( \beta k - k - m = \ell > 0 \)

\[ E \left( \frac{1}{X + k} \right) = \frac{1}{\theta^k} \left[ B(k, \ell) - \beta B(k + 1, \ell) \right] \] (2.5.4)

Where \( B(p, q) = \int_0^\theta x^{p-1} (1 - x)^{q-1} dx \) is incomplete beta function.

ii) \( \beta k - k - m = \ell < 0; \) the integral in (2.5.3) can be evaluated in the form of a convergent series and the first negative moment becomes

\[ E \left[ \frac{1}{X + k} \right] = \frac{1}{(1 - \theta)^{\ell}} \sum_{i=0}^\infty \left( -\ell - i \right) \left( \frac{1}{k + i} \frac{\theta \beta}{k + i + 1} \right)^i \] (2.5.5)

The first negative moment of the truncated generalized negative binomial distribution (TGNBD) truncated on the left at \( x = 0 \), is given by recursive formula (2.5.2), for \( k = 0 \) as

\[ E \left( \frac{1}{X} \right) = \left\{ \frac{1}{(1 - \theta)^m - 1} \right\} \int_0^\theta \left\{ \frac{1}{\theta (1 - \theta)^{m+1}} - \frac{1}{\theta (1 - \theta)^{m+1}} + \frac{\beta}{1 - \theta} \right\} d\theta \]

Which reduced to

\[ E \left( \frac{1}{X} \right) = \frac{(1 - \theta)^m}{1 - (1 - \theta)^m} \left[ \sum_{i=1}^m \left( \frac{m}{i} \right) \left( \frac{\theta}{1 - \theta} \right)^i - \beta \frac{1 - (1 - \theta)^m}{(1 - \theta)^m} - \beta \log(1 - \theta) \right] \] (2.5.6)

The identity

\[ \sum_{i=1}^m \left( \frac{m}{i} \right) \alpha^i = \sum_{i=1}^m \left( (1 + \alpha)^i - 1 \right) \]

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transforms the above result to the form given by Gupta (1974a). On substituting the result (2.5.6) in (2.5.1), for \( k = 0 \) and \( r = 2 \) the second negative moments of the decapitated GNBD is obtained as

\[
E \left[ \frac{1}{X^2} \right] = \frac{(1-\theta)^m}{1-(1-\theta)^m} \int_0^\theta \left( 1-\theta \theta^i \right) \left( \frac{\theta}{1-\theta} \right)^i \left[ \frac{m}{i} \left( \frac{\theta}{1-\theta} \right)^i \right] \left( -\beta \left( 1-\theta \right)^{m-i} \right) d\theta 
\]

(2.5.7)

2.6: Incomplete Moments of GNBD

On substitution of (2.3.2) and (2.3.4) in (1.3.10) one gets

i) The recurrence relation for incomplete moments about \( k \) for GNBD as

\[
u_{r+1}' = \frac{\theta}{1-\theta \beta} \left[ m \nu_r' + \left( 1-\theta \right) \frac{d}{d\theta} \nu_r' + \frac{dk}{d\theta} \nu_r'_{r-1} \right] - k\nu_r' \]

(2.6.1)

if \( k \) is a constant, independent of \( \theta \), (2.6.1) reduced to

\[
u_{r+1}' = \frac{\theta}{1-\theta \beta} \left[ m \nu_r' + \left( 1-\theta \right) \frac{d}{d\theta} \nu_r' \right] - k\nu_r'
\]

if \( k = 0 \), (2.6.1) reduces to

\[
u_{r+1}' = \frac{\theta}{1-\theta \beta} \left[ m \nu_r' + \left( 1-\theta \right) \frac{d}{d\theta} \nu_r' \right]
\]

ii) In particular, if \( k = \mu \), by virtue of relation (1.3.11), the recurrence relation for incomplete moments about \( \mu \) is given as

\[
u_{r+1} = \frac{\theta(1-\theta)}{(1-\theta \beta)^3} \left[ (1-\theta \beta)^3 \frac{d}{d\theta} \nu_r + r \cdot m \cdot \nu_{r-1} \right]
\]

(2.6.2)

For \( \beta = 1 \), the results (2.6.1) and (2.6.2) gives recurrence relations for negative integer moments for negative binomial distribution, obtained independently by Romanovskiy (1923).
2.7: Moments of Decapitated GNBD

Suppose $X$ has a decapitated generalized negative binomial distribution (1.6.4). It is MPSD with $g(\theta) = \theta(1-\theta)^{\beta-1}$ and $f(\theta) = (1-\theta)^{-m-1}$

and $g'(\theta) = (1-\theta)^{\beta-2}(1-\theta\beta)$, $f'(\theta) = m(1-\theta)^{-m-1}$

Gupta (1974b) obtained the mean and variance of decapitated GNBD as

\[
\mu'_1 = \frac{g(\theta) \cdot f'(\theta)}{f(\theta) \cdot g'(\theta)} = \frac{m\theta}{(1-\theta\beta)[1-(1-\theta)^m]} \tag{2.7.1}
\]

\[
\mu'_2 = \frac{m\theta(1-\theta)}{(1-\theta\beta)^3[1-(1-\theta)^m]} - \frac{m^2\theta^2(1-\theta)^m}{(1-\theta\beta)^2[1-(1-\theta)^m]^2} \tag{2.7.2}
\]

for $\beta = \theta$, (2.7.2) reduces to

\[
\frac{m\theta(1-\theta)}{1-(1-\theta)^m} - \frac{m^2\theta^2(1-\theta)^m}{[1-(1-\theta)^m]^2}
\]

which is variance of decapitated binomial distribution as obtained by Stephan (1945), and for $\beta = 1$, it gives variance of decapitated negative binomial distribution as,

\[
\frac{m\theta}{(1-\theta)^2[1-(1-\theta)^m]} - \frac{m^2\theta^2(1-\theta)^{m-2}}{[1-(1-\theta)^m]^2}
\]

2.8: Negative Integer Moments of a Decapitated GNBD

The first negative moment of a decapitated generalized negative binomial distribution was obtained by Gupta (1974b). Since decapitated GNBD is also member of a MPSD with

\[
g(\theta) = \theta(1-\theta)^{\beta-1} \quad \text{and} \quad f(\theta) = (1-\theta)^{-m-1}
\]

Taking $r = -1$ in relation (1.3.3), we have,

\[
1 = \frac{\theta(1-\theta)}{1-\theta\beta} \cdot \frac{d\mu'_{-1}}{d\theta} + \frac{m\theta}{(1-\theta\beta)[1-(1-\theta)^m]} \cdot \mu'_{-1}
\]
Letting \( \mu'_{-1} = y \), and solving the differential equation

\[
\frac{dy}{d\theta} + \frac{m}{(1-\theta)(1-(1-\theta)^m)} \cdot y = \frac{1 - \theta \beta}{\theta(1-\theta)}
\] (2.8.1)

The first negative moment for (DGNBD) is given as

\[
E \left( \frac{1}{X} \right) = y = \frac{(1-\theta)^m}{1-(1-\theta)^m} \sum_{k=1}^{m} \frac{1}{k} \left[ \frac{1}{(1-\theta)^k} - 1 \right] \frac{\beta}{m} \frac{(1-\theta)^m}{1-(1-\theta)^m} \log(1-\theta)
\] (2.8.2)

2.9: Characterization of Generalized Negative Binomial Distribution

**Additive Property**

Consider two independent GNB variate \( X_1 \) and \( X_2 \) with same parameters, \( \theta \) and \( \beta \) but indices \( m \) and \( n \). The probability mass function of sum of \( X_1 \) and \( X_2 \) is then given by

\[
P(X_1 + X_2 = x) = \sum_{j=0}^{x} P[j; m, \theta, \beta] \cdot P[x - j; n, \theta, \beta]
\]

\[
= \sum_{j=0}^{x} \frac{m}{m + \beta j} \left[ \frac{m + \beta_{j}}{j} \right] \theta^j (1-\theta)^{m+\beta_{j}} \cdot \frac{n}{n + \beta(x - j)} \left( \frac{m + \beta(x - j)}{x - j} \right)^{x - j} (1-\theta)^{n+\beta(x-j)-(x-j)}
\]

\[
= \theta^x (1-\theta)^{m+n+\beta x-x} \sum_{j=0}^{x} \frac{m}{m + \beta j} \left[ \frac{m + \beta_{j}}{j} \right] \frac{n}{n + \beta(x - j)} \left( \frac{n + \beta(x - j)}{x - j} \right)^{x - j}
\] (2.9.1)

Since \( \sum_{x=0}^{\infty} \frac{m}{m + \beta j} \left[ \frac{m + \beta_{j}}{j} \right] \theta^j (1-\theta)^{m+\beta_{j}j} = 1 \)

rewrite the expression we have

\[
(1-\theta)^{-m} = \sum_{j=0}^{\infty} \left( \frac{m}{m + \beta j} \right)^{j} \theta^j (1-\theta)^{\beta_{j}j}
\]
Equating co-efficients of $\theta^x (1-\theta)^{(\beta-1)x}$ on both sides of the identity

$$(1-\theta)^{-m-n} = (1-\theta)^{-m} (1-\theta)^{-n},$$

we get

$$P(X_1 + X_2 = x) = \theta^x (1-\theta)^{m+n+\beta x-x} \cdot \frac{m+n}{m+n+\beta x} \binom{m+n+\beta x}{x}$$

It follows that the sum of $k$ independent GNB variate with indices $n_i, i = 1,2 \ldots k$ is also GNB variate with index $\sum_{i=1}^{k} n_i$.

**Distribution of Difference of Two GNB Variates**

The difference of two random variables is required in many statistical problems. The distribution of difference of two GNBD variates was the given by Consul (1989).

Let $X$ and $Y$ be two independent random variables having generalized negative binomial distribution $P[x; m, \theta, \beta]$ and $P[y; n, \theta, \beta]$ respectively, where $P(x; m, \theta, \beta)$ is as defined in (1.6.1).

Since the joint probability distribution of $X$ and $Y$ is given by product

$$P[x; m, \theta, \beta] P[y; n, \theta, \beta].$$

The probability distribution of the difference $D = Y - X$ is given by

$$P(Y - X = d) = \sum_{x=0}^{\infty} P[x; m, \theta, \beta] \cdot P[x + d; n, \theta, \beta]$$

$$= \sum_{x=0}^{\infty} \left( \frac{m}{m+\beta x} \right)^x (1-\theta)^{m+\beta x-x} \frac{n}{n+\beta(x+d)} \binom{n+\beta(x+d)}{x+d}$$

$$= (1-\theta)^{m+n} \sum_x (m,\beta)_x \cdot (n,\beta)_{d+x} \theta^{2x+d} (1-\theta)^{2\beta x+\beta d-2x-d}$$

$$= (1-\theta)^{m+n} \sum_x (m,\beta)_x \cdot (n,\beta)_{d+x} \left[ \theta(1-\theta)^{\beta-1} \right]^{2x-d} \quad (2.9.2)$$
where

\[
(m, \beta)_x = \left( \frac{m}{m + \beta x} \right) \left( \frac{m + \beta x}{x} \right)
\]

and 'd' takes all integral values from $-\infty$ to $\infty$. When 'd' is a negative integer such that $x + d < 0$, all these terms under summation in (2.9.2) for which $x-d$ is negative integer, will vanish, then

\[
P(D = d) = \begin{cases} 
(l - \theta)^{m+n} \sum_x (m, \beta)_x (n, \beta)_{d+x} [\theta(1-\theta)^{\beta-1}]^{2x+d} & \text{if } d \geq 0 \\
(l - \theta)^{m+n} \sum_y (n, \beta)_y (m, \beta)_{y-d} [\theta(1-\theta)^{\beta-1}]^{2y-d} & \text{if } d < 0
\end{cases}
\] (2.9.3)

The probability generating function of distribution of random variable $D = Y - X$ was obtained by Consul (1989), by the bivariate Lagrangian expansion of function

\[f(t_1, t_2) = (1 - \theta)^{m+n}(1 - \theta t_1)^{-n}(1 - \theta t_2)^{-m}\]

under the transformation

\[t_1 = u \cdot (1 - \theta)^{-\beta+1}(1 - \theta t_1)^{-\beta+1} \quad \text{and} \quad t_2 = v \cdot (1 - \theta)^{-\beta+1}(1 - \theta t_2)^{-\beta+1}\]

The recurrence relation for the cumulants of the probability distribution of the random variable $D = Y - X$ is given by

\[
(1 - \theta \beta)K_{r+1} = 2m\theta(1-\theta)\frac{\partial^2 K_r}{\partial \theta \partial m} - \theta(1-\theta)\frac{\partial K_r}{\partial \theta}
\]

\[
= \theta(1-\theta)\frac{\partial}{\partial \theta} \left[ 2m \frac{\partial K_r}{\partial m} - K_r \right]
\] (2.9.4)

\[r = 1, 2, 3, \ldots\]

and

\[K_1 = \frac{(m-n)\theta}{(1-\theta \beta)}\]
Using (2.9.4), the first four Cumulants of the probability distribution of the difference $D = Y - X$ are obtained.

\[
K_1 = \frac{(m-n)\theta}{(1-\theta\beta)} \tag{2.9.5}
\]

\[
K_2 = \frac{(m+n)\theta(1-\theta)}{(1-\theta\beta)^3} \tag{2.9.6}
\]

\[
K_3 = \frac{(m-n)\theta(1-\theta)}{(1-\theta\beta)^5} \left[1 - 6\theta + 18\theta^2 - 6\theta^3 + \theta\beta^2(6 - 6\theta + \theta^2)\right] \tag{2.9.7}
\]

\[
K_4 = \frac{(m+n)\theta(1-\theta)}{(1-\theta\beta)^7} \left[1 - 6\theta + 6\theta^2 + \theta\beta(8 - 18\theta + 8\theta^2) + \theta^2\beta^2(6 - 6\theta + \theta^2)\right] \tag{2.9.8}
\]

which provides the co-efficient of skewness $(\beta_1)$ and the co-efficient of Kurtosis $(\beta_2)$ as

\[
\beta_1 = \frac{(m-n)^2(1 - 2\theta + 2\theta\beta - \theta^2\beta^2)}{(m+n)^3 \theta(1-\theta)(1-\theta\beta)} \tag{2.9.9}
\]

\[
\beta_2 = 3 + \frac{1 - 6\theta + 6\theta^2 + \theta\beta(8 - 8\theta + 8\theta^2) + \theta^2\beta^2(6 - 6\theta + \theta^2)}{(m+n)\theta(1-\theta)(1-\theta\beta)} \tag{2.9.10}
\]

**Unimodality**

The property of unimodality plays an important role in the problem of density estimation and in the plausible inference model. Consul and Famoye (1986) proved the unimodality of GNBD by using two results viz. Steutel and Van Hem's (1979) Lemma and Lagrange's expansion, they showed that

i) when $m(1-\theta)^\theta > 1$, the mode is not at point $x = 0$, and for that case they obtained lower and upper bounds of the mode
ii) when \( m \theta (1-\theta)^{m-1} < 1 \), the distribution is non-increasing and so the mode is at \( x = 0 \)

iii) when \( m \theta (1-\theta)^{m-1} = 1 \), the mode is at dual points \( x = 0 \) and \( x = 1 \) as both have the same probability mass.

**Conditional Distribution**

i) The conditional distribution of \( X_1 = x \) given that \( X_1 + X_2 = Z \) where \( X_1 \) and \( X_2 \) are two independent GNB variates, with indices \( m_1 \) and \( m - m_1 \), is a generalized negative hyper-geometric distribution with parameters \( m, m_1, Z \) and \( \beta \) and having probability mass function as

\[
P[X_1 = x/X_1 + X_2 = Z] = \frac{b(x; m_1, \theta, \beta) \cdot b(Z - X; m - m_1, \theta, \beta)}{\sum_{x=0}^{Z} b(x; m_1, \theta, \beta) b(Z - X; m - m_1, \theta, \beta)}
\]

where \( b(x; n, p, q) = \frac{n}{n + qx} \binom{n + qx}{x} p^x (1 - p)^{n+qx-x} \)

which gives

\[
P[X_1 = x/X_1 + X_2 = Z] = \frac{m_1 (m + \beta x - 1)}{m_1 + \beta x - x} \cdot \frac{m - m_1}{m - m_1 + \beta (z-x) - (z-x)} \cdot \frac{m - m_1 + \beta (z-x) - 1}{z - x} \]

\[
\frac{m}{m + \beta z - z}
\]

(2.9.11)

ii) Jain and Consul (1971) proved that the two independent discrete random variates \( X_1 \) and \( X_2 \), where the sum \( X_1 + X_2 = Z \), \( Z = 0, 1, 2, \ldots \), then if the conditional distribution of \( X_1 = x \) given \( X_1 + X_2 = Z \) is generalized negative hyper-geometric distribution (2.9.11), then each of \( X_1 \) and \( X_2 \) is a GNB variate.

Let \( P(x_1) \) and \( q(x_2) \) denotes the distribution functions of \( X_1 \) and \( X_2 \) respectively then
\[
\frac{p(x)q(z-x)}{\sum_{x=0}^{\infty} p(x)q(z-x)} = \frac{\frac{m_1}{m_1 + \beta x - x} \left( \frac{m_1 + \beta x - 1}{x} \right)}{\frac{m-m_1}{m-m_1 + \beta (z-x)-(z-x)} \frac{m-m_1 + \beta (z-x)-1}{z-x}}
\]

therefore

\[
\frac{p(x)q(z-x)}{p(x-1)q(z-x+1)} = \frac{\frac{m_1}{m_1 + \beta x - x} \left( \frac{m_1 + \beta x - 1}{x} \right)}{\frac{m_1}{m_1 - \beta + \beta x - x + 1} \left( \frac{m_1 + \beta x - \beta - 1}{x - 1} \right)}
\]

\[
= \frac{\frac{m-m_1}{m-m_1 + \beta (z-x)-(z-x)} \frac{m-m_1 + \beta (z-x)-1}{z-x}}{\frac{m-m_1}{m-m_1 + \beta (z-x)-(z-x)+\beta-1} \frac{m-m_1 + \beta + \beta (z-x)-1}{z+x-1}}
\]

writing \(z - x = 1\), we have

\[
\frac{p(x)}{p(x-1)} = \frac{\theta \frac{m_1}{m_1 + \beta x - x} \left( \frac{m_1 + \beta x - 1}{x} \right)}{\frac{m_1}{m_1 + \beta x - \beta - x + 1} \left( \frac{m_1 + \beta x - \beta - 1}{x - 1} \right)}
\]

where \(\theta\) is a constant and independent of \(x\) then

\[
p(x) = \theta p(\theta) \cdot \frac{m_1}{m_1 + \beta x - x} \left( \frac{m_1 + \beta x - x}{x} \right)
\]

since \(p(x) > 0\), and \(\sum_{x=0}^{\infty} p(x) = 1\), \(\theta P(\theta) = 1\), therefore

\[
p(x) = \frac{m_1}{m_1 + \beta x - x} \left( \frac{m_1 + \beta x - x}{x} \right)
\]
ii) Das et.al (1995) showed that if \(X_i (i = 1, 2, \ldots, n)\) be \(n\) independent discrete random variables such that the conditional distribution of \(X_1 = x_1 \cup X_2 = x_2 \cup \ldots \cup X_{n-1} = x_{n-1}\) for given \(\sum_{i=1}^{n} X_i = m\), is a mixed quasi multivariate hyper geometric distribution, with probability function

\[
P_n(x) = \frac{\left(\frac{a + nt}{a}\right) \prod_{i=1}^{n} \left(\frac{a}{a_i + x_i t}\right) \prod_{i=1}^{n} \left(\frac{a_i + x_i t}{x_i}\right)}{\left(\frac{a + nt}{n}\right)}
\]

then each \(X_i (i = 1, 2, \ldots, n)\) follow a generalized negative binomial distribution GNBD with pmf,

\[
f_i(x_i) = \frac{a_i x_i^{a + tx_i - 1}}{a_i + x_i t} \left(1 - a_i \right)^{x_i - 1}.
\]

\[
2.10 \text{ Relationship With Other Distributions}
\]

**Generalized Logarithmic Series Distribution (GLSD) as a limiting case of zero Truncated GNBD**

Zero truncated GNBD with parameters \(m, \theta\) and \(\beta\) is given by probability mass function

\[
P(X = x) = P(x; m, \theta, \beta) = \frac{\frac{m}{m + \beta x} \left(\frac{m + \beta x}{x}\right)^x (1 - \theta)^{m + \beta x - x}}{1 - (1 - \theta)^m}
\]

\[
= \frac{\frac{m(m + \beta x - 1)}{x!(m + \beta x - x)} \cdot \theta^x (1 - \theta)^{m + \beta x - x}}{1 - \left\{1 - \left(\frac{m}{1}\right) \theta^1 + \left(\frac{m}{2}\right) \theta^2 - \ldots \ldots (-1)^m \theta^m\right\}}
\]

\[
= \frac{(m + \beta x - 1)! \theta^x (1 - \theta)^{m + \beta x - x}}{x!(m + \beta x - x)!} \cdot \frac{m}{m \theta - \frac{(m-1)}{2} \theta^2 + \ldots \ldots (-1)^m \theta^m}
\]
\[ (m + \beta x - 1)! \theta^x (1 - \theta)^{m+\beta x=x} \]
\[ \frac{1}{x!(m + \beta x - x)!} \left[ \frac{-m-1}{2} \theta^2 + \ldots \right] \]

Now taking limit \( m \to 0 \)

\[ P(X = x) = \frac{\Gamma(x\beta)}{x! \Gamma(x\beta - x + 1)} \cdot \theta^x (1 - \theta)^{\beta x-x} \cdot \frac{1}{[-\log(1-\theta)]} \]

\[ = \frac{\Gamma(x\beta)}{x! \Gamma(x\beta - x + 1)} \cdot a \left\{ [\theta(1-\theta)]^{b-1} \right\}^x \]

which is the pmf of generalized Logarithmic series distribution [1.9.1].

**Relation Between Moments of GLSD and Those of GNBD**

Generalized negative binomial distribution is given by its probability mass function as (1.6.1), it can also be put in the form

\[ P(x, m, \theta, \beta) = \frac{m \Gamma(m + \beta x)}{x! \Gamma(m + \beta x - x + 1)} \theta^x (1 - \theta)^{m+\beta x-x} \quad x = 0, 1, 2, \ldots \]

\[ 0 < \theta < 1, \quad |\theta\beta| < 1 \quad \text{and} \quad m > 0. \]

The \( r \)th moments of the GLSD is given by

\[ \mu'_r = \alpha \sum_{x=1}^{\infty} x^r \frac{\Gamma(x\beta)}{x! \Gamma(x\beta - x + 1)} \theta^x (1 - \theta)^{\beta x-x} \]

where \( \alpha = -[\log(1-\theta)]^{-1} \)

\[ = \theta \alpha \sum_{x=0}^{\infty} (1+x)^{r-1} \frac{\Gamma(x\beta + \beta)}{x! \Gamma(x\beta + \beta - x)} \theta^x (1 - \theta)^{x\beta-x+\beta-1} \]

\[ = \theta \alpha \sum_{j=0}^{r-1} \binom{r-1}{j} \sum_{x=0}^{\infty} x^j \frac{(x\beta + \beta - 1)}{\beta-1} \cdot \frac{\beta-1}{(x\beta + \beta - 1)} \cdot \frac{\Gamma(x\beta + \beta)}{x! \Gamma(x\beta + \beta - x)} \theta^x (1 - \theta)^{x\beta-x+\beta-1} \]
the expression within brackets is the sum of the \( j \) th and \((j + 1)\)th moments of the GNBD with \((\beta - 1)\) in place of \(m\). Denoting the \( r \) th moments of GNBD (1.6.1) by \( M'_r(m) \) we have \( r \) th moment of the GLSD

\[
\mu'_r = \theta \alpha \cdot \sum_{j=0}^{r-1} \binom{r-1}{j} M'_j(\beta - 1) + \frac{\beta}{(\beta - 1)} M'_{j+1}(\beta - 1), \quad r = 1, 2, \ldots \quad (2.10.1)
\]

In particular, we have

\[
\mu'_1 = \theta \alpha (1 + \lambda M'_1) \quad (2.10.2)
\]
\[
\mu'_2 = \theta \alpha (1 + \lambda + 1 \cdot M'_1 + \lambda M'_2) \quad (2.10.3)
\]
\[
\mu'_3 = \theta \alpha (1 + \frac{\beta}{\beta - 1} 2M'_1 + 2\lambda + 1 \cdot M'_2 + \lambda M'_3) \quad (2.10.4)
\]
\[
\mu'_4 = \theta \alpha (1 + \lambda + \frac{3\beta}{\beta - 1} M'_1 + 3\lambda + 1 \cdot M'_2 + 3\lambda + 1 \cdot M'_3 + M'_4) \quad (2.10.5)
\]

where \( \lambda = \frac{\beta}{(\beta - 1)} \) and \( M'_r = M'_r(\beta - 1) \)

Jain and Consul (1971) obtained the first four moments of GNBD, in which substituting \( m = \beta - 1 \), \( M'_r \) may be obtained. These relationships may be used for obtaining the moments of GLSD knowing the moments of the GNBD
The Poisson Type Approximation

For large \( m \) such that \( m \theta = \lambda_1 \) and \( \theta \beta = \lambda_2 \). The generalized negative binomial distribution gives a Poisson type approximation.

\[
P(X = x) = \frac{\lambda_1 (\lambda_1 + x \lambda_2)^{x-1} \exp[-(\lambda_1 + x \lambda_2)]}{x!}
\]

\( x = 0, 1, 2, \ldots \)

Mixture Distribution

Taking a mixture on \( \theta \) of GNBD with parameter \( m = \lambda - k + 1, \theta \) and \( \beta \), where \( \theta \) has a beta distribution with parameters 1 and \( k - 1 \), the generalized negative binomial-beta distribution, also known as generalized factorial distribution is obtained whose probability function is given by [Jain and Consul (1971)]

\[
P(X = x) = \frac{(k-1)!(\lambda + x\beta - x - 1)^{(k-1)}}{((\lambda + x\beta)^{(k)}} \cdot \frac{\lambda - k + 1}{\lambda - k + 1 + \beta x - x}
\]

\[ (2.10.6) \]

Where \( y^{(k)} = y(y - 1)(y - 2) \cdots (y - k + 1) \)

2.11: Correlation Co-efficient (\( \rho \))

It has been shown by Kojima and Kelleher (1962) that in case of boys and girls in a family, negative binomial distribution is appropriate as a distribution of the family size. Gupta (1976) obtained a general expression for the correlation coefficient \( \rho \) between number of boys and girls in a family considering that family size has Gupta’s (1974a) modified power series distribution.

In certain application involving discrete data, it is sometimes found that \( X = 0 \) is observed with a frequency significantly higher than predicted by assumed model, this has been a major motivating force behind the development of many distributions, that have been used as model in applied statistics.
Considering Inflated Modified Power Series Distribution (IMPSD) introduced in literature by Gupta, Gupta, and Tripathi (1995) as distribution for family size, which includes among others, the inflated generalized negative binomial and inflated generalized Poisson distribution, and hence negative binomial, binomial and Poisson distribution, we have obtained the general expression for the correlation coefficient for boys and girls in a family, and is then specialized to generalized negative binomial distribution and generalized Poisson distribution, and their particular cases. The expression thus obtained is more general in nature and a refinement of the results obtained by Gupta (1976).

**GENERAL EXPRESSION FOR CORRELATION COEFFICIENT ($\rho$)**

Let $X$ be a binomial random variable with parameters $N$ and $p$, and we are interested in correlation coefficient $\rho$ between $X$ and $Y = N - X$, where $N$ is family size. In case $N$ is constant $\rho = -1$, considering $N$ itself as a random variable. Let $N$ be a discrete random variable having Inflated Modified Power Series Distribution (IMPSD) given by

$$P(N = 0) = \phi + (1 - \phi) a(0)/f(0)$$

$$P(N = x) = (1 - \phi) a(x)(g(0))^x/f(\theta) \quad x = 1, 2, 3, \ldots \quad (2.11.1)$$

where $0 < \phi \leq 1$ and $f(\theta) = \sum_x a(x)((g(0))^x)$ and $g(\theta)$ are positive, finite and differentiable and coefficients $a(x)$ are nonnegative and free of $\theta$ when $\phi = 0$ it confirms to MPSD given by Gupta (1974a) with its pmf

$$P(Z = z) = a(z)(g(\theta))^z/f(\theta) \quad z = 0, 1, 2, 3, \ldots$$

$a(z) \geq 0$, $g(\theta)$ and $f(\theta)$ are positive finite and differentiable.

Let $E(Z)$ and $E(N)$ denotes the non central moments of MPSD and IMPSD respectively, then
\[ E(N) = (1 - \phi)E(Z) \]  
(2.11.2)

and \[ V(N) = (1 - \phi)\left[ \text{var}(Z) + \phi(E(Z))^2 \right] \]  
(2.11.3)

where \( E(Z) \) and \( V(Z) \) are given by

\[ E(Z) = \frac{f'(\theta)g(\theta)}{f(\theta)g'(\theta)} \]  
(2.11.4)

\[ V(Z) = \frac{g(\theta)}{g'(\theta)} \int E(Z) \, d\theta \]  
(2.11.5)

Roa et al (1973) showed that correlation coefficient \( \rho \) between \( X \) and \( Y = N - X \) is given by

\[ \rho = \frac{(pq)^{1/2}[V(N) - E(N)]}{[pV(N) + qE(N)]^{1/2}[qV(N) = pE(N)]^{1/2}} \]  
(2.11.6)

\( q = 1 - p \), when \( p = q = \frac{1}{2} \), (1.11.6) reduces to

\[ \rho = \frac{V(N) - E(N)}{V(N) + E(N)} \]  
(2.11.7)

using (2.11.2) and (2.11.3) in (2.11.6) we get,

\[ \rho = \frac{(pq)^{1/2}\left(1 - \phi\left[V(Z) + \phi(E(Z))^2\right] - (1 - \phi)E(Z)\right)}{p(1 - \phi)\left[V(Z) + \phi(E(Z))^2\right] + q(1 - \phi)\left[V(Z) + \phi(E(Z))^2\right] + p(1 - \phi)E(Z)}^{1/2} \]

\[ \rho = \frac{(pq)^{1/2}\left[V(Z) + \phi(E(Z))^2\right] - E(Z)}{p\left[V(Z) + \phi(E(Z))^2\right] + qE(Z)}^{1/2} \]  
(2.11.8)

Substituting the values from (2.11.4) and (2.11.5) in (2.11.8) we get,

\[ \rho = \frac{(pq)^{1/2}\left\{ g(\theta) \left[ E'(Z) - (1 - \phi(E(Z)) \right] f'(\theta)g(\theta) \right\}}{p\left[ g(\theta) \left[ E'(Z) + \phi(E(Z))^2 \right] + qE(Z) \right]^{1/2} \left\{ q\left[ g(\theta) \left[ E'(Z) + \phi(E(Z))^2 \right] + pE(Z) \right]^{1/2} \right\}^{1/2}} \]
\[
\rho = \frac{(pq)^{1/2} \{f(\theta)E'(Z) - [1 - \phi E(Z)]f'(\theta)\}}{\{pf(\theta)E'(Z) + [p\phi E(Z) + q]f'(\theta)\}^{1/2}\{qf(\theta)E'(Z) + [q\phi(E(Z) + p)f'(\theta)\}^{1/2}} \tag{2.11.9}
\]

When \(\phi = 0\), one immediately gets the results obtained by Gupta(1976), when \(\phi = 0\) and \(p = q = 1/2\) we get,

\[
\rho = \frac{f(\theta)E'(Z) - f'(\theta)}{f(\theta)E'(Z) + f'(\theta)} \tag{2.11.10}
\]

Letting \(\lambda = (1-\phi)(1-a(0)/f(\theta))\), the model (2.11.1) can be written as

\[
P(N=0) = 1 - \lambda
\]

\[
P(N = x) = \frac{\lambda a(x)(g(\theta))^{x}}{(1-a(0)/f(\theta))f(\theta)} \quad x = 1,2,3,... \tag{2.11.11}
\]

Let \(x_1, x_2, x_3, ..., x_n\) be a random sample from (2.11.1) and where \(n = \sum_i n_i\), the number of observations which is equal to \(i\). Gupta, Gupta and Tripathi (1995) showed that maximum likelihood estimator of \(\phi\) is given by

\[
\hat{\phi} = 1 - \frac{\hat{\lambda}f(\hat{\theta})}{f(\hat{\theta}) - a(0)} \tag{2.11.12}
\]
### SOME SPECIAL DISTRIBUTIONS OF N

<table>
<thead>
<tr>
<th>Distribution of N</th>
<th>Probability function</th>
<th>$g(\theta)$</th>
<th>$f(\theta)$</th>
<th>$\rho$</th>
</tr>
</thead>
</table>
| Generalized Negative Binomial Distribution | $\frac{m}{m+\beta x} \left( \frac{m+\beta x}{x} \right) \theta^x (1-\theta)^{m+\beta x-2}$ | $\theta (1-\theta)^{\beta-1}$ | $(1-\theta)^{-m}$ | \[
\frac{(pq)^{1/2} \left[ \theta(2\beta-1) - \theta^3 \beta^2 + \phi(m\theta)(1-\theta\beta) \right]}{\left[ 1 - \theta(p + 2\beta q) + q \theta^2 \beta^2 + p \phi(m\theta)(1-\theta\beta) \right]^{1/2} \left[ 1 - \theta(q + 2\beta q) + p \theta^2 \beta^2 + q \phi(m\theta)(1-\theta\beta) \right]^{1/2}}
\]
when $p = q = \frac{1}{2}$ |
| Special Cases | | | | |
| i) Negative binomial Distribution $\beta = 1$ | | | | \[
\frac{(pq)^{1/2} \left[ \theta(1-\theta) + \phi(m\theta)(1-\theta) \right]}{\left[ 1 - \theta(p + 2\beta q) + q \theta^2 \beta^2 + p \phi(m\theta)(1-\theta\beta) \right]^{1/2} \left[ 1 - \theta(q + 2\beta q) + p \theta^2 \beta^2 + q \phi(m\theta)(1-\theta\beta) \right]^{1/2}}
\]
| ii) Binomial distribution $\beta = 0$ | | | | \[
\frac{(pq)^{1/2} \left[ -\theta + \phi(m\theta) \right]}{\left[ 1 - p \theta + p \phi(m\theta) \right]^{1/2} \left[ 1 - q \theta + q \phi(m\theta) \right]^{1/2}}
\]

Continued ->
| iii) Generalized Geometric Series Distribution | \( \frac{1}{1 + \beta x} \left( \frac{1 + \beta x}{x} \right)^{\theta} (1 - \theta)^{\beta x - x} \) | \( x = 0, 1, 2, 3, \ldots \) |
|-------------------------------------------------|-------------------------------------------------|
| iv) Geometric Series Distribution m=1,\( \beta = 1 \) | \( \frac{(pq)^{1/2} \left[ \theta(2\beta - 1) - \theta^2\beta^2 + \phi(\theta)(1 - \theta\beta) \right]}{\left[ 1 - \theta(p + 2\beta q) + q\theta^3\beta^2 + p\phi(\theta)(1 - \theta\beta) \right]^{1/2} \left[ 1 - \theta(q + 2\beta q) + p\theta^3\beta^2 + q\phi(\theta)(1 - \theta\beta) \right]^{1/2}} \) |
| When \( p = q = 1/2 \) | \( \frac{\theta(2\beta - 1) + \theta^2\beta^2 + \phi(\theta)(1 - \theta\beta)}{2 - \theta(2\beta + 1) + \theta^2\beta^2 + \phi(\theta)(1 - \theta\beta)} \) |
| | \( \frac{(pq)^{1/2} \left[ \theta - \theta^2 + \phi(\theta)(1 - \theta) \right]}{\left[ 1 - \theta(p + 2q) + q\theta^2 + p\phi(\theta)(1 - \theta) \right]^{1/2} \left[ 1 - \theta(q + 2q) + p\theta^2 + q\phi(\theta)(1 - \theta) \right]^{1/2}} \) |
| Generalized Poisson distribution Consul and Jain (1973) | \( \frac{(1 + \beta x)^{x-1} (\theta e^{-\theta})^x}{x!e^\theta} \) | \( x = 0, 1, 2, 3, \ldots \) |
| (1) Poisson Distribution \( \beta = 0 \) | \( \theta e^{-\theta} \) | \( e^\theta \) |
| | \( \frac{(pq)^{1/2} \left[ \theta\beta(2 - \theta\beta) + \phi(\theta)(1 - \theta\beta) \right]}{\left[ 1 - 2\theta\beta + q\theta^3\beta^2 + p\phi(\theta)(1 - \theta\beta) \right]^{1/2} \left[ 1 - 2p\theta\beta + p\theta^3\beta^2 + q\phi(\theta)(1 - \theta\beta) \right]^{1/2}} \) |
| | \( \frac{(pq)^{1/2} \left[ \phi \theta \right]}{\left[ 1 + p\phi \theta \right]^{1/2} \left[ 1 + q\phi \theta \right]^{1/2}} \) |
2.12: Size Biased Generalized Negative Binomial Distribution (SBGNBD)

In this section a size biased generalized negative binomial distribution (SBGNBD), a particular case of the weighted generalized negative binomial, taking weights as the variate value has been defined and moments and recurrence relation between the moments about the origin of SBGNBD have also been obtained.

The probability mass function of the GNBD is given by

\[
P(X = x) = \frac{m}{m + \beta x} \left( \frac{m + \beta x}{x} \right)^x (1 - \theta)^{m + \beta x - x}
\]

\[0 < \theta < 1, \quad |\theta \beta| < 1 \quad \text{and} \quad m > 0, \quad x = 0, 1, 2, \ldots\]

we have from (2.3.2)

\[
\sum_{x=0}^{\infty} x \cdot P(X = x) = \frac{m \theta}{(1 - \theta \beta)}
\]

\[
= \frac{1 - \theta \beta}{m \theta} \sum_{x=0}^{\infty} x \cdot P(X = x) = 1
\]

thus, \( P_1(X = x) = x \cdot P(X = x) \cdot \frac{(1 - \theta \beta)}{m \theta} \)

represents a probability distribution. This gives probability mass function of size-biased generalized negative binomial distribution (SBGNBD) as

\[
P_1(X = x) = (1 - \theta \beta) \cdot \frac{m + \beta x - 1}{x - 1} \theta^{x-1} (1 - \theta)^{m + \beta x - x}
\]

\[0 < \theta < 1, \quad |\theta \beta| < 1, \quad m > 0, \quad x = 1, 2, 3, \ldots
\]

\[= 0 \quad \text{for} \quad x \geq t \quad \text{such that} \quad m + \beta t < 0.\]
Moments of SBGNBD

The $r^{th}$ moment $M'_r$ about origin of the size-biased GNBD (2.12.1) can be defined as

$$M'_r = \sum_{x=1}^{\infty} x^r \cdot P(X = x), \quad r = 1, 2, 3, \ldots$$

obviously $M'_0 = 1$, and for $r \geq 1$

$$M'_r = \frac{1 - \theta \beta}{m \theta} \sum_{x=0}^{\infty} x^{r+1} P(X = x)$$

$$M'_r = \frac{1 - \theta \beta}{m \theta} \mu'_{r+1} \quad (2.12.3)$$

where $\mu'_{r+1}$ is the $(r + 1)$th moment about origin of the GNBD (1.6.1). For $r = 1$, we have from (2.12.3)

$$M'_1 = \frac{1 - \theta \beta}{m \theta} \mu'_2$$

Using (2.3.4), the first moment about origin of the size-biased GNBD is obtained as

$$M'_1 = \frac{(1 - \theta \beta)}{m \theta} \left[ \frac{(m \theta)^2}{(1 - \theta \beta)^2} + \frac{m \theta (1 - \theta)}{(1 - \theta \beta)^3} \right]$$

$$= \frac{m \theta}{1 - \theta \beta} + \frac{1 - \theta}{(1 - \theta \beta)^2} \quad (2.12.4)$$

which is mean of size biased GNBD (2.12.1).

Similarly for $r = 2$

$$M'_2 = \frac{1 - \theta \beta}{m \theta} \mu'_3$$

make use of relation (2.3.5) we have
Thus variance $M_2$ of SBGNBD is obtained as

$$M_2 = \frac{(m\theta)^2}{(1-\theta\beta)^2} + \frac{3m\theta(1-\theta)}{(1-\theta\beta)^3} + \frac{(1-\theta)}{(1-\theta\beta)^4} \left[ 1 - 2\theta - \theta\beta(2-\theta) \right]$$

which on simplification gives

$$M_2 = \frac{m\theta(1-\theta)}{(1-\theta\beta)^3} + \frac{\theta(1-\theta)}{(1-\theta\beta)^4} \left[ \beta(2-\theta) - 1 \right]$$

and for $r = 3$ using relation (2.3.6), we have

$$M_3 = \frac{(m\theta)^3}{(1-\theta\beta)^3} + \frac{6(m\theta)^2(1-\theta)}{(1-\theta\beta)^4} + \frac{m\theta(1-\theta)}{(1-\theta\beta)^5} \left[ 7 - 11\theta - 4\theta\beta(2-\theta) \right]$$

$$+ \frac{1-\theta}{(1-\theta\beta)^6} \left[ 1 - 6\theta + 6\theta^2 + 29\beta(4 - 9\theta + 4\theta^2) + \theta^2\beta^2(6 - 6\theta + \theta^2) \right]$$

The higher moments can be obtained similarly using (2.12.3).

**Recurrence Relationship for moments about origin of SBGNBD**

We have $r$ th moment of SBGNGD about origin as

$$M'_r = E(X^r) = \sum_{x=1}^{\infty} x^r (1-\theta\beta) \binom{m + \beta x - 1}{x-1} \theta^{x-1}(1-\theta)^{m+\beta x-x}$$
\[
\frac{\partial M'_r}{\partial \theta} = \sum_{x=1}^{\infty} x\left(\frac{m + \beta x - 1}{x - 1}\right) \left(\frac{\partial}{\partial \theta} \left(1 - \theta \beta\right)^{x - 1} (1 - \theta)^{m + \beta x - x} \right)
\]

\[
= \sum_{x=1}^{\infty} x\left(\frac{m + \beta x - 1}{x - 1}\right) \left(1 - \theta \beta\right)^{x - 1} (1 - \theta)^{m + \beta x - x - 1} (-1) + (1 - \theta)^{m + \beta x - x}
\]

\[
\left(1 - \theta \beta\right)(x - 1)\theta^{x-2} + \theta^{x-1} (-\beta)\right]\]

\[
= \sum_{x=1}^{\infty} x\left(\frac{m + \beta x - 1}{x - 1}\right) \left[-(1 - \theta \beta)^{x - 1} (1 - \theta)^{m + \beta x - x} \left(\frac{m + \beta x - x}{1 - \theta}\right) + (1 - \theta \beta)^{x - 1} (1 - \theta)^{m + \beta x - x} \left(1 - \theta \beta\right)^{\frac{\beta}{1 - \theta \beta}}\right]
\]

\[
= \sum_{x=1}^{\infty} x\left(\frac{m + \beta x - 1}{x - 1}\right) \left(1 - \theta \beta\right)^{x - 1} (1 - \theta)^{m + \beta x - x} \left[-\frac{m + \beta x - x}{1 - \theta} + \frac{x - 1}{1 - \theta} - \frac{\beta}{1 - \theta \beta}\right]
\]

\[
\frac{\partial M'_r}{\partial \theta} = \frac{(1 - \theta \beta)^{x - 1} (1 - \theta)^{m + \beta x - x - 1} (-1) + (1 - \theta)^{m + \beta x - x}}{(1 - \theta)(1 - \theta \beta)}
\]

\[
M'_{r+1} = \frac{\theta(1 - \theta)}{1 - \theta \beta} \frac{\partial M'_r}{\partial \theta} \frac{(m^2 \beta + \theta - m^2 - 1)}{(1 - \theta \beta)^2} M'_r
\]

\[
M'_{r+1} = \frac{\theta(1 - \theta)}{1 - \theta \beta} \left[\left(\frac{(1 - \theta)}{(1 - \theta \beta)^2} + \frac{m^2 \beta + \theta - m^2 - 1}{(1 - \theta \beta)^2}ight)\right]
\]

(2.12.9)

If we put \( r = 0 \), we have

\[
M'_1 = \left[\frac{(1 - \theta)}{(1 - \theta \beta)^2} + \frac{m^2 \beta + \theta - m^2 - 1}{(1 - \theta \beta)^2}\right]
\]

\[
= \frac{m^2 \beta + \theta - m^2 - 1}{(1 - \theta \beta)^2}
\]

The higher moment of SBGNBD (2.12.1) about origin can be obtained from the relation (2.12.9) for different value of \( r \).
2.13: A General Class of Generalized Negative Binomial Distribution

In this section a general class of GNBD’s which represent a wider class and generates various GNBD’s as its particular cases have been presented. Some of the characterization properties have been discussed and a general expression for moments of this class of NBD’s has been obtained through which the moments of various particular GNBD’s can easily be obtained.

For a discrete random variable $X$, a class of GNBD’s with three parameters $m, \theta$, and $\beta$ and a given integer $j$, is defined by probability function.

$$f_j(x,m,\theta,\beta) = \frac{1}{B_j(m,0,\beta)} \frac{\Gamma(m+\beta x)}{\Gamma(m+\beta x-x+j)} \frac{\theta^x}{x!} (1-\theta)^{m+\beta x-x+j-1}$$

where

$$B_j(m,0,\beta) = \sum_{x=0}^{\infty} \frac{\Gamma(m+\beta x)}{\Gamma(m+\beta x-x+j)} \frac{\theta^x}{x!} (1-\theta)^{m+\beta x-x+j-1}$$

Choosing different value of $j$, different GNBD’s can be obtained from (2.13.1) if the values of $B$’s and known.

**Recurrence Relation for $B_j(m,\theta,\beta)$**

We can write (2.13.2) as

$$B_j(m,\theta,\beta) = \sum_{x=0}^{\infty} (m+\beta x - 1)^{(x-j)} \frac{\theta^x}{x!} (1-\theta)^{m+\beta x-x+j-1}$$

where we denote $m^{(n)} = m(m-1)(m-2) \ldots (m-n+1)$

$$= \frac{m+j}{1-\theta} \sum_{x=0}^{\infty} (m+\beta x - 1)^{(x-j-1)} \frac{\theta^x}{x!} (1-\theta)^{m+\beta x-x+j}$$

$$+ \frac{\theta(\beta-1)}{1-\theta} \sum_{x=1}^{\infty} (m+\beta x - 1)^{(x-j-1)} \frac{\theta^{x-1}}{(x-1)!} (1-\theta)^{m+\beta x-x+j}$$
which gives the recurrence relation as

$$B_j(m, \beta) = \frac{1}{(1-\theta)} \left[ (m+j)B_{j-1}(m, \theta, \beta) + \theta(\beta - 1)B_j(m+\beta, \theta, \beta) \right]$$  \hspace{0.5cm} (2.13.3)

The repeated use of which may give $B_j(m, \theta, \beta)$ if $B_{j+1}(m, \theta, \beta)$ is known.

The equation (2.13.3) can also be written as

$$B_{j+1}(m, \theta, \beta) = \frac{1}{m+j} \left[ (1-\theta)B_j(m, \theta, \beta) - \theta(\beta - 1)B_j(m+\beta, \theta, \beta) \right]$$  \hspace{0.5cm} (2.13.4)

which may be used to find $B_{j+1}(m, \theta, \beta)$ provided $B_j(m, \theta, \beta)$ is known.

**Factorial Moments of the Class of Generalized Negative Binomial Distribution’s (GNBD’s)**

The $r^{th}$ factorial moments of the class of GNBD’s (2.13.1) is given by

$$\mu_r' = \frac{1}{B_j(m, \theta, \beta)} \sum_{x=0}^{w} \frac{\Gamma(m + \beta x)}{\Gamma(m + \beta x - x + j + 1)} \frac{\theta^x}{(1-\theta)^{m+\beta x + j+1}}$$  \hspace{0.5cm} (2.13.5)

$$= \frac{1}{B_j(m, \theta, \beta)} \sum_{x=0}^{w} \frac{\Gamma(m + i\beta + \beta x)}{\Gamma(m + i\beta + \beta x - i + j + 1)} \frac{\theta^{x+i}}{x! (1-\theta)^{m+i\beta + \beta x - i + j+1}}$$  \hspace{0.5cm} (2.13.5)

$$= \left( \frac{\theta}{1-\theta} \right)^{i} \frac{B_{j-1}(m+i\beta, \theta, \beta)}{B_j(m, \theta, \beta)} \hspace{0.5cm} i=1,2,3,\cdots$$

**Some particular GNBD’s**

From Jain and Consul’s (1971) GNBD, it is clear that $B_1(m, \theta, \beta) = \frac{1}{m}$ and we may call this GNBD as GNBD I. Using this and making suitable use of (2.13.3) and (2.13.4) we find the expressions for some B’s as summarized in the following table which provide some particular GNBD’s.
Expression for $1/B_j(m,\theta,\beta)$ in the class of GNBD’s for some particular values of $j$.

<table>
<thead>
<tr>
<th>GNBD</th>
<th>$J$</th>
<th>$1/B_j(m,\theta,\beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>0</td>
<td>$(1 - \theta\beta)$</td>
</tr>
<tr>
<td>III</td>
<td>-1</td>
<td>$(1 - \theta)(1 - \theta\beta)^3/(m - 1)((1 - \theta\beta) - \theta(\theta + \beta - 2\theta\beta))$</td>
</tr>
<tr>
<td>IV</td>
<td>2</td>
<td>$m(m + 1)(m + \beta)/(m + \beta - \theta\beta(m + 1))$</td>
</tr>
</tbody>
</table>
| V    | 3  | $m(m + 1)(m + 2)(m + \beta)(m + \beta + 1)(m + 2\beta)/(m + \beta + 1)
- m(m + 1)\theta(\beta - 1)(m + 2\beta - \theta\beta(m + 1))$ |

The expression for $B$’s for $j$ other than those for which $B$’s have been obtained are coming in lengthy forms and hence they are not given here. However, if desired, they can be obtained in the similar way.

Some properties of the class of GNBD’s

Jain and Consul (1971) proved some properties of the GNBD I. A question arises whether there exists a wider class of distributions which possess these properties. The class of GNBD’s is such a class which possesses these properties. We are proving here some of the following properties.

a) Let $X_k$ have the distribution $f_{jk}(x_k; m_k, \theta, \beta)$ ($k = 1,2$) and $X_1, X_2$ be independent then the sum $X_1 + X_2$ has the distribution $f_{j_1 + j_2}(m_1 + m_2, \theta, \beta)$

Proof: It can be easily seen that the equation (2.13.1) can also be written as

$$f_j(x; m, \theta, \beta) = \frac{\theta^j(1-\theta)^{m+\beta x-x+j-1}}{B_j(m,\theta,\beta)(m+\beta x)^{ij}}$$ (2.13.6)

we have
\[ P(X_1 + X_2 = x) = \sum_{r=0}^{\infty} f_{h_r}(r, m_1, \alpha, \beta) f_{h}(x-r, m_2, \alpha, \beta) \]

\[ = \theta^x (1-\theta)^{\sum_{k=1}^{r} (j_k-1) + \beta x - x} \sum_{r=0}^{\infty} \left( \prod_{k=1}^{r} \frac{1}{B_{h_k}(m_{1_k}, \theta, \beta)(m_{2_k} + \beta)} \right) \frac{(m_1 + \beta r)^r}{r!} \left( \frac{m_2 + x - r \beta - 1}{x - r} \right) \]

Equating the coefficient of \( \theta^x (1-\theta)^{\sum_{k=1}^{r} (j_k-1)} \) on both sides of the identity

\[ (1-\theta)^{-\sum_{k=1}^{r} (j_k-1)} = (1-\theta)^{-m_1 - m_2} (1-\theta)^{-m_2 - j_2 + 1} \]

and writing (2.13.1) as

\[ (1-\theta)^{-m_1 - m_2} = \sum_{r=0}^{\infty} \frac{\binom{m_1 + j_1 - 1 + \beta r}{r} \theta^r (1-\theta)^{\beta r - r}}{B_{h}(m_1, \theta, \beta)(m_1 + \beta r)^{h}} \]

we have

\[ P(X_1 + X_2 = x) = \theta^x (1-\theta)^{\sum_{k=1}^{r} (j_k-1) + \beta x - x} \frac{\sum_{k=1}^{r} B_{h_k}(m_{1_k}, \theta, \beta)(m_{2_k} + \beta) x^{h_k}}{B_{h_k}(m_{1_k} + m_{2_k}, \theta, \beta)(m_{1_k} + m_{2_k} + \beta x)^{h_k}} \]

which is the GNBD with parameters \( m_1 + m_2, \theta, \beta \) and the index as \( j_1 + j_2 \). This can be easily generalized for any finite number of variates.

b) Let \( X_k \) have the distribution \( f_{h_k}(x_k, m_k, \theta, \beta), (k = 1, 2) \) and \( X_1, X_2 \) be independent. The conditional distribution of \( X_1 = x \) given that \( X_1 + X_2 = z \) is a \( m_1 + m_2, m_1, z \) and \( \beta \) and indices \( j_1 \) and \( j_2 \).

Proof: We have

\[ P(X_1 = x/X_1 + X_2 = z) = \frac{f_{h}(x, m_1, \theta, \beta) f_{h}(z-x, m_2, \theta, \beta)}{\sum_{x=0}^{z} f_{h}(x, m_1, \theta, \beta) f_{h}(z-x, m_2, \theta, \beta)} \]

using (2.13.5) this can be written as
\[
\frac{\binom{m_1 + j_1 + \beta x - 1}{x} \binom{m_1 + j_2 + (z-x)\beta - 1}{z-x}}{(m_1 + z\beta)^{j_1} (m_2 + z-x \beta)^{j_2}} C_{j_1j_2}(m_1 + m_2, \theta, \beta) \left( \sum m_k + \sum (j_k - 1) + \beta x - x \right)
\]

(2.13.11)

where

\[C_{j_1j_2}(m_1 + m_2, \theta, \beta) = \frac{B_{j_1}(m_1, \theta, \beta)B_{j_2}(m_2, \theta, \beta)}{B_{j_1j_2}(m_1 + m_2, \theta, \beta)}\]

c) As \(m \to 0\), the zero-truncated class of GNBD's tends to a class of Jain and Gupta's (1973) type generalized logarithmic series distribution (GLSD).

**Proof:** We have from (2.13.2) as \(m \to 0\)

\[
\sum_{x=1}^{\infty} \frac{\Gamma(\beta x)}{\Gamma(\beta x - x + j)} \frac{\theta^x}{x} (1-\theta)^{\beta x - x + j - 1}
\]

\[
= \lim_{m \to 0} \left[ B_j(m, \theta, \beta) - \frac{(1-\theta)^{m+j-1}}{m(m+1)...(m+j-1)} \right] \text{ if } j > 0
\]

\[
= \lim_{m \to 0} \left[ B_j(m, \theta, \beta) - (1-\theta)^{m-1} \right] \text{ if } j = 0
\]

\[
= \lim_{m \to 0} \left[ B_j(m, \theta, \beta) - (m+j)(m+j-1)...(m-1)(1-\theta)^{m+j-1} \right] \text{ if } j < 0
\]

denoting the right hand side by \(M_j(\theta, \beta)\) we get a class of generalized logarithmic series distribution (GLSD) as

\[
p_j(x, \theta, \beta) = \frac{1}{M_j(\theta, \beta)} \frac{\Gamma(\beta x)}{x\Gamma(\beta x - x + j)} \frac{\theta^x}{x} (1-\theta)^{\beta x - x + j - 1} \quad x=1,2,...
\]