CHAPTER – IV

Generalized Geometric Series Distribution
4.1: Introduction

Mishra (1982) using lattice path analysis obtained the generalized geometric series distribution (GGSD). This distribution has two parameters $\theta$ and $\beta$, and reduces to the geometric series distribution with parameter $\theta$, if $\beta = 1$. Needless to say that the GGSD is a member of Lagrangian distribution and can be obtained by taking $g(t) = (1 - \theta + \theta t)^\beta$ and $f(t) = (1 - \theta + \theta t)$. It is also obtained as Lagrangian expansion of function $f(z) = (1 - \theta) (1 - \theta)^{-1}$ under the transformation $z = u g(z)$ where $g(z) = (1 - \theta)^{\beta - 1} (1 - \theta z)^{-\beta + 1}$ in power of $u$.

The various aspects, interesting properties and fields of application of Generalized Geometric Series Distribution (GGSD) have been studied by Mishra (1982), Singh (1989) Mishra and Singh (1992), Hassan (1995) and Mishra and Singh (2000). They found this distribution to provide much closer fits to all those observed distribution where the geometric distribution and the various compound geometric distributions were fitted earlier. A brief account of various properties and methods of estimation for GGSD is presented in this chapter.

We have presented a simple and quick method for estimating the parameters of $\theta$ and $\beta$ of the GGSD. This method requires estimating only one parameter $\theta$ which is based on the mean of the observed distribution. The parameter $\beta$ is obtained just by counting the number of non-zero frequency classes. To know how efficient the estimator of $\theta$ so obtained is, the variance of
the estimator of $\theta$ has been obtained and its values computed for certain selected values of $\theta$ and the number of non-zero frequency classes. Using this method a study on the goodness of fit of the distribution has also been done. The new estimator gives the better fits. [Hassan, Mishra and Jan (2002)].

No study on Bayesian estimation of the GGSD and truncated GGSD seems to appear in statistical literature. An attempt has also been made to find the Bayes estimator of $\theta$, for GGSD and for truncated GGSD, assuming other parameter $\beta$ known. In addition we have also defined the size-biased generalized geometric series distribution and studied its structural properties.

4.2: Moments of Generalized Geometric Series Distribution (GGSD)

The moments of GGSD can be obtained by simply putting $m = 1$ in the expression for moments of GNBD. Thus first four moments about origin of GGSD are as follows

$$
\mu_1' = \frac{\theta}{1 - \theta \beta} \quad (4.2.1)
$$

$$
\mu_2' = \frac{\theta(1 - \theta)}{(1 - \theta \beta)^3} + \frac{\theta^3}{(1 - \theta \beta)^2} \quad (4.2.2)
$$

$$
\mu_3' = \frac{\theta^3}{(1 - \theta \beta)^3} + \frac{3\theta^2(1 - \theta)}{(1 - \theta \beta)^4} + \frac{\theta(1 - \theta)}{(1 - \theta \beta)^5} \left[1 - 2\theta + \theta \beta (2 - \theta)\right] \quad (4.2.3)
$$

$$
\mu_4' = \frac{\theta^4}{(1 - \theta \beta)^4} + \frac{6\theta^3(1 - \theta)}{(1 - \theta \beta)^5} + \frac{\theta^2(1 - \theta)\left[7 - 11\theta + 4\theta \beta (2 - \theta)\right]}{(1 - \theta \beta)^6} \\
+ \frac{\theta(1 - \theta)\left[1 - 6\theta + 6\theta^2 + 2\theta \beta (4 - 9\theta + 4\theta^2) + \theta^2 \beta^2 (6 - 6\theta + \theta^2)\right]}{(1 - \theta \beta)^7} \quad (4.2.4)
$$

Jain and Consul also obtained the central moments of GNBD. From those moments, the central moments of the GGSD are obtained as
The moments about origin of the zero truncated GGSD \((1.7.2)\) may be obtained by just dividing the corresponding moments of GGSD \((1.7.1)\) by \(\theta\), we get

\[
\mu'_1 = \frac{1}{(1-\theta\beta)} \quad (4.2.8)
\]

\[
\mu'_2 = \frac{\theta}{(1-\theta\beta)^2} + \frac{(1-\theta)}{(1-\theta\beta)^3} \quad (4.2.9)
\]

\[
\mu'_3 = \frac{3\theta(1-\theta)}{(1-\theta\beta)^4} + \frac{\theta^2}{(1-\theta\beta)^3} + \frac{(1-\theta)}{(1-\theta\beta)^5} \left[1 - 2\theta + \theta\beta(2-\theta)\right] \quad (4.2.10)
\]

The central moments of the zero truncated GGSD may be obtained by multiplying the corresponding central moments of GGSD \((1.7.1)\) by \(\beta\). We get

\[
\mu_2 = \frac{\theta\beta(1-\theta)}{(1-\theta\beta)^3} \quad (4.2.11)
\]

\[
\mu_3 = \frac{\theta\beta(1-\theta)}{(1-\theta\beta)^5} \left[1 - 2\theta + \theta\beta(2-\theta)\right] \quad (4.2.12)
\]

The first negative integer moments (NIM) about \(-k\) \((k > 0)\) for GGSD is given by

\[
M(1,k) = E \left( \frac{1}{X + k} \right) = \frac{(1-\theta)^{1+k-\beta k}}{\theta^k} \int_0^\theta \beta^{k-1} (1-\theta)^{\beta k-k-2} d\theta \quad (4.2.13)
\]

Using relation (2.8.2) and taking \(m = 1\), and \(k = 0\) we can get the expression for negative integer moments for truncated GGSD as
\[ E \left( \frac{1}{X} \right) = -\beta - \frac{1-\theta}{\theta} \log(1-\theta). \] (4.2.14)

### 4.3: Characterization of Geometric Series Distribution

If there are two independent identically distributed geometric variates then the conditional distribution of one given the sum of the two variates is a discrete uniform distribution. Mishra and Singh (1992) found that this property holds true in case of GGSD, also conditional distribution gives rise to a uniform type distribution which is more general in nature and of which classical uniform distribution is a particular case. They termed the resulting distribution as quasi uniform distribution (QUD). If \( X \) and \( Y \) are two independent random variables each following GGSD \((1.7.1)\) with parameters \( \theta \) and \( \beta \), the conditional distribution of \( X \) given \( X + Y = n \) \((n \text{ non negative integer})\) is given by

\[
P(X = x/X + Y = n) = \frac{P(X = x, Y = n - x)}{\sum_{r=0}^{n} P(X = r, Y = n - r)} = \frac{\left(1 + \beta x\right)\left(1 + (n - x)\beta\right)}{\left(1 + x\beta\right)\left(1 + (n - x)\beta\sum_{r=0}^{n} \frac{1}{1 + x\beta} \left(\frac{1 + x\beta}{r}\right)\frac{1}{1 + (n - r)\beta} \left(\frac{1 + (n - r)\beta}{r - 1}\right)\right)} = \frac{\left(1 + \beta x\right)\left(1 + (n - x)\beta\right)}{\left(1 + x\beta\right)\left(1 + (n - x)\beta\right)} \cdot \frac{1}{\sum_{r=0}^{n} A_r(1, \beta)A_{n-r}(1, \beta)}
\]

(4.3.1)

where 

\[ A_k(\alpha, \beta) = \frac{\alpha}{\alpha + k\beta} \binom{\alpha + k\beta}{k} \]

The sum \( \sum_{r=0}^{n} A_r(1, \beta)A_{n-r}(1, \beta) \) can be obtained by using generalization of the Vandermonde convolution given by Gould (1958) and also in Riordon (1968) which is as

\[ \sum_{r=0}^{n} A_r(\alpha, \beta)A_{n-r}(\delta, \beta) = A_n(\alpha + \delta, \beta) \]
Selecting $\alpha = \delta = 1$, one gets the sum in (4.3.1) as $A_n(2, \beta)$ which gives conditional probability as

$$P(X = x/X + Y = n) = f(x; n, \beta) = \frac{\left(1 + x\beta\right)^{x}}{(1 + x\beta)(1 + (n - x)\beta)} \cdot \frac{2 + n\beta}{(2 + n\beta)^{n}}$$

Its sum over $x = 0, 1, 2 \ldots n$ is unity and hence (4.3.2) represent a probability distribution. It can be easily seen that at $\beta = 1$, this distribution reduces to classical uniform distribution.

4.4: Estimation of Generalized Geometric Series Distribution by Method of Moments

Suppose a random sample of size $n$ is taken from GGSD model (1.7.1). Let the observed frequencies be $n_0, n_1, \ldots n_k$ where $k$ is the largest value of $x$ in sample such that $n = \sum_{i=1}^{k} n_x$. Let the first two sample moments for GGSD model be denoted as

$$m_1 = \frac{1}{n} \sum_{x=0}^{k} x n_x$$

and

$$m_2 = \frac{1}{n} \sum_{x=0}^{k} x^2 n_x$$

First Two Moment Method (TMM)

By using elimination between the expression (4.2.1) and (4.2.5) and replacing $\mu_1$ and $\mu_2$ by respective sample moments, we get from (4.2.1)

$$\hat{\beta} = \frac{(m_1 - \theta)}{m_1 \theta} \quad \text{(4.4.1)}$$

Also

$$\frac{(m_2 - m_1^2)}{m_1^3} = \frac{1 - \theta}{\theta^2}$$

which gives quadratic equation in $\theta$ as

$$m_1^3 (m_2 - m_1^2) \theta^2 + \theta - 1 = 0$$

The admissible roots of $\theta$ is given by
\[ \hat{\theta} = \frac{-1 + (1 + 4k)^{1/2}}{2k} \]  

(4.4.2)

where \( k = (m_2 - m_1^2) \eta_i^3 \)

**Zero frequency and first moment method (ZFFM)**

Let \( P_0 \) be the probability of the zero class in GGSD (1.7.1)

\[ P_0 = f_0 n^{-1} \]  

(4.4.3)

We equate \( P_0 \) to the corresponding sample proportion of zeros to get

\[ f_0 n^{-1} = (1 - \theta) \]

which gives \( \hat{\theta} = (1 - f_0 n^{-1}) \)

In addition we have

\[ \mu'_i = \frac{\theta}{1 - 0\beta} \]

Replacing \( \mu'_i \) by corresponding sample moments \( m_i \) and after simplification we get the estimate of \( \beta \) as

\[ \hat{\beta} = \frac{m_i - \hat{\theta}}{m_i \hat{\theta}} \]  

(4.4.4)

**First two moments and Ratio of first two frequencies (MORA)**

Let \( P_1 \) be the probability of the “one” class and \( P_0 \) be the probability of “zero” class in GGSD (1.7.1). The ration of “one” class to the “zero” class is given by

\[ \frac{P_1}{P_0} - \theta(1 - \theta)^{n-1} = f_r \]  

(4.4.5)

Squaring (4.4.5), we get

\[ \theta^2(1 - \theta)^{2n-2} = f_r^2 \]
which gives $\theta^2 = \frac{f_t^2}{(1-\theta)^{2p-2}}$ \hspace{1cm} (4.4.6)

Also we have

$$\tilde{\mu}_1' = \frac{\theta}{(1-\theta\beta)}$$ \hspace{1cm} (4.4.7)

Substituting the value of $(1-\theta \beta)^3$ in (4.2.5) we have

$$\tilde{\mu}_2 = \frac{(1-\theta)\mu_1^3}{\theta^3}$$

Which gives $\theta^2 = (1-\theta)\mu_1^3 \cdot \mu_2^{-1}$ \hspace{1cm} (4.4.8)

on combining (4.4.6) and (4.4.8) we have

$$\frac{f_t^2}{(1-\theta)^{2p-2}} = (1-\theta)\mu_1^3 \mu_2^{-1}$$

$$= \mu_2 f_t^2 \mu_1^{-3} = (1-\theta)^{2p-1}$$ \hspace{1cm} (4.4.9)

Applying log, we get

$$2\beta - 1 \log(1-\theta) - \log(\mu_2 f_t^2 \mu_1^{-3})$$ \hspace{1cm} (4.4.10)

Also ratio of first two moments gives

$$\frac{\mu_1}{\mu_2} = \frac{\theta(1-\theta\beta)^2}{(1-\theta)}$$

which on simplification gives

$$\hat{\beta} = \frac{1}{\theta} \left[ 1 - \frac{\mu_1 (1-\theta)}{\mu_2} \right]^{1/2}$$ \hspace{1cm} (4.4.11)

using relation (4.4.11) in (4.4.10) on simplification, we have

$$f(\theta) = \left( \frac{2}{\theta} - 2 \left[ \frac{\mu_1 (1-\theta)}{\mu_2} \right]^{1/2} \right) \log(1-\theta) - \log(\mu_2 f_t^2 \mu_1^{-3}) = 0$$
Replacing the first two sample moments to their corresponding population moments, we have

\[
f(\theta) = \left( \frac{2}{\theta} - \frac{2}{\theta} \left[ \frac{m_1 (1 - \theta)}{m_2 - m_1^2} \right]^{1/2} \right) \log(1 - \theta) - \log \left( \frac{m_2 - \theta m_2^2}{m_1^2} \right) = 0
\]

We solve \( f(\theta) \) iteratively to obtain \( \hat{\theta} \), the MORA estimator of parameter \( \theta \). The estimate of \( \beta \) can be obtained by using (4.4.4).

**Estimation of Truncated Generalized Geometric Series Distribution (TGGSD)**

For estimating the parameters of the zero truncated GGSD, first two moments about origin are used, we have for zero truncated GGSD (1.7.2)

\[
\mu_1' = \frac{1}{1 - \theta \beta} \quad (4.4.12)
\]

\[
\mu_2' = \frac{1 - \theta}{(1 - \theta \beta)^3} + \frac{\theta}{(1 - \theta \beta)^2} \left( \mu_1' \right)^2 \quad (4.4.13)
\]

expression (4.4.13) gives

\[
\mu_2' = (1 - \theta) \mu_1'^3 + \theta \mu_2'^2
\]

and \( \theta = \frac{\mu_2' - \mu_1'^3}{\mu_1'^2 (1 - \mu_1')} \)

Replacing \( \mu_2' \) by \( m_2 \) and \( \mu_1' \) by \( m_1 \), the estimate of \( \theta \) in obtained us

\[
\hat{\theta} = \frac{m_2 - m_1^3}{m_1^2 (1 - m_1)} \quad (4.4.14)
\]

Substituting this value of \( \hat{\theta} \) in (4.4.12) and replacing \( \mu_1' \) by \( m_1 \), the estimate of \( \beta \) is as

\[
\hat{\beta} = \frac{m_1 (m_1 - 1)^2}{(m_1^3 - m_2)} \quad (4.4.15)
\]
4.5: Maximum Likelihood Estimation

The likelihood function of GGSD based on the random sample \( x_1, \ldots, x_n \) is given by

\[
L = \frac{\theta^{nx} (1 - \theta)^{n+n(\beta-1)x} \prod_{x=1}^{k} \prod_{j=1}^{x-1} (1 + \beta x - j)^{nx}}{\prod_{x=0}^{k} (x_i)^{f_x}}
\]

The two likelihood equation are obtained as

\[
\frac{\partial}{\partial \theta} \log L = \frac{n\bar{x}}{\theta} - \frac{n[1 + (\beta - 1)\bar{x}]}{1 - \theta} = 0 \tag{4.5.1}
\]

\[
\frac{\partial}{\partial \beta} \log L = n\bar{x} \log(1 - \theta) + \sum_{x=2}^{k} \sum_{j=1}^{x-1} \frac{nx}{(1 + \beta x - j)} = 0 \tag{4.5.2}
\]

from (4.5.1) we have

\[
\hat{\theta} = \frac{\bar{x}}{1 + \beta \bar{x}} \tag{4.5.3}
\]

when substituted in (4.5.2) gives

\[
= n\bar{x} \log \left[ \frac{1 + (\beta - 1)\bar{x}}{1 + \beta \bar{x}} \right] + \sum_{x=2}^{k} \sum_{j=1}^{x-1} \frac{nx}{(1 + \beta x - j)} = 0
\]

The equation (4.5.4) can be solved for \( \beta \) applying some iteration technique. The estimate of \( \beta \) when substituted in (4.5.3) gives an estimate of \( \theta \).


In this section we present a quick method for estimating the parameters of generalized geometric series distribution (GGSD) for the case when non-zero frequencies are found only up to a finite number of values of the variable. In such cases only one parameter \( \theta \) is estimated which is based on the mean of the observed distribution, the parameter \( \beta \) being obtain just by counting the number of non-zero frequency classes, and also the variance of the estimators has been obtained.
Let \( t - 1 \) be the highest observed value having non-zero frequency. From the condition of GGSD (1.7.1)

\[
P(X = x) = 0 \quad \text{for} \quad x \leq t \quad \text{if} \quad 1 + \beta t - t \leq 0
\]

we may have \( 1 + \beta t - t = 0 \), which gives minimum value of \( \beta \), say \( \beta_0 \) as

\[
\beta_0 = \frac{t - 1}{t} \tag{4.6.1}
\]

Substituting this values of \( \beta \) in the expression for the mean of GGSD (4.2.1) and replacing \( \mu' \) by sample mean \( \bar{x} \) we get the estimate of \( \theta \), \( \hat{\theta} \) as

\[
\hat{\theta} = \frac{\bar{x}}{1 + \bar{x}\beta_0} \tag{4.6.2}
\]

which is same if \( \beta \) in replaced by \( \beta_0 \) in maximum likelihood estimator of \( \theta \) (4.5.3)

The value of \( \beta_0 \) is obtained directly from the non-zero frequency classes and may be treated as predetermined as \( n \) in case of binomial distribution.

The variance of the estimator \( \hat{\theta} \) can be obtained by using differential method [Kendall and Stuart (1969)] as

\[
V(\hat{\theta}) = \frac{\mu'_2}{n} \left[ \frac{1 + \beta(\mu'_1 - 1)}{(1 + \beta\mu'_1)^2} \right]^2_{\beta=\beta_0} \tag{4.6.3}
\]

where \( n \), \( \mu'_1 \) and \( \mu'_2 \) are sample size, mean and variance, respectively.

Substituting the expression for \( \mu'_1 \) and \( \mu'_2 \) from (4.2.1) and (4.2.5) in (4.6.3), we get

\[
V(\hat{\theta}) = \frac{\theta(1 - \theta)}{n(1 - \theta\beta_0)} [1 + \beta_0(1 - \theta\beta_0)]^2 \tag{4.6.4}
\]

The values of \( n \) times \( V(\hat{\theta}) \) for some selected sets of values of \( t - 1 \) and \( \theta \) have been shown in table (4.6.1).
Table 4.6.1
Values of n times V(\(\hat{\theta}\)) at Certain Selected Points of \(\theta\) and t – 1

<table>
<thead>
<tr>
<th>(t-1)</th>
<th>(\beta_0)</th>
<th>(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>5</td>
<td>0.8333</td>
<td>0.00547</td>
</tr>
<tr>
<td>6</td>
<td>0.8571</td>
<td>0.00461</td>
</tr>
<tr>
<td>7</td>
<td>0.8750</td>
<td>0.00401</td>
</tr>
<tr>
<td>8</td>
<td>0.8888</td>
<td>0.00357</td>
</tr>
<tr>
<td>9</td>
<td>0.9000</td>
<td>0.00324</td>
</tr>
<tr>
<td>10</td>
<td>0.9091</td>
<td>0.00298</td>
</tr>
<tr>
<td>11</td>
<td>0.9166</td>
<td>0.00278</td>
</tr>
<tr>
<td>12</td>
<td>0.9231</td>
<td>0.00261</td>
</tr>
</tbody>
</table>

This table reveals that with increase in the value of \(\theta\). There is an increase in \(V(\hat{\theta})\), too. A peculiar phenomenon can also be observed from the table, whereas for \(\theta \geq 0.5\), \(V(\hat{\theta})\) increases as \(\beta_0\) assumes higher values, for \(\theta < 0.5\); \(V(\hat{\theta})\) shows a decline in its values of \(\beta_0\). This suggests that this estimator may be preferred in the case where \(\beta\) in large and \(\theta\) in small. Also we find that for fixed \(\bar{x}\) large value of \(\beta\) would give smaller values of \(\theta\). This suggests that for making an efficient use of this estimator we should only confirm that \(\beta\) is large, as for large \(\beta\), \(\theta\) is expected to be small but again \(\beta\) is large if the number of non-zero frequency classes in large. This implies that this estimator is expected to be sufficiently efficient if the number of non-zero frequency classes is large.

4.7: Goodness of Fit

The GGSD is fitted to some observed distributions, estimating the parameters \(\theta\) and \(\beta\) first by method of moments and then by the suggested method. Singh (1989) used GGSD for explaining some sports data. He used the method of
moments for the purpose. GGSD is fitted to these sports data after estimating the parameters $\theta$ and $\beta$ by suggested method. The expected frequencies according to both the method along with the estimates of the parameters and the value of Chi-square are shown in the following tables.

Table 4.7.1

Runs Scored by Gavanskar in 136 Completed Innings

<table>
<thead>
<tr>
<th>Runs (Units of 30)</th>
<th>Observed frequency</th>
<th>Expected Frequency</th>
<th>Method of moments</th>
<th>Suggested method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>64</td>
<td>59.2</td>
<td>56.1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>27</td>
<td>34.7</td>
<td>35.4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>19.4</td>
<td>20.3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>10.6</td>
<td>12.3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5.7</td>
<td>5.8</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>6.4</td>
<td>6.1</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>136</td>
<td>136</td>
<td>136</td>
<td></td>
</tr>
<tr>
<td>$\overline{X}$ (mean)</td>
<td>1.22794</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S^2$ (Variance)</td>
<td>2.52893</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\theta}$</td>
<td>0.5646005</td>
<td>0.5871091</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{\beta}$</td>
<td>0.9567918</td>
<td>0.88888</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>2.74</td>
<td>3.39</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d.f</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4.7.2
Wickets Taken by Sobers in 158 Completed Innings

<table>
<thead>
<tr>
<th>Runs (Units of 30)</th>
<th>Observed frequency</th>
<th>Expected Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Method of moments</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Suggested method</td>
</tr>
<tr>
<td>0</td>
<td>49</td>
<td>51.1</td>
</tr>
<tr>
<td>1</td>
<td>41</td>
<td>43.0</td>
</tr>
<tr>
<td>2</td>
<td>31</td>
<td>29.2</td>
</tr>
<tr>
<td>3</td>
<td>23</td>
<td>17.4</td>
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<tr>
<td>4</td>
<td>28</td>
<td>9.4</td>
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<tr>
<td>5</td>
<td>5</td>
<td>6.3</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1.6</td>
</tr>
<tr>
<td>Total</td>
<td>158</td>
<td>158</td>
</tr>
<tr>
<td>( \bar{x} ) (mean)</td>
<td>1.4873417</td>
<td></td>
</tr>
<tr>
<td>( S^2 ) (Variance)</td>
<td>1.97135876</td>
<td></td>
</tr>
<tr>
<td>( \hat{\theta} )</td>
<td>0.674130</td>
<td>0.6538156</td>
</tr>
<tr>
<td>( \hat{\beta} )</td>
<td>0.8060461</td>
<td>0.8571429</td>
</tr>
<tr>
<td>( \chi^2 )</td>
<td>2.69</td>
<td>3.89</td>
</tr>
<tr>
<td>d.f.</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

The table reveals that the suggested estimator is giving a close fit to the sports data. This method may be preferred over others because of its simplicity and quickness in obtaining the estimators.

4.8: Bayesian Estimation of Parameter \( \theta \) of GGSD

The Bayesian estimation of parameters of Generalized Geometric Series Distribution (GGSD) and Zero truncated GGSD does not seem to appear in literature so far. We have made an attempt to obtain Bayesian estimation of parameters of GGSD. Since \( 0 < \theta < 1 \), for GGSD it is assumed that prior information of \( \theta \) is given by beta distribution.
The probability mass function of GGSD is given as,

\[ P(X = x) = \frac{1}{1 + \beta x} \left( \frac{1 + \beta x}{x} \right)^x (1 - \theta) \left(1 + \beta x \right)^{y} \quad x = 0, 1, 2, \ldots \quad 0 < \theta < 1 \]

The likelihood function is obtained as

\[ L(x \mid \theta, \beta) = \prod_{i=1}^{n} \left( \frac{1}{1 + \beta x_i} \right)^x (1 - \theta) \left(1 + \beta x_i \right)^{y} \left(1 + \beta \Sigma x_i, (1 - \theta)^{y} \left(1 + \beta \Sigma x_i, (1 - \theta)^{n} \right) \right) \]

\[ \text{where } K = \prod_{i=1}^{n} \left( \frac{1}{1 + \beta x_i} \right) \left(1 + \beta x_i \right) \]

and \[ y = \sum_{i=1}^{n} x_i \]

Since \( \theta < \theta < 1 \), it is assumed that prior information of \( \theta \) is given by a beta distribution with density function

\[ g(\theta; a, b) = \frac{\theta^{a-1} (1 - \theta)^{b-1}}{B(a, b)} \quad (4.8.2) \]

\[ a, b > 0, \quad 0 < \theta < 1 \]

The posterior distribution of \( \theta \) is defined as \( \pi(\theta \mid y) = \frac{L(x \mid \theta, \beta)g(\theta; a, b)}{\int L(x \mid \theta, \beta)g(\theta; a, b) d\theta} \)

\[ \pi(\theta \mid y) = \frac{\theta^{y} (1 - \theta)^{n + (\beta - 1)y} \cdot \theta^{a-1} (1 - \theta)^{b-1}}{\int \theta^{y} (1 - \theta)^{n + (\beta - 1)y} \cdot \theta^{a-1} (1 - \theta)^{b-1} d\theta} \]

\[ \pi(\theta \mid y) = \frac{\theta^{a+y-1} (1 - \theta)^{n + (\beta - 1)y + b-1}}{\int \theta^{y+a-1} (1 - \theta)^{n + (\beta - 1)y + b-1} d\theta} \]

\[ \pi(\theta \mid y) = \frac{\theta^{a+y-1} (1 - \theta)^{n + (\beta - 1)y + b-1}}{B(y + a, n + (\beta - 1)y + b)} \]
The Bayes estimator for parametric function $\phi(\theta)$

$$
\phi(\theta) = \int_0^1 \phi(\theta) \pi(\theta | y) d\theta \\
\phi(\theta) = \frac{\int_0^1 \phi(\theta) \theta^{y-1} (1 - \theta)^{(\beta-1)y+b-1} d\theta}{B(y + a, n + (\beta - 1)y + b)}
$$

(4.8.3)

If we take $\phi(\theta) = \theta$ then Bayes estimator $\theta$ is

$$
\theta = \frac{\int_0^1 \theta^{y-1} (1 - \theta)^{(\beta-1)y+b-1} d\theta}{B(y + a, n + (\beta - 1)y + b)}
$$

$$
= \frac{B(y + a + 1, n + (\beta - 1)y + b)}{B(y + a, n + (\beta - 1)y + b)}
$$

$$
\theta = \frac{a + y}{n + a + b + \beta y}
$$

(4.8.4)

If $a = b = 0$, (4.8.4) is identical to mle of GGSD.

4.9: Bayesian Estimation of Parameter $\theta$ of Truncated GGSD

The zero truncated form of GGSD is given by its mass function as

$$
P(X = y) = \frac{1}{\beta x + 1} \left( \frac{\beta x + 1}{x} \right)^x (1 - \theta)^{\beta - x + 1} 
\quad x = 1, 2, ...
$$

Let $x_1, x_2, \ldots, x_n$ be a random sample from the GGSD. The likelihood function is given by

$$
L(x/\theta, \beta) = \prod_{i=1}^n \frac{1}{\beta x_i + 1} \left( \frac{\beta x_i + 1}{x} \right)^{x_i} (1 - \theta)^{\beta x_i - x_i + 1}
\quad = C \theta^{\sum x_i - n} (1 - \theta)^{\beta \sum x_i - \sum x_i + n}
\quad = C \theta^{n\bar{x} - n} (1 - \theta)^{\beta n\bar{x} - n\bar{x} + n}
$$

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Since $0 < \theta < 1$, the prior in function on $\theta$ can be summarized by beta distribution $B(a, b)$. With probability density function

$$g(\theta; a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a, b)}$$

where $a$ & $b$ are unknown.

This gives pdf of $(x_1 \ldots x_n)$, and $\theta$ as

$$H(X_1 \ldots X_n / \theta) = L(x/\theta) \cdot g(\theta : a, b)$$

Then using Bayes theorem, the posterior distribution of $\theta$ becomes

$$\pi(\theta / x) = \frac{L(x/\theta) \cdot g(\theta : a, b)}{\Omega}$$

$$= \frac{\theta^{n\bar{x}_n - n + a - 1}(1-\theta)^{\beta n\bar{x}_n - n\bar{x} + b - 1}}{1}$$

$$= \frac{\int_0^1 \theta^{n\bar{x}_n - n + a - 1}(1-\theta)^{\beta n\bar{x}_n - n\bar{x} + b - 1} d\theta}{B(n\bar{x} - n + a, \beta n\bar{x} - n\bar{x} + n + b)}$$

under squared loss function, Bayes estimator of $\theta$ is

$$\hat{\theta} = \frac{1}{B(n\bar{x}_n - n + a, \beta n\bar{x}_n - n\bar{x} + n + b)}$$

$$= \frac{\int_0^1 \theta^{n\bar{x}_n - n + a}(1-\theta)^{\beta n\bar{x}_n - n\bar{x} + b - 1} \cdot \beta n\bar{x}_n - n\bar{x} + n + b - 1}{B(n\bar{x} - n + a, \beta n\bar{x} - n\bar{x} + n + b)}$$

which on simplification gives
\[ \hat{\beta} = \frac{n\overline{x} - n + a}{\beta n\overline{x} + a + b} \] (4.9.1)

4.10: Some Other Generalizations of Geometric Distribution

Generalized Geometric Series Distribution I (GGSD-I)

To find another form of GGSD, the GGSD-I, Singh (1982) used the second form of the Lagrange’s expansion given as

\[ \frac{f(z)}{1 - \frac{zg'(z)}{g(z)}} = \sum_{x=0}^{\infty} \frac{1}{x!} \frac{\partial^x}{\partial z^x} \left[ f(z) \left\{ g(z)^x \right\} \right] \left[ \frac{z}{g(z)} \right]^x \] (4.10.1)

where \( f(z) \) and \( g(z) \) are positive continuous functions and \( f(z)g(z) \) is differentiable.

Let \( f(z) = 1 \) and \( g(z) = (1 + z)^\theta \) where \( z = \frac{\theta}{1 - \theta} \) which gives

\[ \frac{\partial^x}{\partial z^x} \left[ f(z) \{g(z)^x\} \right]_{z=0} = x\beta(x\beta - 1)(x\beta - 2)\cdots(x\beta - x + 1) \] (4.10.2)

\[ \frac{f(z)}{1 - \frac{zg'(z)}{g(z)}} = \frac{1 + z}{1 + z(1 - \beta)} \] (4.10.3)

\[ \left( \frac{z}{g(z)} \right)^x = z^x (1 + z)^{-x\beta} \] (4.10.4)

Substituting these expression in the Lagrange’s expansion (4.10.1), we get

\[ \sum_{x=0}^{\infty} (1 - \theta \beta) \cdot \left( \frac{x\beta}{x} \right)^x (1 - \theta)^{x \beta - x} = 1 \]

which shows that the function

\[ P(X = x) = (1 - \theta \beta) \left( \frac{x\beta}{x} \right)^x (1 - \theta)^{x \beta - x}, \quad x = 0, 1, 2, \cdots \] (4.10.5)

where \( 0 < \theta < 1, |\theta \beta| < 1 \)
represents a true probability distribution. At $\beta = 1$, the distribution defined in (4.10.5) reduces to the geometric series distribution. Thus the distribution defined in (4.10.5) is another generalization of geometric series distribution.

As (4.10.5) gives $P(X = 0) = 1 - \theta \beta$, the zero-truncated GGSD-I is given by the probability mass function as

$$P(X = x) = \frac{1 - \theta \beta}{\theta \beta} \left(\frac{x^\beta}{x}\right) \theta^x (1 - \theta)^{\theta - x}$$

(4.10.6)

$0 < \theta < 1$, $|\theta \beta| < 1$ and $x = 1, 2, ...$

Moments of GGSD-I

GGSD-I belong to the class of MPSD defined in (1.3.1) with

$$a(x) = \left(\frac{x^\beta}{x}\right)$$

(4.10.7)

$$g(\theta) = \theta (1 - \theta)^{\theta - 1}$$

$$f(\theta) = (1 - \theta \beta)^{-1}$$

The mean and variance of GGSD-I may be obtained using (4.10.7) in the relation (1.3.2) and (1.3.5).

The mean of GGSD-I is given as

$$\mu'_1 = \frac{\theta \beta (1 - \theta)}{(1 - \theta \beta)^2}$$

(4.10.8)

and variance $\mu_2$ of GGSD-I is obtained as

$$\mu_2 = \frac{\theta (1 - \theta) \beta (1 - 2\theta + \theta \beta)}{(1 - \theta \beta)^4}$$

(4.10.9)

which may also be put in the form

$$\mu_2 = \frac{\mu'_1 (1 - 2\theta + \theta \beta)}{(1 - \theta \beta)^2}$$

(4.10.10)
this shows that mean is greater than, equal to and less than the variance as 0 is respectively greater than, equal to and less than \((3\beta - 2)/\beta^2\).

**Estimation of GGSD-I**

The two parameters of GGSD-I can be estimated using the mean and variance of the distribution. Writing \(\alpha\) for \((1 - \theta)\), mean and variance can be expressed as

\[
\mu'_1 = \frac{(1-\theta)(1-\alpha)}{\alpha^2}
\]

(4.10.11)

and

\[
\mu'_2 = \frac{\mu'_1[2(1-\theta) - \alpha]}{\alpha^2}
\]

(4.10.12)

from these one gets the value of \(\theta\) as

\[
\hat{\theta} = 1 - \frac{\mu'_1\alpha}{1 - \alpha}
\]

(4.10.13)

and

\[
\frac{\mu'_2}{\mu'_1} = \frac{2 - 2\theta + \alpha}{\alpha^2}
\]

(4.10.14)

\[
\frac{\alpha\mu'_2}{\mu'_1} = \frac{2\mu'_1\alpha}{1 - \alpha} - 1
\]

which gives a quadratic equation in \(\alpha\) as

\[
\mu_2\alpha^2 + \left(2\mu'_1\alpha + \mu'_1 - \mu_2\right)\alpha - \mu'_1 = 0
\]

(4.10.15)

Hence

\[
\alpha = -A \pm \left(A^2 + 4\mu'_1\mu_2\right)^{1/2}/2\mu_2
\]

where

\[
A = 2\mu'_1\alpha + \mu'_1 - \mu_2.
\]

The mean \(\mu'_1\) and the variance \(\mu_2\) are replaced by the sample mean \(\bar{x}\) and sample variance \(S_2\) respectively. The value of \(\alpha\) is obtained.
Substituting the value of \( \alpha \) in (4.10.13) and replacing \( \mu \) by \( \bar{x} \), the estimated value of \( \theta \) is obtained as

\[
\hat{\theta} = 1 - \frac{\bar{x} \alpha^2}{1 - \alpha}
\]  
(4.10.16)

Also \((1 - \theta \beta) = \alpha\) we get

\[
\hat{\beta} = \frac{1 - \alpha}{\hat{\theta}}
\]  
(4.10.17)

**Generalized Geometric Series Distribution – II (GGSD-II)**

Tripathi, Gupta (1987) obtained some generalization of geometric distribution by length biasing some generalized versions of the log series distribution.

The probability function of Kempton’s (1975) generalized log series distribution-I (GLSD-I) given by

\[
P_r = \frac{\alpha}{r!} \int_0^{\infty} (1 + bx)^{-q} x^{r-1} e^{-x} dx, \quad r = 1, 2, \ldots; \quad b > 0, q > 0
\]  
(4.10.18)

where \( \alpha^{-1} = \sum_{r=1}^{\infty} \frac{1}{r!} \int_0^{\infty} (1 + bx)^{-q} x^{r-1} e^{-x} dx \).

The generalized geometric distribution based on (4.10.18) is derived by considering its length-biased version, \( X^* \) denotes the resulting random variable. With associated probability function \( p_r^* \), then

\[
p_r^* = \frac{r p_r}{\sum_{r=1}^{\infty} r p_r} = \frac{\alpha^*}{(r-1)!} \int_0^{\infty} (1 + bx)^{-q} x^{r-1} e^{-x} dx
\]  
(4.10.19)

\[r = 1, 2, \ldots; \quad b > 0, \quad l > 0\]
where \((\alpha')^{-1}\) is given by
\[
\sum_{r=1}^{\infty} \frac{1}{(t-1)!} \int_0^\infty (1 + bx)^{-t} x^{-t} e^{-x} dx
\]
\[
= \frac{1}{b(q-1)}, \quad q > 1
\]
equation (4.10.19) gives the probability function of a two parameter generalized geometric distribution with parameters \(b\) and \(q\). Integrating (4.10.19) by parts gives
\[
p_r^* = \sum_r \frac{r+1}{r} p_{r+1} + \frac{qba}{r!} \int_0^\infty \frac{x^r e^{-x}}{(1 + bx)^{q+1}} dx
\]
\[
= p_{r+1}^* + \frac{qba(r-1)}{r(r+1)!} \int_0^\infty \frac{x^{r-1} e^{-x}}{(1 + bx)^{q+1}} dx
\]
(4.10.20)
on taking limit of both sides as \(b \to 0, q \to \infty\), such that \(bq = \phi\) remains fixed,
\[
p_r^* = (1 + \phi)p_{r+1}^* \quad \text{or} \quad \frac{p_{r+1}^*}{p_r} = \theta
\]
with \(\theta = (1 + \phi)^{-1}\) which is recurrence relation of the ordinary geometric distribution, which shows (4.10.19) is indeed a generalized geometric series distribution.

Moments of GGSD-II

In deriving the GLSD-I, Kempton (1975) first considered a Poisson random variable \(Y\) with mean \(\theta\). Then, he assumed \(\theta\) to have the Beta type II distribution with probability density function.
\[
f(x, b, k, q) = \frac{b^k}{B(k, q)} \cdot \frac{x^{k-1}}{(1 + bx)^{k+q}}, \quad x > 0
\]
(4.10.21)
where
\[
B(k, q) = \frac{\Gamma(k)\Gamma(q)}{\Gamma(k+q)}
\]
GLSD-I is obtained by taking the limit of the truncated version of unconditional distribution of \(Y\) as \(k \to 0\). If \(\nu_{ij}\) denotes the \(j\)th factorial moment of unconditional distribution of \(Y\) then
\[ \mu_{(j)} = \frac{\text{EE}(Y_{(j)}/\theta)}{\text{E}(\theta')} = \frac{1}{b^j} \frac{B(j+k,q-j)}{B(k,q)}, \quad q \geq j \]

If \( \mu_{(j)} \) denotes the \( j \)th factorial moment of GLSD-I and \( \mu_j^* \) be that of GGSD-II, then from the basic definition, \( \mu_{(j)} \) can be obtained as the limit of the \( j \)th factorial moment of the unconditional distribution of \( Y \) truncated at zero as \( k \to 0 \). Thus, it denotes the probability function of the unconditional distribution of \( Y \).

\[ \mu_{(j)} = \lim_{k \to 0} \sum_{r=1}^{\infty} \frac{1}{b^j} \frac{B(j+k,q-j)}{B(k,q)} \]

\[ = \lim_{k \to 0} \frac{1}{b^j} \frac{B(j+k,q-j)}{B(k,q)} \sum_{r=1}^{\infty} \frac{b^k}{r!B(k,q)} \int_0^{\infty} x^{r+k-1}e^{-x}dx \]

\[ = \frac{\alpha}{b^j} B(j,q-j), \quad j \leq q \]

Then \( \mu_j = \frac{\mu_{(j)} + j\mu_{(j)}}{\mu_{(j)}} \)

\[ = \frac{B(j,q-j-1)}{b^j} (j+b(q-j-1)), \quad j \leq q-1 \]

**iii) Generalized Geometric Series Distribution-III (GGSD-III)**

The probability function of GLSD-II defined by Tripathi and Gupta (1985) is given by

\[ p_r = C \cdot \frac{(r-1)!\theta^{r-1}}{(\lambda + 1)_{r-1}}, \quad \lambda > -1, \ 0 < \theta \leq 1, \ r = 1,2, \ldots \]  

(4.10.22)

where \( (a)_r = (a+1) \ldots (a+r-1) \) and \( C = p_1 \) with

\[ p_1 = \left[ \frac{\sum_{r=1}^{\infty} (r-1)!\theta^{r-1}}{\sum_{r=1}^{\infty} (\lambda + 1)_{r-1}} \right]^{-1} \]
The length biased version of (4.10.22) gives
\[
p_i^* = \frac{r p_r}{\sum_r p_r} = C^* \frac{r! \theta^{r-1}}{(\lambda + 1)_{r-1}} \tag{4.10.23}
\]
where \( C^* = p_1^* = \left[ \sum_{r=0}^{\infty} \frac{r! \theta^{r-1}}{(\lambda + 1)_{r-1}} \right]^{-1} \).

The probability given by (4.10.23) corresponds to another two parameters generalized geometric distribution with parameter \( \lambda \) and \( \theta \).

**Moments**

If \( G(z) \) is the probability generating function corresponding to (4.10.23) and if \( u_j^* \) be the \( j \)th factorial moment, then
\[
u_{(j)}^* = \left[ \frac{d^j}{dz^j} G(z) \right]_{z=1} \tag{4.10.24}
\]
The \( u_{(j)}^* \) can also be obtained from the following recurrence relation
\[
(1 - \theta) u_{(j+1)}^* + (\lambda + j - 1 - \theta(2j + 1)) u_j^* - \theta^2 u_{(j-1)}^* = \begin{cases} \lambda + j - 1 \binom{j}{1} + 1^{(j+1)} p_1^* & , j = 1, 2, \ldots \\
(a^{(j)} = a(a + 1)(a - 2) \ldots (a - j + 1) \end{cases} \tag{4.10.25}
\]
The above recurrence relation is derived by multiplying both sides of
\[
(\lambda + r)p_{r+1}^* - (r + 1)\theta p_r^* = 0
\]
by \( (r + 1)^{a} \) and summing over \( r \), thus, moments of GGSD-III can be obtained by finding the first two factorial moments using (4.10.24) and then using (4.10.25).

**4.11: Size Biased Generalized Geometric Series Distribution (SBGGSD)**

In this section a Size biased GGSD has been defined and is obtained by taking the weight of the GGSD (1.7.1) as \( X \). the moments and the recurrence relation between the moments about the origin of SBGGSD has been obtained.
Mishra (1982) defined GGSD as

\[ P(X = x) = \frac{1}{1 + \beta x} \left(1 + \beta x\right) \theta^x (1 - \theta)^{1+\beta x-x} \]  
\[ x = 0, 1, 2, \ldots \]

He also obtained first four moments about origin of GGSD as given in (4.2.1), (4.2.2), (4.2.3) and (4.2.4) respectively.

we have from (4.2.1)

\[ \sum_{x=0}^{\infty} x \cdot P(X = x) = \frac{\theta}{1 - \theta \beta} \]

where \( P(X = x) \) is defined in (1.7.1)

\[ \frac{\theta}{1 - \theta \beta} = \sum_{x=0}^{\infty} x \cdot \frac{1}{1 - \beta x} \left(1 - \beta x\right) \theta^x (1 - \theta)^{1+\beta x-x} \]

\[ = \sum_{x=1}^{\infty} \frac{1}{(x - 1)! (1 + \beta x - x)!} (1 + \beta x)! \theta^x (1 - \theta)^{1+\beta x-x} \]

\[ = \sum_{x=1}^{\infty} \frac{(\beta x)!}{(x - 1)! (1 + \beta x - x)!} \theta^x (1 - \theta)^{1+\beta x-x} \]

\[ = \sum_{x=1}^{\infty} \left( \frac{\beta x}{x - 1} \right) \theta^x (1 - \theta)^{1+\beta x-x} \]

\[ = \sum_{x=1}^{\infty} (1 - \theta \beta) \left( \frac{\beta x}{x - 1} \right) \theta^x (1 - \theta)^{1+\beta x-x} = 1; \quad x = 1, 2, \ldots \]

\[ \Rightarrow \sum_{x=1}^{\infty} P_X[X = x] = 1 \]

where \( P_X[X = x] \) represents a probability distribution. This gives the probability mass function of size-biased generalized geometric series distribution (SBGGSD) as

\[ P_X[X = x] = (1 - \theta \beta) \left( \frac{\beta x}{x - 1} \right) \theta^{x-1} (1 - \theta)^{1+\beta x-x} \]  
\[ x = 1, 2, \ldots \]

\( 0 < \theta < 1, \quad |\theta \beta| < 1, \quad 0 < \theta \beta < t \quad \text{whenever} \quad 1 + \beta t < 0 \)
When $\beta = 1$, the SBGGSD (4.11.1) reduce to size biased geometric series distribution with probability mass function as

$$P_2[X = x] = x(1-\theta)^2 \theta^{x-1}, \quad x = 1, 2, \ldots$$

The $r^{th}$ moment $S'_r$ about origin of SBGGSD (4.11.1) is defined as

$$S'_r = E(X^r) = \sum_{x=1}^{\infty} x^r P_1[X = x], \quad r = 1, 2, 3, \ldots$$

$$\sum_{x=1}^{\infty} x^r \frac{1-\theta \beta}{\theta} \left( \frac{\beta x}{x-1} \right) \theta^x (1-\theta)^{1+\beta x-x}$$

obviously $S'_0 = 1$ and for $r \geq 1$

$$= \frac{1-\theta \beta}{\theta} \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x-1} \right) \theta^x (1-\theta)^{1+\beta x-x}$$

$$= \frac{1-\theta \beta}{\theta} \sum_{x=1}^{\infty} x^{r+1} \cdot \frac{1}{1+\beta x} \left( \frac{1+\beta x}{x} \right) \theta^x (1-\theta)^{1+\beta x-x}$$

$$= \frac{1-\theta \beta}{\theta} \sum_{x=1}^{\infty} x^{r+1} P[X = x]$$

$$S'_r = \frac{1-\theta \beta}{\theta} \mu'_{r+1}$$

(4.11.2)

where $\mu'_{r+1}$ is the $(r + 1)^{th}$ moments about origin of GGSD

For $r = 1$, we have from (4.11.2)

$$S'_1 = \frac{1-\theta \beta}{\theta} \mu'_2$$

Using relation (4.2.2), we have

$$S'_1 = \frac{1-\theta \beta}{\theta} \left[ \frac{\theta^2}{(1-\theta \beta)^2} + \frac{\theta(1-\theta)}{(1-\theta \beta)^3} \right]$$

$$= \left[ \frac{\theta}{1-\theta \beta} + \frac{1-\theta}{(1-\theta \beta)^2} \right]$$
\[ S'_r = \frac{(1 - \theta^2 \beta)}{(1 - \theta \beta)^2} \quad (4.11.3) \]

For \( r = 2 \), we have
\[ S'_2 = \frac{1 - \theta \beta}{\theta} \mu'_s \quad (4.11.4) \]

Using relation (4.2.3) in (4.11.4), we get
\[
\frac{1 - \theta \beta}{\theta} \left[ \frac{\theta^4}{(1 - \theta \beta)^4} + \frac{3\theta^3(1-\theta)}{(1 - \theta \beta)^5} + \frac{\theta(1-\theta)}{(1 - \theta \beta)^6} \left[ 1 - 2\theta + \theta \beta(2 - \theta) \right] \right]
\]
\[
= \frac{1}{(1 - \theta \beta)^2} \left[ \theta^2 + \frac{3\theta(1-\theta)}{(1 - \theta \beta)^3} + \frac{1-\theta}{(1 - \theta \beta)^4} \left[ 1 - 2\theta + \theta \beta(2 - \theta) \right] \right]
\]
\[
= \frac{1}{(1 - \theta \beta)^2} \left[ \theta^2(1 - \theta \beta)^2 + 3\theta(1-\theta)(1 - \theta \beta)^2 + (1-\theta) \left[ 1 - 2\theta + \theta \beta(2 - \theta) \right] \right]
\]
\[
S'_2 = \frac{1}{(1 - \theta \beta)^4} \left[ \theta^4 \beta^2 + 2\theta^3 \beta - 6\theta^2 \beta + 2\theta \beta + 1 \right] \quad (4.11.5)
\]

Hence variance of \( S_2 \) of SBGGSD
\[
S_2 = S'_2 - S'_1^2
\]
\[
= \frac{1}{(1 - \theta \beta)^4} \left[ \theta^4 \beta^2 + 2\theta^3 \beta - 6\theta^2 \beta + 2\theta \beta + 1 - (1 - \theta^2 \beta)^2 \right]
\]
\[
S_2 = \frac{1}{(1 - \theta \beta)^4} \left[ 2\theta^3 \beta - 4\theta^2 \beta + 2\theta \beta \right] \quad (4.11.6)
\]

For \( r = 3 \), we have
\[ S'_3 = \frac{1 - \theta \beta}{\theta} \mu'_s \]
\[
= \frac{1 - \theta \beta}{\theta} \left[ \frac{\theta^4}{(1 - \theta \beta)^4} + \frac{6\theta^3(1-\theta)}{(1 - \theta \beta)^5} + \frac{\theta^2(1-\theta)}{(1 - \theta \beta)^6} \left[ 7 - 11\theta - 4\theta \beta(2 - \theta) \right] + \frac{\theta(1-\theta)}{(1 - \theta \beta)^7} \left[ 1 - 6\theta + 6\theta^2 + 2\theta \beta(4 - 9\theta + 4\theta^3) + \theta^2 \beta^2 (6 - 6\theta + \theta^2) \right] \right]
\]
The higher moments of SBGGSD (4.11.1) about origin can be similarly be obtained using (4.11.2) if so desired.

**Recurrence Relationship of Moments about Origin of Size Biased GGSD**

The rth moment about origin of size biased GGSD (1.7.1) is defined as

$$S_r' = E(X^r) = \sum_{x=1}^{\infty} x^r P_1[X = x]$$

$$= \sum_{x=1}^{\infty} x^r (1 - \theta) \left\{ \frac{\beta x}{x - 1} \right\} \theta^{x-1} (1 - \theta)^{1+\beta x-x}$$

(4.11.8)

The recurrence relation can be obtained by differentiating (4.11.8) w.r.t $\theta$, which gives

$$\frac{\partial S_r'}{\partial \theta} = \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x - 1} \right) \left\{ \frac{\partial}{\partial \theta} \left\{ \theta^{x-1} (1 - \theta)^{1+\beta x-x} \right\} \right\}$$

$$= \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x - 1} \right) \left\{ (1 - \theta) \theta^{x-2} + \theta^{x-1} (-\beta) (1 - \theta)^{1+\beta x-x} \right.$$

$$+ (1 - \theta) \theta^{x-1} (1 + \beta x - x) (1 - \theta)^{\beta x-x} (-1) \right\}$$

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\[
\sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x-1} \right)^{(1-\theta)\theta x-x} (1-\theta)^{\beta x-x} \left[ \left( \frac{x-1}{\theta} \right) - (1 + \beta x - x) \right] - \beta \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x-1} \right)^{(1-\theta)\theta x-x} \left[ \left( \frac{x}{\theta} \right) - 1 \right] \]
\]
\[
= \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x-1} \right)^{(1-\theta)\theta x-x} (1-\theta)^{\beta x-x} \left[ \left( \frac{x(1-\theta\beta)}{\theta} \right) - 1 \right] - \beta \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x-1} \right)^{(1-\theta)\theta x-x} \left[ \left( \frac{x}{\theta} \right) - 1 \right]
\]
\[
= \sum_{x=1}^{\infty} x^{r+1} \left( \frac{(1-\theta\beta)^2}{\theta} \right) \theta x-x (1-\theta)^{\beta x-x} - \frac{1}{\theta} \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x-1} \right)^{(1-\theta)\theta x-x} (1-\theta)^{\beta x-x} \left[ \left( \frac{x}{\theta} \right) - 1 \right]
\]
\[
= \sum_{x=1}^{\infty} x^{r+1} \left( \frac{(1-\theta\beta)^2}{\theta} \right) \theta x-x (1-\theta)^{\beta x-x} - \beta \sum_{x=1}^{\infty} x^r \left( \frac{\beta x}{x-1} \right)^{(1-\theta)\theta x-x} (1-\theta)^{\beta x-x} \left[ \left( \frac{x}{\theta} \right) - 1 \right]
\]
\[
= \frac{(1-\theta\beta)}{\theta(1-\theta)} S_\gamma^{r+1} - \frac{1}{\theta(1-\theta)} S_r - \frac{\beta}{(1-\theta\beta)} S_r
\]
\[
\frac{\partial S_r}{\partial \theta} = \frac{1}{\theta(1-\theta)(1-\theta\beta)} \left[ (1-\theta\beta)^2 S_\gamma^{r+1} - 1(1-\theta)^2 S^r_r \right]
\]
\[
\Rightarrow (1-\theta\beta)^2 S_\gamma^{r+1} = \theta(1-\theta)(1-\theta\beta) \frac{\partial S_r}{\partial \theta} + (1-\theta\beta) S_r
\]
\[
\Rightarrow S_\gamma^{r+1} = \frac{\theta(1-\theta)}{(1-\theta\beta)} \frac{\partial S_r}{\partial \theta} + (1-\theta\beta) S_r
\]
(4.11.9)

If we put \( r = 0 \)

\[
S_0' = \frac{(1-\theta\beta)}{(1-\theta\beta)^2} \quad , \quad S_0 = 1
\]

The second moment about origin can be obtained by taking \( r = 2 \) in (4.11.9)

\[
S_2' = \frac{1}{(1-\theta\beta)^4} \left[ \theta^4 \beta^2 - 6\theta^2 \beta + 2\theta^3 \beta + 2\theta + 1 \right]
\]

Similarly, we can obtain the higher moments of SBGGSD.