Chapter 4

Morita Context and Generalized $(\alpha, \beta)$—Derivation in Prime Rings

4.1 Introduction

A classical problem in ring theory is to study and generalize conditions under which a ring is forced to become commutative. Stimulated from Jacobson’s famous result, several techniques are developed to achieve this goal. For instance, generalizing Herstein’s conditions, using restrictions on polynomials, introducing derivations and generalized derivations on rings, looking special properties for rings, etc. We can also achieve this goal by comparing two rings and impose conditions on them. Here we present an alternate treatment of the theory in which we involve a pair of rings. If one of the ring is commutative, in compatible way, the other ring will also become commutative. In order to explore these ideas Morita theory is found to be a suitable tool.

A study of Morita theory of equivalence and duality was introduced by K. Morita himself in 1958 in a long article [65]. After that, this work was developed by many authors. Now there is an abundance amount of literature on Morita theory, its generalizations, extensions and applications in many fields, including theoretical physics. About the physical applications of Morita theory see detailed references in [?] and [86].
Let us assume that rings $R$ and $S$ are ingredients of a Morita context (see Definition (4.2.1)). It is observed in [67] that if a Morita context is semi-projective, in the sense that the Morita map on $S$ is epic, and if $R$ is commutative and $S$ is reduced, then $S$ becomes commutative. In section 4.2, we weaken the condition on the ring $R$ and assume that if one of the ring is prime with the generalized $(\alpha, \beta)$—derivation that satisfy certain conditions on the trace ideal of the ring, and the other ring is reduced, then the trace ideal of the reduced ring is contained in the center of the ring. As an outcome, in case of a semi-projective Morita context, the reduced ring becomes commutative. Finally, in section 4.3, some consequences are studied and effects of various types of derivations on these rings are listed. If such a ring is an ingredient of a semi-projective Morita context then the reduced ring will also become commutative and if the context is strict then both rings become isomorphic fields. Some consequences related to domains and division rings are stated and proved.

### 4.2 Reduced rings with Generalized $(\alpha, \beta)$-Derivations

We begin our discussion with the definition of Morita context.

**Definition 4.2.1** (Morita Context). Let $R$, $S$ be rings, $M$ an $(S,R)$-bimodule and $N$ an $(R,S)$-bimodule. Then the datum

$$K(R,S) = \{R, S, M, N, \mu_R, \tau_S, I, J\}$$

is said to be a **Morita context** if the maps $\mu_R : N \otimes_S M \to R$ and $\tau_S : M \otimes_R N \to S$ are bimodule morphisms satisfying the following associativity conditions:

$$m_1 \mu_R(n \otimes m) = \tau_S(m_1 \otimes n)m$$

and

$$\mu_R(n \otimes m)n_1 = n\tau_S(m \otimes n_1)$$
\( \mu_R \) and \( \tau_S \) are called Morita maps (or MC maps). The images \( \mu_R := I \) and \( \tau_S := J \) are two-sided ideals of \( R \) and \( S \), respectively, and are called the trace ideals of the MC.

**Remark 4.2.1.** If both Morita maps are epimorphisms then \( K(R, S) \) is said to be a projective Morita context (or PMC). If one of the MC maps is an epimorphism, then \( K(R, S) \) is said to be semi-projective Morita context or semi-PMC. If \( K(R, S) \) is a PMC of rings, then the rings \( R \) and \( S \) are said to be Morita similar (or Morita equivalent). Common properties shared by Morita similar rings are termed as Morita invariant. For instance, being prime or semiprime are Morita invariant, while being reduced, commutative, domain, division rings or Fields are not Morita invariant.

The proof of Remark 4.2.2 is rather elementary and is based on the fact that a group cannot be written as the set-theoretic union of its two proper subgroups.

**Remark 4.2.2.** Let \( R \) be a prime ring and \( S \) an additive subgroups of \( R \). Let \( f : S \rightarrow R \) and \( g : S \rightarrow R \) be additive functions such that \( f(s)Rg(s) = \{0\} \) for all \( s \in S \). Then either \( f(s) = 0 \) for all \( s \in S \), or \( g(s) = 0 \) for all \( s \in S \).

We begin with some results due to N.M. Muthana and S. K. Nauman [67] which will be used extensively to prove our results.

**Lemma 4.2.1.** Let \( R \) and \( S \) be rings of semi-PMC \( K(R, S) \) in which \( \tau_S \) is epic. If \( R \) is commutative and \( S \) is reduced, then \( S \) is also commutative.

**Lemma 4.2.2.** Let \( K(R, S) \) be a PMC of rings in which \( R \) is commutative. Then

(i) If \( S \) is a reduced ring, the \( R \) is also reduced and \( R \cong S \).

(ii) If \( S \) is a domain, then both \( R \) and \( S \) become isomorphic integral domains.

(iii) If \( S \) is division rings, then both \( R \) and \( S \) an isomorphic field.
Lemma 4.2.3 ([58, Theorem 3.1]). Let R be a prime ring of characteristic different from two and I be a nonzero square-closed Lie ideal of R. Suppose that \( \alpha, \beta \) are automorphisms of R. If R admits a generalized \((\alpha, \beta)\)-derivation \( F \) with an associated nonzero \((\alpha, \beta)\)-derivation \( d \) such that \( [F(x), x]_{\alpha, \beta} = 0 \), for all \( x \in I \), then \( I \subseteq Z(R) \).

Lemma 4.2.4 ([58, Lemma 2.4]). Let R be a prime ring of characteristic different from two and I be a nonzero square closed Lie ideal of R. Let \( \alpha, \beta \) be automorphisms of R. If \( [x, y]_{\alpha, \beta} = 0 \), for all \( x, y \in I \), then \( I \subseteq Z(R) \).

In view of Lemma 4.2.4, we get the following corollary:

Corollary 4.2.1. Let R be a prime ring of characteristic different from two and I be a nonzero ideal of R. Let \( \alpha, \beta \) be automorphisms of R. If \( [x, y]_{\alpha, \beta} = 0 \), for all \( x, y \in I \), then R is commutative.

We begin our discussion with the following lemmas.

Lemma 4.2.5. Let R be a prime ring of characteristic different from two and I be a nonzero ideal of R. Let \( \alpha, \beta \) be automorphisms of R. If \( [x, y]_{\alpha, \beta} \in Z(R) \), then R is commutative.

Proof. For any \( x, y \in I \), we have

\[
[x, y]_{\alpha, \beta} \in Z(R). \tag{4.2.1}
\]

Replacing \( x \) by \( xa(y) \) in (4.2.1), we get \( [x, y]_{\alpha, \beta}a(y) \in Z(R) \), this implies that \( [[x, y]_{\alpha, \beta}a(y), r] = 0 \) for all \( x, y \in I, r \in R \). Thus, application of (4.2.1), we find \( [x, y]_{\alpha, \beta}a(y), r] = 0 \). Again, replacing \( r \) by \( r(a(m)) \) and using the last expression, we get \( [x, y]_{\alpha, \beta}R[a(y), a(m)] = \{0\} \), for all \( x, y, m \in I \). Thus, by Remark 4.2.2, either \( a([y, m]) = 0 \) for all \( y, m \in I \), or \( [x, y]_{\alpha, \beta} = 0 \) for all \( x, y \in I \). In the first case, R is commutative by Lemma 2.2.3. In the second case, R is commutative by Corollary 4.2.1. \[\blacksquare\]
Lemma 4.2.6. Let \( R \) be a prime ring of characteristic different from two and \( I \) be a nonzero ideal of \( R \). Let \( \alpha, \beta \) be automorphisms of \( R \). If \( (x \circ y)_{\alpha, \beta} \in Z(R) \), then \( R \) is commutative.

Proof. For all \( x, y \in I \), we have

\[
(x \circ y)_{\alpha, \beta} \in Z(R). \tag{4.2.2}
\]

Replacing \( x \) by \( \beta(y)x \), we get \( \beta(y)(x \circ y)_{\alpha, \beta} \in Z(R) \), which implies that, 
\[
[\beta(y)(x \circ y)_{\alpha, \beta}, r] = 0, \quad \text{for all } r \in R.
\]
Hence, by equation (4.2.2), we get \([\beta(y), r](x \circ y)_{\alpha, \beta} = 0\), for all \( x, y \in I \), and \( r \in R \). Now replace \( r \) by \( \beta(m)r \), to get \([y, m]R\beta^{-1}((x \circ y)_{\alpha, \beta}) = \{0\}\), for all \( x, y, m \in I \). By Remark 4.2.2, we conclude that either \([y, m] = 0\) for all \( y, m \in I \), or \( \beta^{-1}((x \circ y)_{\alpha, \beta}) = 0\) for all \( x, y \in I \). In the first case, \( R \) is commutative by Lemma 2.2.3. On the second case, if \( \beta^{-1}((x \circ y)_{\alpha, \beta}) = 0 \) for all \( x, y \in I \), then \((x \circ y)_{\alpha, \beta} = 0\). Replacing \( y \) by \( ym \), and using the last expression, we get \( \beta(y)[x, m]_{\alpha, \beta} = 0 \). Again, replacing \( y \) by \( yr \) for all \( r \in R \), we get \( IR\beta^{-1}([x, m]_{\alpha, \beta}) = 0 \). Since \( I \) is nonzero ideal and \( R \) is prime which yields that \([x, m]_{\alpha, \beta} = 0\) for all \( x, m \in I \), and hence \( R \) is commutative by Lemma 4.2.5.

Theorem 4.2.1. Let \( K(R, S) \) be a semi-PMC in which the trace ideal \( I \) is nonzero and \( \tau_S \) is epic. Suppose that \( \alpha, \beta \) are automorphisms of \( R \), and \( R \) admits a generalized \((\alpha, \beta)\)-derivation \( F \) with associated \((\alpha, \beta)\)-derivation \( d \) such that \( F = 0 \) or \( d \neq 0 \) and \( R \) satisfies any one of the following conditions:

(i) \( F([x, y]) = (x \circ y)_{\alpha, \beta} \) for all \( x, y \in I \), or

(ii) \( F(x \circ y) = [x, y]_{\alpha, \beta} \) for all \( x, y \in I \).

Further, if \( R \) is a prime ring of characteristic different from two and \( S \) is reduced, then \( S \) is commutative.

Proof. (i) Let \( F \) be a generalized \((\alpha, \beta)\)-derivation of \( R \) such that

\[
F([x, y]) = (x \circ y)_{\alpha, \beta} \quad \text{for all } x, y \in I. \tag{4.2.3}
\]
If \( F = 0 \), then \((x \circ y)_{\alpha, \beta} = 0\) for all \( x, y \in I \), and hence \( R \) is commutative by Lemma 4.2.6. Since \( S \) is reduced, so by Lemma 4.2.1, \( S \) is commutative.

Therefore, we shall assume that \( d \neq 0 \). Replacing \( y \) by \( yx \) in (4.2.3) and using (4.2.3), we obtain

\[
[x, y]d(x) = -\beta(y)[x, x]_{\alpha, \beta} \quad \text{for all } x, y \in I. \tag{4.2.4}
\]

Again replace \( y \) by \( zy \) in (4.2.4), to get

\[
\beta([x, z]) \beta(y)d(x) = 0 \quad \text{for all } x, y, z \in I. \tag{4.2.5}
\]

This implies that, \([x, z]I\beta^{-1}(d(x)) = \{0\}\) for all \( x, z \in I \); and applying Lemma 2.3.1 and the fact that \((I, +)\) is not the union of its two proper subgroups shows that either \( d(x) = \{0\} \) or \([x, z] = 0\) for all \( x, z \in I \). If \( d(x) = 0 \) for all \( x \in I \), then \( d(xr) = 0 \) for all \( r \in R \). Hence, it follows that \( \beta(x)d(r) = 0 \) that is, \( \beta(I)Rd(r) = 0 \). Since \( \beta \) is an automorphism on \( R \) and \( I \neq 0 \), the primeness of \( R \) yields that \( d(r) = 0 \) for all \( r \in R \), a contradiction. On the other hand, if \([x, z] = 0\) for all \( x, z \in I \) then by Lemma 2.2.3, \( R \) is commutative and hence \( S \) is commutative by Lemma 4.2.1.

(ii) It is given that \( F \) is a generalized derivation of \( R \) such that \( F(x \circ y) = [x, y]_{\alpha, \beta} \) for all \( x, y \in I \). If \( F = 0 \), then \([x, y]_{\alpha, \beta} = 0\) for all \( x, y \in I \). Thus, by Lemma 4.2.5, \( R \) is commutative. Since \( S \) is reduced, by Lemma 4.2.1, \( S \) is commutative.

Hence, onward we shall assume that \( d \neq 0 \). For any \( x, y \in I \), we have

\[
F(x \circ y) = [x, y]_{\alpha, \beta}. \tag{4.2.6}
\]

Replacing \( y \) by \( yx \) in (4.2.6) and using (4.2.6), we get

\[
\beta(x \circ y)d(x) = \beta(y)[x, x]_{\alpha, \beta}. \tag{4.2.7}
\]

Now, replace \( y \) by \( xy \) in (4.2.7), to get \( \beta([x, z]) \beta(y)d(x) = 0 \) for all \( x, y, z \in I \). The last expression is same as the equation (4.2.5) and hence the result follows.
Theorem 4.2.2. Let $K(R, S)$ be a semi-PMC in which the trace ideal $I$ is nonzero and $\tau_S$ is epic. Suppose that $\alpha, \beta$ are automorphisms of $R$, and $R$ admits a generalized $(\alpha, \beta)$-derivation $F$ with associated $(\alpha, \beta)$-derivation $d$ such that $R$ satisfies any one of the following properties:

(i) $[F(x), y]_{\alpha, \beta} = (F(x) \circ y)_{\alpha, \beta}$ for all $x, y \in I$,

(ii) $F([x, y]) = [F(x), y]_{\alpha, \beta}$ for all $x, y \in I$,

(iii) $F(x \circ y) = (F(x) \circ y)_{\alpha, \beta}$ for all $x, y \in I$.

Further, if $R$ is a prime ring of characteristic different from two and $S$ is reduced, then $S$ is commutative.

Proof. (i) For any $x, y \in I$, we have

$$[F(x), y]_{\alpha, \beta} = (F(x) \circ y)_{\alpha, \beta}. \tag{4.2.8}$$

Replacing $y$ by $yx$ in (4.2.8) and using (4.2.8), we find that $\beta(y)[F(x), x]_{\alpha, \beta} = 0$ for all $x, y \in I$. This implies that, $\beta^{-1}([F(x), x]_{\alpha, \beta})I\beta^{-1}([F(x), x]_{\alpha, \beta}) = \{0\}$ for all $x \in I$. Thus, by Lemma 2.3.1 and Lemma 4.2.3, respectively, $R$ is commutative. Since $S$ is reduced, by Lemma 4.2.1, $S$ is commutative.

(ii) For any $x, y \in I$, we have $F([x, y]) = [F(x), y]_{\alpha, \beta}$. This can be rewritten as

$$F(x)\alpha(y) + \beta(x)d(y) - F(y)\alpha(x) - \beta(y)d(x) = [F(x), y]_{\alpha, \beta}. \tag{4.2.9}$$

Replacing $y$ by $yx$ in (4.2.9), we get

$$\beta([x, y])d(x) = \beta(y)[F(x), x]_{\alpha, \beta}. \tag{4.2.10}$$

Again replace $y$ by $zy$ in (4.2.10), to get

$$\beta(z)\beta([x, y]d(x) + \beta([x, z])\beta(y)d(x) = \beta(z)\beta(y)[F(x), y]_{\alpha, \beta}. \tag{4.2.11}$$
Combining (4.2.10) and (4.2.11), we find that $\beta([x, z])\beta(y)d(x) = 0$ for all $x, y, z \in I$, that is, $[x, z]I\beta^{-1}(d(x)) = \{0\}$ for all $x, y \in I$. Notice that the arguments given in the proof of Theorem 4.2.1 after equation (4.2.5), are still valid in the present situation and hence repeating the same process, we get the required result.

(iii) For any $x, y \in I$, we have $F(x \circ y) = (F(x) \circ y)_{\alpha, \beta}$. This can be rewritten as

$$F(x)\alpha(y) + \beta(x)d(y) + F(y)\alpha(x) + \beta(y)d(x) = (F(x) \circ y)_{\alpha, \beta}. \tag{4.2.12}$$

Replacing $y$ by $yx$ in (4.2.12), we get

$$\beta(x \circ y)d(x) = -\beta(y)[F(x), x]_{\alpha, \beta}. \tag{4.2.13}$$

Again replace $y$ by $zy$ in (4.2.13), to get $\beta([x, z])\beta(y)d(x) = 0$ for all $x, y, z \in I$, that is, $[x, z]I\beta^{-1}(d(x)) = \{0\}$ for all $x, z \in I$. Now, application of similar arguments as used after equation (4.2.5) in the proof of Theorem 4.2.1, yields the required result.

**Theorem 4.2.3.** Let $K(R, S)$ be a semi-PMC in which the trace ideal $I$ is nonzero and $\tau_S$ is epic. Suppose that $\alpha, \beta$ are automorphisms of $R$, and $R$ admits a generalized $(\alpha, \beta)$-derivations $F$ and $G$ with associated $(\alpha, \beta)$-derivations $d$ and $g$, respectively, such that either

(i) $F([x, y]) = [\alpha(y), G(x)]$ for all $x, y \in I$, or

(ii) $F(x \circ y) = (\alpha(y) \circ G(x))$ for all $x, y \in I$.

Further, if $R$ is a prime ring of characteristic different from two and $S$ is reduced, then $S$ is commutative.

**Proof.** (i) For any $x, y \in I$, we have

$$F([x, y]) = [\alpha(y), G(x)]. \tag{4.2.14}$$

Replacing $y$ by $yx$ in (4.2.14) and using (4.2.14), we get $\beta([x, y])d(x) = \alpha(y)[\alpha(x), G(x)]$ for all $x, y \in I$. Again replace $y$ by $zy$, to get $\beta([x, z])\beta(y)d(x) = 0$, that is,
Thus, for each \( x \in I \), by Lemma 2.3.1, we find that either \([x, z] = 0\) or \( \beta^{-1}(d(x)) = 0\). Now using similar arguments as used in the proof of Theorem 4.2.1, we get the required result.

(ii) For any \( x, y \in I \), we have

\[
F(x \circ y) = (\alpha(y) \circ G(x)).
\] (4.2.15)

Replace \( y \) by \( yx \) in (4.2.15), to get \( \beta(x \circ y)d(x) = \alpha(y)[\alpha(x), G(x)] \) for all \( x, y \in I \).

Again replacing \( y \) by \( zy \), we find that \( \beta([x, z])\beta(y)d(x) = 0 \), and hence \([x, z]I\beta^{-1}(d(x)) = \{0\}\). Now an application of similar arguments as used after equation (4.2.5) in the proof of Theorem 4.2.1, yields the required result.

**Corollary 4.2.2.** In each of the above Theorem 4.2.1 to Theorem 4.2.3, if \( R \) and \( S \) are Morita similar ring and \( S \) is division ring, then \( R \) and \( S \) are isomorphic field.

**Proof.** If \( R \) and \( S \) are Morita similar rings, then by Lemma 4.2.2 (a), \( R \) is also reduced and \( Z(R) \cong Z(S) \). Since \( R \) and \( S \) are commutative, \( R = Z(R) \) and \( S = Z(S) \) and hence \( R \cong S \). If \( S \) is division ring then \( S \) is a field. Since \( S \) is commutative division ring, by Lemma 4.2.2 (c), \( R \) and \( S \) are becomes isomorphic filed.

### 4.3 Generalized \((\alpha, \beta)\)-Derivations via Morita Contexts

This section devoted to study of commutativity of rings satisfying several conditions on prime rings by using two different ways. Firstly, we use derivation on ring and finally Morita theory has been used. More precisely, it is proved that if two rings are ingredients of a semi-projective Morita context, in which one is commutative and the other is reduced, then the reduced ring also become commutative. Some consequences related to domain, division rings, coalgebra, and cauchy module are stated and proved.
We begin our discussion with the following definitions:

**Definition 4.3.1 (R-Algebra).** Let $R$ be any ring. An $R$-algebra is an $(R, R)$-bimodule $M$ together with module morphisms (we will also call them linear maps):

$$
\mu : M \otimes_R M \longrightarrow M, \text{ and } \eta : R \longrightarrow A,
$$

such that

$$
M \otimes_R M \otimes_R M \xrightarrow{\mu \otimes 1_M} M \otimes_R M \xrightarrow{\mu} M, \text{ associativity}
$$

with $\mu \circ (\mu \otimes 1_M) = \mu \circ (1_M \otimes \mu)$, and

$$
R \xrightarrow{\eta \otimes 1_M} M \otimes_R M \xrightarrow{\mu} M, \text{ unit}
$$

with $\mu \circ (\eta \otimes 1_M) = \mu \circ (1_M \otimes \eta)$.

**Definition 4.3.2 (Co-Algebra).** Let $R$ be a commutative ring. An $R$-coalgebra is an $(R, R)$-bimodule $C$, with $R$-linear maps:

$$
\Delta : C \longrightarrow C \otimes_R C \text{ and } \varepsilon : C \longrightarrow R,
$$

such that

$$
C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{1_C \otimes \Delta} C \otimes_R C \otimes_R C, \text{ coassociativity}
$$

with $(1_C \otimes \Delta) \circ \Delta = (\Delta \otimes 1_C) \circ \Delta$ and

$$
C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{\varepsilon \otimes 1_C} R, \text{ counit}
$$

with $(1_C \otimes \varepsilon) \circ \Delta = 1_C = (\varepsilon \otimes 1_C) \circ \Delta$.

**Definition 4.3.3 (Bi-Algebra).** For a commutative ring $R$, an $R$-bialgebra $B$ is an $R$-module which is an algebra $(B, \mu, \eta)$ and a coalgebra $(B, \Delta, \varepsilon)$ such that $\Delta$ and $\varepsilon$ are algebra morphisms or, equivalently, $\mu$ and $\eta$ are coalgebra morphisms.
Definition 4.3.4 (Cauchy Module). Let $R$ and $S$ be rings and $M$ an $(R, S)$—bimodule. Then the dual of $M$ which is denoted by $M^* = \text{Hom}_R(M, R)$ is an $(S, R)$—bimodule, and for every Left $R$-module $L$ there is a canonical module morphism

$$\varphi^M_L : M^* \otimes_R L \longrightarrow \text{Hom}_R(M, L)$$

defined by

$$\varphi^M_L(m^* \otimes l)(m) = m^*(m)l \in L \text{ for all } m \in M, m^* \in M^*, l \in L.$$ 

If $\varphi^M_L$ is an isomorphism for each left $R$—module $L$, then $RMS$ is called a Cauchy module.

The proof of Remark (4.3.1) is clearly by using elementary properties of bimodules and definition of Morita contexts.

Remark 4.3.1. Let $R$ and $S$ be rings of a Morita context $K(R, S) = \{R, S, M, N, \mu_R, \tau_S\}$ such that $R$ is commutative and $R \cong S$. Then $M \otimes_R N \cong N \otimes_R M$ and the datum $\{R, M, N, \mu_R\}$ is Morita context where the map $\mu_R : M \otimes_R N \longrightarrow R$ satisfies the associative condition

$$\mu_R(m \otimes n)m_1 = m\mu_R(n \otimes m_1).$$

Lemma 4.3.1 ([1, Theorem 3.7]). Let $R$ be a commutative ring, $M$ and $N$ Cauchy $R$—modules. Then the datum $\{R, M, N, \mu_R\}$ is Morita context if and only if $M \otimes_R N$ is an $R$—bialgebra.
Theorem 4.3.1. Let $K(R, S)$ be a semi-PMC in which the trace ideal $I$ is nonzero and $\tau_S$ is epic. Suppose that $\alpha, \beta$ are automorphisms of $R$, and $R$ admits a generalized $(\alpha, \beta)$-derivation $F$ with associated $(\alpha, \beta)$-derivation $d$ with $\{0\} \neq d(Z(R)) \subseteq Z(R)$, such that either

(i) $[F(x), x]_{\alpha, \beta} \in Z(R)$ for all $x \in I$, or

(ii) $(F(x) \circ x)_{\alpha, \beta} \in Z(R)$ for all $x \in I$.

Further, if $R$ is a prime ring of characteristic different from two and $S$ is reduced, then $S$ is commutative.

Proof. (i) For all $x \in I$, we have

$$[F(x), x]_{\alpha, \beta} \in Z(R). \quad (4.3.1)$$

Linearizing (4.3.1), we get

$$[F(x), y]_{\alpha, \beta} + [F(y), x]_{\alpha, \beta} \in Z(R) \quad \text{for all } x, y \in I. \quad (4.3.2)$$

For any $z \in Z(R)$, replacing $y$ by $yz$ in (4.3.2), using (4.3.2), and Lemma 2.3.4, we get

$$\beta(y)[F(x), z]_{\alpha, \beta} + [\beta(y), x]_{\alpha, \beta} d(z) \in Z(R) \quad \text{for all } x, y \in I.$$

Again, replacing $y$ by $my$ and using the above expression, we get

$$\beta(m)[\beta(y)[F(x), z]_{\alpha, \beta} + [\beta(y), x]_{\alpha, \beta} d(z)] + \beta([m, x]) \beta(y) d(z) \in Z(R).$$

Thus, in particular, we have

$$[\beta(m)[\beta(y)[F(x), z]_{\alpha, \beta} + [\beta(y), x]_{\alpha, \beta} d(z)] + \beta([m, x]) \beta(y) d(z), \beta(m)] = 0.$$

This gives

$$[\beta([m, x]) \beta(y) d(z), \beta(m)] = 0 \quad \text{for all } x, y, m \in I. \quad (4.3.3)$$
Since \( R \) is prime and \( \{0\} \neq d(Z(R)) \subseteq Z(R) \), we find that \( \beta([m, x, y, m]) = 0 \) for all \( x, y, m \in I \), that is \( [m, x][y, m] + [m, x, m]y = 0 \). Again, replacing \( y \) by \( yx \) and using the above expression, we get \( [m, x][y, m] = 0 \), for all \( x, y, m \in I \). That is \( [m, x][m, x] = 0 \) for all \( m, x \in I \). Thus, Lemma 2.3.1, forces that \( [m, x] = 0 \), and hence \( R \) is commutative by Lemma 2.2.3. Since \( S \) is reduced, we get the required result by Lemma 4.2.1.

(ii) For all \( x \in I \), we have

\[
(F(x) \circ x)_{a, \beta} \in Z(R). \tag{4.3.4}
\]

Linearizing (4.3.4), we get

\[
(F(x) \circ y)_{a, \beta} + (F(y) \circ x)_{a, \beta} \in Z(R) \text{ for all } x, y \in I.
\]

For any nonzero \( z \in Z(R) \), replacing \( y \) by \( yz \) in the last expression and using Lemma 2.3.4, we get \(-\beta(y)[x, z]_{a, \beta} + (\beta(y) \circ x)_{a, \beta}d(z) + \beta(y)[d(z), \alpha(x)] \in Z(R)\). Since \( \{0\} \neq d(Z(R)) \subseteq Z(R) \), then

\[
-\beta(y)[x, z]_{a, \beta} + (\beta(y) \circ x)_{a, \beta}d(z) \in Z(R).
\]

Again replacing \( y \) by \( my \), we get

\[
\beta(m)\{-\beta(y)[x, z]_{a, \beta} + (\beta(y) \circ x)_{a, \beta}d(z)\} - [\beta(m), \beta(x)]\beta(y)d(z) \in Z(R) \text{ for all } x, y, m \in I.
\]

Thus, in particular, we have

\[
[\beta(m)\{-\beta(y)[x, z]_{a, \beta} + (\beta(y) \circ x)_{a, \beta}d(z)\} - [\beta(m), \beta(x)]\beta(y)d(z), \beta(m)] = 0.
\]

This gives

\[
[[\beta(m), \beta(x)]\beta(y)d(z), \beta(m)] = 0 \text{ for all } x, y, m \in I.
\]

Now using similar arguments as used in the proof of (i) after equation (4.3.3), we get the required result.
Theorem 4.3.2. Let $K(R, S)$ be a semi-PMC in which the trace ideal $I$ is nonzero and $\tau_S$ is epic. Suppose that $\alpha, \beta$ are automorphisms of $R$, and $R$ admits a generalized $(\alpha, \beta)$-derivation $F$ with associated $(\alpha, \beta)$-derivation $d$ with $\{0\} \neq d(Z(R)) \subseteq Z(R)$, such that $F = 0$ or $d \neq 0$ and $R$ satisfies any one of the following conditions:

(i) $F([x, y]) - [x, y]_{\alpha, \beta} \in Z(R)$ for all $x, y \in I$, or

(ii) $F(x \circ y) - (x \circ y)_{\alpha, \beta} \in Z(R)$ for all $x, y \in I$.

Further, if $R$ is a prime ring of characteristic different from two and $S$ is reduced, then $S$ is commutative.

Proof. (i) For all $x, y \in I$, we have

$$F([x, y]) - [x, y]_{\alpha, \beta} \in Z(R). \quad (4.3.5)$$

If $F = 0$, then $[x, y]_{\alpha, \beta} \in Z(R)$ for all $x, y \in I$, thus $R$ is commutative by Lemma 4.2.5. Since $S$ is reduced, by Lemma 4.2.1, $S$ is commutative.

Therefore, we shall assume that $d \neq 0$. Since $\alpha(z) \in Z(R)$ by Lemma 2.3.4, for any nonzero $z \in Z(R)$, replacing $y$ by $yz$ in (4.3.5) and using (4.2.5), we get

$$\beta([x, y])d(z) - \beta(y)[x, z]_{\alpha, \beta} \in Z(R). \quad (4.3.6)$$

Again, replacing $y$ by $my$ in (4.3.6), we get

$$\beta(m)\{\beta([x, y])d(z) - \beta(y)[x, z]_{\alpha, \beta}\} + \beta([x, m])\beta(y)d(z) \in Z(R) \text{ for all } x, y, m \in I.$$ 

Thus, in particular

$$[\beta(m)\{\beta([x, y])d(z) - \beta(y)[x, z]_{\alpha, \beta}\} + \beta([x, m])\beta(y)d(z), \beta(m)] = 0.$$ 

This implies $[\beta([x, m])\beta(y)d(z), \beta(m)] = 0$ for all $x, y, m \in I$. Notice that the arguments used in the proof of Theorem 4.3.1 after equation (4.3.3), are still valid in the present situation, and hence repeating the same process, we get the required result.
(ii) It is given that \( F \) is a generalized \((\alpha, \beta)\)-derivation on \( R \). If \( F = 0 \), then 
\((x \circ y)_{\alpha, \beta} \in Z(R)\), thus, \( R \) is commutative by Lemma 4.2.6. Since \( S \) is reduced, we get the required result by Lemma 4.2.1.

Therefore, we shall assume that \( d \neq 0 \). Now for all \( x, y \in I \), we have

\[
F(x \circ y) - (x \circ y)_{\alpha, \beta} \in Z(R). \tag{4.3.7}
\]

Since \( \alpha(z) \in Z(R) \) by Lemma 2.3.4, replacing \( y \) by \( yz \) for any \( z \in Z(R) \) in (4.3.7) and using (4.3.7), we get

\[
\beta(x \circ y)d(z) + \alpha(x, z)_{\alpha, \beta} \in Z(R). \tag{4.3.8}
\]

Again, replacing \( y \) by \( my \) in (4.3.8), we get

\[
\beta(m)\{\beta(x \circ y)d(z) + \alpha(x, z)_{\alpha, \beta}\} + \beta([[x, m]])\beta(y)d(z) \in Z(R) \text{ for all } x, y, m \in I.
\]

Thus, in particular

\[
[[\beta(m)\{\beta(x \circ y)d(z) + \alpha(x, z)_{\alpha, \beta}\} + \beta([[x, m]])\beta(y)d(z), \beta(m)]] = 0.
\]

Hence, we obtain \([\beta([[x, m]])\beta(y)d(z), \beta(m)] = 0 \) for all \( x, y, m \in I \). Using similar arguments as used in the proof of Theorem 4.3.1 that follows (4.3.3), we get the required result. \( \blacksquare \)

**Theorem 4.3.3.** Let \( K(R, S) \) be a semi-PMC in which the trace ideal \( I \) is nonzero and \( \tau_S \) is epic. Suppose that \( \alpha, \beta \) are automorphisms of \( R \), and \( R \) admits a generalized \((\alpha, \beta)\)-derivation \( F \) with associated \((\alpha, \beta)\)-derivation \( d \) with \( \{0\} \neq d(Z(R)) \subseteq Z(R) \), such that \( F = 0 \) or \( d \neq 0 \) and \( R \) satisfies any one of the following conditions:

(i) \((F(x) \circ F(y)) - [x, y]_{\alpha, \beta} \in Z(R) \text{ for all } x, y \in I,\)

(ii) \([F(x), d(y)] - [x, y]_{\alpha, \beta} \in Z(R) \text{ for all } x, y \in I,\)

(iii) \([F(x), F(y)] \in Z(R) \text{ for all } x, y \in I.\)
Further, if $R$ is a prime ring of characteristic different from two and $S$ is reduced, then $S$ is commutative.

**Proof.** (i) It is given that $F$ is a generalized $(\alpha, \beta)$-derivation. If $F = 0$, then $[x, y]_{\alpha, \beta} \in Z(R)$, for all $x, y \in I$, and hence $R$ is commutative by Lemma 4.2.5. Since $S$ is reduced, we get the required result by Lemma 4.2.1.

Therefore, we shall assume that $d \neq 0$. Now for all $x, y \in I$, we have

$$ (F(x) \circ F(y)) - [x, y]_{\alpha, \beta} \in Z(R). \quad (4.3.9) $$

For any $z \in Z(R)$, replace $y$ by $yz$, in (4.3.9) and use Lemma 2.3.4, we get

$$ (F(x) \circ \beta(y))d(z) - \beta(y) [x, z]_{\alpha, \beta} \in Z(R). $$

Again, replacing $y$ by $my$ in the last expression, we get

$$ \beta(m)\{(F(x) \circ \beta(y))d(z) - \beta(y) [x, z]_{\alpha, \beta}\} + [F(x), \beta(m)]\beta(y)d(z) \in Z(R). $$

Thus, in particular

$$ [\beta(m)\{(F(x) \circ \beta(y))d(z) - \beta(y) [x, z]_{\alpha, \beta}\} + [F(x), \beta(m)]\beta(y)d(z), \beta(m)] = 0, $$

and hence

$$ [[F(x), \beta(m)]\beta(y)d(z), \beta(m)] = 0 \text{ for all } x, y, m \in I. \quad (4.3.10) $$

Since $R$ is prime and $\{0\} \neq d(Z(R)) \subseteq Z(R)$, we find that

$$ [[F(x), \beta(m)], \beta(m)]\beta(y) + [F(x), \beta(m)][\beta(y), \beta(m)] = 0. $$

For any $t \in I$ replacing $y$ by $yt$ in the last expression, we get $[F(x), \beta(m)]\beta(y)\beta([t, m]) = 0$ and hence $\beta^{-1}([F(x), \beta(m)])y[t, m] = 0$. That is $\beta^{-1}([F(x), \beta(m)])I[t, m] = 0$ for all $x, m, t \in I$. Thus, by Lemma 2.3.1, either $\beta^{-1}([F(x), \beta(m)]) = 0$ for all $x, m \in I$ or $[t, m] = 0$ for all $t, m \in I$. If $[t, m] = 0$, then $R$ is commutative by Lemma 2.2.3. Now since $S$ is reduced, we get the required result by Lemma 4.2.1. On the other hand, if $\beta^{-1}([F(x), \beta(m)]) = 0$, then $[F(x), \beta(m)] = 0$ for all $x, m \in I$. For any nonzero
$z \in Z(R)$ replacing $x$ by $xz$ in the last expression and using Lemma 2.3.4, we get \\
$\beta([x,m])d(z) = 0$, for all $x,m \in I$. Since $\{0\} \neq d(Z(R)) \subseteq Z(R)$ and $R$ is prime, we find that $\beta([x,m]) = 0$ for all $x,m \in I$, and hence $R$ is commutative by Lemma 2.2.3. Since $S$ is reduced, by Lemma 4.2.1, $S$ is commutative.

(ii) It is given that $F$ is a generalized $(\alpha, \beta)$-derivation. If $F = 0$, then $[x,y]_{\alpha,\beta} \in Z(R)$, for all $x,y \in I$, and thus, $R$ is commutative by Lemma 4.2.5. Since $S$ is reduced then by Lemma 4.2.1, $S$ is commutative.

Therefore, we shall assume that $d \neq 0$. For all $x,y \in I$, we have

$$[F(x),d(y)] - [x,y]_{\alpha,\beta} \in Z(R). \quad (4.3.11)$$

For any nonzero $z \in Z(R)$ replacing $y$ by $yz$ in (4.3.11) and using Lemma 2.3.4, we get

$$[F(x),\beta(y)]d(z) - \beta(y)[x,z]_{\alpha,\beta} \in Z(R) \text{ for all } x,y \in I.$$ 

Replacing $y$ by $my$ in the last expression, we find

$$\beta(m){[F(x),\beta(y)]d(z) - \beta(y)[x,z]_{\alpha,\beta} + [F(x),\beta(m)]\beta(y)d(z) \in Z(R).}$$

Hence, in particular

$$[\beta(m){[F(x),\beta(y)]d(z) - \beta(y)[x,z]_{\alpha,\beta} + [F(x),\beta(m)]\beta(y)d(z)\beta(m)] = 0.$$ 

This implies $[[F(x),\beta(m)]\beta(y)d(z), \beta(m)] = 0$ for all $x,y,m \in I$. Now using the same arguments as used after equation (4.3.10), we get the required result.

(iii) For all $x,y \in I$, we have

$$[F(x),F(y)] \in Z(R). \quad (4.3.12)$$

For any nonzero $z \in Z(R)$, replacing $y$ by $yz$ in (4.3.12), using (4.3.12) and Lemma 2.3.4, we get

$$[F(x),\beta(y)]d(z) \in Z(R).$$
Since \( \{0\} \neq d(Z(R)) \subseteq Z(R) \) and \( R \) is prime, we get

\[
[F(x), \beta(y)] \in Z(R), \text{ for all } x, y \in I.
\]

Again since \( \{0\} \neq d(Z(R)) \subseteq Z(R) \), then for any nonzero \( z \in Z(R) \), replacing \( x \) by \( xz \) in the above expression and using Lemma 2.3.4, we find \( \beta([x, y])d(z) \in Z(R) \) for all \( x, y \in I \). But \( \{0\} \neq d(Z(R)) \subseteq Z(R) \) and \( R \) is prime, so we get \( \beta([x, y]) \in Z(R) \) that is, \( [x, y] \in Z(R) \) for all \( x, y \in I \), and hence \( R \) is commutative by Lemma 2.2.3.

Since \( S \) is reduced, so by Lemma 4.2.1, \( S \) is commutative.

**Theorem 4.3.4.** Let \( K(R, S) \) be a semi-PMC in which the trace ideal \( I \) is nonzero and \( \tau_S \) is epic. Suppose that \( \alpha, \beta \) are automorphisms of \( R \), and \( R \) admits a generalized \((\alpha, \beta)\)-derivations \( F \) and \( G \) with associated \((\alpha, \beta)\)-derivations \( d \) and \( g \), respectively, with \( \{0\} \neq g(Z(R)) \subseteq Z(R) \), such that \( F = 0 \) (or \( G = 0 \)) or \( d \neq 0 \) (or \( g \neq 0 \)) and \( R \) satisfy the condition \( [F(x), G(y)] - [x, y]_{\alpha, \beta} \in Z(R) \) for all \( x, y \in I \). Further, if \( R \) is a prime ring of characteristic different from two and \( S \) is reduced, then \( S \) is commutative.

**Proof.** It is given that \( F \) and \( G \) are generalized \((\alpha, \beta)\)-derivations on \( R \). If \( F = 0 \) (or \( G = 0 \)) then \( [x, y]_{\alpha, \beta} \in Z(R) \), for all \( x, y \in I \), and hence by Lemma 4.2.5, \( R \) is commutative. Since \( S \) is reduced, we get the required result by Lemma 4.2.1.

Therefore, we shall assume that \( g \neq 0 \). Then for all \( x, y \in I \), we have

\[
[F(x), G(y)] - [x, y]_{\alpha, \beta} \in Z(R). \tag{4.3.13}
\]

For any nonzero \( z \in Z(R) \) replacing \( y \) by \( yz \), in (4.3.13), using (4.3.13) and Lemma 2.3.4, we get

\[
[F(x), \beta(y)]g(z) - \beta(y)[x, z]_{\alpha, \beta} \in Z(R). \tag{4.3.14}
\]

Again, replacing \( y \) by \( my \) in (4.3.14), we get

\[
\beta(m)\{[F(x), \beta(y)]g(z) - \beta(y)[x, z]_{\alpha, \beta}\} + [F(x), \beta(m)]\beta(y)g(z) \in Z(R).
\]
Thus, in particular

\[
\beta(m)\{[F(x), \beta(y)]g(z) - \beta(y)[x, z]_{\alpha, \beta}\} + [F(x), \beta(m)]\beta(y)g(z), \beta(m) = 0
\]

and hence \([F(x), \beta(m)]\beta(y)g(z), \beta(m) = 0\) for all \(x, y, m \in I\). Using the same arguments as used in the proof of Theorem 4.3.3 after equation (4.3.10), we get the required result.

In view of these results, we get the following corollaries:

**Corollary 4.3.1.** In each of the above, from Theorem 4.3.1 to Theorem 4.3.4, if \(K(R, S)\) be a PMC in which \(M\) and \(N\) are Cauchy modules, then \(M \otimes_R N\) is an \(R\)-bialgebra if and only if the datum \(K(R, S)\) is Morita context.

**Proof.** Suppose that \(M \otimes_R N\) is an \(R\)-bialgebra, since \(R\) is commutative and \(R \cong S\) from the above theorems, then by Remark 4.3.1 and Lemma 4.3.1, respectively, the datum \(\{R, M, N, \mu_R\}\) is Morita context. On the other hand, if the datum \(K(R, S)\) is Morita context, \(R\) is commutative and \(R \cong S\) by the above theorems, then by Remark 4.3.1 the datum \(\{R, M, N, \mu_R\}\) is Morita context, thus by Lemma 4.3.1 \(M \otimes_R N\) is an \(R\)-bialgebra.

**Corollary 4.3.2.** By the same argument as Corollary 4.2.2, in the cases, from Theorem 4.3.1 to Theorem 4.3.4, if \(R\) and \(S\) are Morita similar rings, then by Lemma 4.2.2 (a), \(R\) is also reduced and \(Z(R) \cong Z(S)\). Since \(R\) and \(S\) are commutative, \(R = Z(R)\) and \(S = Z(S)\) and hence \(R \cong S\). If \(S\) is division ring then \(S\) is a field. Since \(S\) is commutative division ring, by Lemma 4.2.2 (c), \(R\) and \(S\) are becomes isomorphic filed.
Corollary 4.3.3. Let $K(R, S)$ be a PMC in which rings $R$ and $S$ are equipped with multiplicative identity $1$. Then $Z(R) \cong Z(S)$. If the conditions of either Theorem 4.3.1, or of Theorem 4.3.2, or of Theorem 4.3.3, or of Theorem 4.3.4 are satisfied, then $S \cong Z(R)$. Hence $R$ can be treated as an $S$-Algebra. Moreover in this case $S$ becomes prime, as being prime is a Morita invariant property.

Corollary 4.3.4. Let $K(R, S)$ be a semi-PMC in which $\tau_S$ is epic. Then the generalized matrix ring $T = \begin{bmatrix} R & M \\ N & S \end{bmatrix}$ and $S$ are Morita equivalent [[68], Theorem 2.1]. Hence, trivially, in this case, if the conditions of either of the Theorems 4.3.1, 4.3.2, 4.3.3, or 4.3.4, are satisfied, then $Z(T) \cong S$. 