Chapter 2

Warped product submanifolds of Kaehler and nearly Kaehler manifolds

In view of the applications of warped product manifolds it is worthwhile to explore them in known spaces. Such investigations were initiated by B.Y. Chen when he considered CR-submanifolds as warped product submanifolds in Kaehler manifolds. Subsequently, B. Sahin extended the study to semi-slant warped products in Kaehler manifolds. From these studies it was revealed that in the class of semi-slant submanifolds, the only non-trivial warped product submanifolds in a Kaehler manifold are CR-warped product submanifolds.

One of the next steps of extending the study is to consider warped product submanifolds with one of the factors a holomorphic submanifold and the second factor not necessarily slant. These submanifolds are generic in the sense of B.Y. Chen [14]. Our aim in this chapter is to investigate the existence of these warped product spaces in Kaehler and nearly Kaehler manifolds.

2.1 Generic warped product submanifolds of a Kaehler manifold

As non-trivial warped product CR-submanifolds are non-existent in a Kaehler manifold (cf. Theorem 1.4.4) B. Sahin investigated semi-slant submanifolds of a Kaehler manifold as warped product submanifolds and established the following:

**Theorem 2.1.1** [53]. Let $\tilde{M}$ be a Kaehler manifold. Then there do not exist warped product submanifolds $M = M_0 \times_f M_T$ in $\tilde{M}$ such that $M_0$ is a proper slant submanifold and $M_T$ is a holomorphic submanifold of $\tilde{M}$. 

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Theorem 2.1.2 [53]. Let $\tilde{M}$ be a Kaehler manifold. Then there do not exist warped product submanifolds $M = M_T \times_f M_\theta$ in $\tilde{M}$ such that $M_T$ is a holomorphic submanifold and $M_\theta$ a proper slant submanifold of $\tilde{M}$.

Note. Theorem 2.1.1 is valid for all $\theta \in [0, \pi/2]$ in view of Theorem 1.4.4, whereas there are many examples of CR-warped product submanifolds $M_T \times_f M_\perp$ in Kaehler manifolds which are not CR-products strengthening the fact that Theorem 2.1.2 is valid for the case of proper semi-slant warped product submanifolds (cf. [19]).

As a step forward, in this section we study proper generic submanifolds of a Kaehler manifold $\tilde{M}$, i.e., warped product submanifolds of the type $M = M_T \times_f M_0$, and $M = M_0 \times_f M_T$, where $M_T$ is a holomorphic submanifold and $M_0$ is any non-totally real submanifold of $\tilde{M}$.

In view of the Remark 1.4.1, throughout, the tangent bundle $TM_T$ of $M_T$ is denoted by $D$ and the tangent bundle $TM_0$ of $M_0$ is denoted by $D^0$.

Theorem 2.1.3. There do not exist non-trivial proper generic warped product submanifold of a Kaehler manifold $\tilde{M}$.

Proof. Let $M_0$ be submanifold of a Kaehler manifold $\tilde{M}$ which is not totally real and $M_T$ a holomorphic submanifold of $\tilde{M}$. Then, first we consider a warped product $M = M_0 \times_f M_T$ in $\tilde{M}$. By Theorem 1.4.1

$$\nabla_X Z = \nabla_Z X = (Z \ln f) X$$

(2.1.1)

for each $X \in D$ and $Z \in D^0$. Thus

$$g(X, \nabla_X Z) = 0.$$  

(2.1.2)

Using Gauss formula and the Kaehler condition, the above equation is written as

$$g(JX, \tilde{\nabla}_X JZ) = 0.$$  

On decomposing $JZ$ into tangential and normal parts and applying Gauss and Weingarten formula, the above equation takes the form

$$g(JX, \nabla_X PZ) - g(h(JX, JX), FZ) = 0,$$
which on using formula (2.1.1) yields
\[ g(h(JX, JX), FZ) = (PZlnf)\|X\|^2. \] (2.1.3)

Now, on making use of formulae (1.3.3), (1.3.12) and (2.1.1), we obtain
\[ (PZlnf)X - (Zlnf)PX = A_{FZ}X + th(X, Z), \]
which on replacing \( X \) by \( JX \) becomes
\[ (PZlnf)JX + (Zlnf)X = A_{FZ}JX + th(JX, Z). \] (2.1.4)

On taking product with \( Y \in D \) in both sides of the above equation while making use of the fact \( tN \in D^0 \) for each \( N \in T^\perp M \), we get
\[ (PZlnf)g(JX, Y) + (Zlnf)g(X, Y) = g(h(JX, Y), FZ). \] (2.1.5)

Interchanging \( X \) and \( Y \) in the above equation and adding the resulting equation in (2.1.5) while taking account of Theorem 1.3.3, we obtain
\[ (Zlnf)g(X, Y) = g(h(JX, Y), FZ). \]

Thus, for \( Y = JX \), the above equation gives
\[ g(h(JX, JX), FZ) = 0. \] (2.1.6)

Now, by (2.1.3) and (2.1.6), it follows that
\[ PZlnf = 0, \]
for each \( Z \in D^0 \). As \( D^0 \) is a proper purely real distribution, the vector field \( PZ \) can not be zero for each \( Z \in D^0 \), therefore \( f \) is constant and \( M \) is a Riemannian product of \( M_0 \) and \( M_T \).

Since non-trivial warped product CR-submanifold of a Kaehler manifold are non-existent, it follows from the above that generic warped product submanifolds of the form \( M_0 \times_f M_T \) are non-existent in a Kaehler manifold. Hence Theorems 1.4.4 and 2.1.1 are extended for generic warped product submanifolds.
Now, let $M$ be a warped product submanifold of the type $M_T \times_f M_0$ in $\tilde{M}$. Then for any $Z \in D^0$ and $X \in D$ by Theorem 1.4.1

$$\nabla_X Z = \nabla_Z X = (Xlnf)Z.$$  \hfill (2.1.7)

By formulae (1.3.3) and (2.1.7) we have

$$(\tilde{\nabla}_X P)Z = 0,$$

and

$$(\tilde{\nabla}_Z P)X = (PXlnf)Z - (Xlnf)PZ.$$  

The above equations in view of the formula (1.3.12) yield

$$A_{FZ}X + th(X, Z) = 0,$$

and

$$(PXlnf)Z - (Xlnf)PZ = th(X, Z).$$

Thus,

$$A_{FZ}X = (Xlnf)PZ - (PXlnf)Z.$$  

Taking product with $PZ$ in both sides of the above equation gives

$$g(h(X, PZ), FZ) = Xlnf\|PZ\|^2.$$  \hfill (2.1.8)

On the other hand, as $\tilde{M}$ is Kaehler

$$g(\tilde{\nabla}_{PZ}JZ, JX) = g(\nabla_{PZ}Z, X)$$

$$= -(Xlnf)g(PZ, Z)$$

$$= 0.$$  

Now, writing $JZ = PZ + FZ$ and using Gauss, Weingarten formulae, we obtain from the above equation that

$$g(\nabla_{PZ}PZ, JX) - g(A_{FZ}PZ, JX) = 0.$$  

Replacing $X$ by $JX$ and using (2.1.7), we get

$$g(h(X, PZ), FZ) = -(Xlnf)\|PZ\|^2.$$  \hfill (2.1.9)
From (2.1.8) and (2.1.9), we have

$$(Xlnf) ||PZ||^2 = 0.$$ 

Since $M$ is a proper generic warped product, $PZ$ can not be zero for each $Z \in D^0$, therefore from (2.1.8) and (2.1.9), we obtain

$$Xlnf = 0.$$ 

That is, $f$ is constant on $M_T$ which proves that the warped product $M_T \times_f M_0$ is trivial and the proof of the Theorem is complete.

**Remark.** The above Theorem is proved under the assumption that $PD^0 \neq 0$. However, as the warped product submanifold $M_\perp \times_f M_T$ is trivial, the generic warped product submanifolds of the form $M_0 \times_f M_T$ are trivial in Kaehler manifolds. On the other hand as the proper generic warped product submanifolds of the form $M_T \times_f M_0$ are non-existent, the only non-trivial generic warped product submanifolds of a Kaehler manifold are CR-warped product submanifolds.

### 2.2 Warped product submanifolds of a nearly Kaehler manifold

Having proved the non-existence of non-trivial proper generic warped product submanifolds in Kaehler manifolds, it is natural to explore these warped product submanifolds in a more general setting of nearly Kaehler manifolds. We start our investigations by exploring doubly warped products and twisted warped products in nearly Kaehler manifolds.

Let $M_1$ and $M_2$ be two non-trivial submanifolds of an almost Hermitian manifold $\tilde{M}$ such that $M = f_\perp M_1 \times f_\parallel M_2$ is a doubly warped product submanifold of $\tilde{M}$. If one of the factors of $M$ is a holomorphic submanifold of $\tilde{M}$, then $M$ is called a doubly warped product generic submanifold of $\tilde{M}$. A doubly warped product CR-submanifold $f_\perp M_T \times f_\parallel M_\perp$ is a particular case of doubly warped product generic submanifold. Our aim in this section is to explore these submanifolds in the setting
of nearly Kaehler manifolds.

Some important formulas are obtained in the following Lemma:

**Lemma 2.2.1.** Let $M = f_0 M_T \times f_T M_0$ be a doubly warped product generic submanifold of a nearly Kaehler manifold $\tilde{M}$. Then we have

1. $g(\nabla_Z X, W) = g(\nabla_X Z, W) = (X \ln f_T) g(Z, W)$ and $g(\nabla_Z X, Y) = g(\nabla_X Z, Y) = (Z \ln f_0) g(X, Y),$
2. $g(h(X, Y), FZ) = (PZ \ln f_0) g(X, Y),$
3. $g(\mathcal{P}_X Y, Z) = (Z \ln f_0) g(J X, Y),$
4. $g(h(PX, Z), FZ) = (X \ln f_T) ||Z||^2$

for each $X, Y \in D$ and $Z, W \in D^0$.

**Proof.** From formula (1.4.4), we have

$$\nabla_X Z = \nabla_Z X = (X \ln f_T) Z + (Z \ln f_0) X.$$

Taking product with $W \in D^0$ on both sides of the above equation we obtain statement (i). Similarly, taking product with $Y \in D$ in the above formula, we obtain

$$g(\nabla_Z X, Y) = g(\nabla_X Z, Y) = (Z \ln f_0) g(X, Y).$$  \hspace{1cm} (2.2.1)

Now, by formula (1.3.8) we have

$$g(\mathcal{P}_X Y, Z) = g((\nabla_X P)Y, Z) + g(h(X, Y), FZ).$$

Applying (1.3.3) in the right hand side, we get

$$g(\mathcal{P}_X Y, Z) = g(\nabla_X PY, Z) - g(P \nabla_X Y, Z) + g(h(X, Y), FZ).$$

Taking account of the orthogonality of the distributions $D$ and $D^0$, the above equation takes the form

$$g(\mathcal{P}_X Y, Z) = -g(PY, \nabla_X Z) - g(Y, \nabla_X PZ) + g(h(X, Y), FZ),$$

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which, on using formula (2.2.1) gives

\[ g(\mathcal{P}_X Y, Z) = (Z\ln f_0)g(PX, Y) - (PZ\ln f_0)g(X, Y) + g(h(X, Y), FZ). \]  

(2.2.2)

\( \mathcal{P}_X Y \) is skew symmetric in \( X \) and \( Y \), whereas \( g \) and \( h \) are symmetric. Taking account of these facts and comparing symmetric and skew symmetric terms in (2.2.2), we get

\[ g(h(X, Y), FZ) = (PZ\ln f_0)g(X, Y), \]  

(2.2.3)

and

\[ g(\mathcal{P}_X Y, Z) = (Z\ln f_0)g(JX, Y). \]  

(2.2.4)

Thus, the statement (ii) and (iii) are proved.

Now, from (1.3.8) we have

\[ \mathcal{P}_X Z = (\nabla_X P)Z - A_{FZ} X - th(X, Z) \]

and

\[ \mathcal{P}_Z X = (\nabla_Z P)X - th(X, Z). \]

As \( \mathcal{P}_X Z + \mathcal{P}_Z X = 0 \) on \( \bar{M} \), using this fact and formula (1.3.3), we have

\[ 0 = \nabla_X PZ + \nabla_Z PX - P(\nabla_X Z + \nabla_Z X) - A_{FZ} X - 2th(X, Z). \]

Making use of (2.2.1) in above equation, we get

\[ (PZ\ln f_0)X - (Z\ln f_0)PX + (PX\ln f_T)Z - (X\ln f_T)PZ = A_{FZ} X + 2th(X, Z). \]

Taking inner product with \( Z \) in both sides of the above equation gives

\[ (PX\ln f_T)\|Z\|^2 = g(h(X, Z), FZ) + 2g(th(X, Z), Z) \]

\[ = g(h(X, Z), FZ) - 2g(h(X, Z), FZ) \]

\[ = -g(h(X, Z), FZ). \]

Replacing \( X \) by \( PX \) in the above equation while noticing that \( JX = PX \) for each \( X \in D \), we obtain

\[ g(h(PX, Z), FZ) = X\ln f_T\|Z\|^2. \]

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This proves part (iv) and completes the proof of the Lemma.

Next, we prove the following:

**Theorem 2.2.1.** There does not exist a proper doubly warped product CR-submanifold in a nearly Kaehler manifold \( \bar{M} \).

**Proof.** Let \( M = f_{\perp} M_{T} \times f_{T} M_{\perp} \) be a doubly warped product CR-submanifold of a nearly Kaehler manifold \( \bar{M} \). As \( PZ = 0 \) for each \( Z \in D_{\perp} \), by formula (2.2.3), \( g(h(X,Y), FZ) = 0 \) for each \( X,Y \in D \). That means \( h(X,Y) \in \nu \). Now, by using formulae (1.3.9), (1.3.4) and the fact that \( Q_{U}V + Q_{V}U = 0 \) for each \( U,V \in T\bar{M} \), we obtain

\[
F(\nabla X Y + \nabla Y X) = h(X, PY) + h(Y, PX) - 2fh(X, Y). 
\]

In view of the observation that \( h(X,Y) \in \nu \), the right hand side of the last equation lies in \( \nu \), whereas, \( F(\nabla X Y + \nabla Y X) \in F(D^{0}) \). Now, by the orthogonality of \( FD^{0} \) and \( \nu \), it follows that \( F(\nabla X Y + \nabla Y X) = 0 \). Further, as \( D \) is integrable on \( M \), it can be deduced from the above observation that \( \nabla X Y \in D \). That is, \( M_{T} \) is totally geodesic in \( M \). Moreover, as \( M_{\perp} \) is totally umbilical in \( M \) with mean curvature \( \nabla \ln f_{T} \) and \( f_{T} \) being a function on \( M_{T} \), \( Z(\ln f_{T}) = 0 \), we deduce that \( M_{\perp} \) is an extrinsic sphere in \( M \). Hence, by Theorem 1.4.3 we get that \( M \) is locally isometric to a warped product \( M_{T} \times f_{T} M_{\perp} \). That means \( M \) is a (single) warped product submanifold.

The above Theorem can further be refined by proving the following Theorem.

**Theorem 2.2.2.** There does not exist a proper doubly warped product submanifold of a nearly Kaehler manifolds \( \bar{M} \) with one of the factors a holomorphic submanifold of \( \bar{M} \).

**Proof.** Let \( M = f_{0} M_{T} \times f_{T} M_{0} \) be a doubly warped product submanifold of a nearly Kaehler manifolds \( \bar{M} \) such that \( M_{T} \) is a holomorphic submanifold and \( M_{0} \) is an arbitrary submanifold of \( \bar{M} \). As \( \bar{M} \) is nearly Kaehler, it follows that \( \mathcal{P}_{X} JX = 0 \) for
any \( X \in D \). Using this property in formula (iii) of Lemma 2.2.1, we get

\[
(Z \ln f_0)\|X\|^2 = 0. 
\]  

(2.2.5)

As \( N_T \) is a non-trivial submanifold, it follows from (2.2.5) that \( Z \ln f_0 = 0 \) for each \( Z \in D^0 \). That is, \( f_0 \) is constant. This proves that \( M \) is simply a generic warped product submanifold, i.e., \( M = M_T \times f M_0 \).

Having established the non-existence of proper doubly warped product generic submanifolds in nearly Kaehler manifolds, we are left with two possible generic submanifolds as (single) warped product submanifolds, namely, warped products of the types (a) \( M_T \times f M_0 \) and (b) \( M_0 \times f M_T \).

However, it follows from (2.2.5) that the non-trivial warped product submanifolds of the types (b) are non-existent in nearly Kaehler manifolds. Hence, as an extension of Theorem 2.1.1 and Theorem 2.1.3 to the setting of nearly Kaehler manifolds, we may state

**Theorem 2.2.3.** Let \( \tilde{M} \) be a nearly Kaehler manifold and \( M = M_0 \times f M_T \) a warped product submanifold of \( \tilde{M} \) where \( M_T \) is a holomorphic and \( M_0 \) is an arbitrary submanifold of \( \tilde{M} \). Then \( M \) is a generic product.

As an immediate consequence of the above Theorem we have

**Corollary 2.2.1.** There does not exist non-trivial warped product CR-submanifold in a nearly Kaehler manifold.

*Note.* The above corollary is an extension of the non-existence Theorem for warped product CR-submanifold in Kaehler manifolds (cf. Theorem 1.4.4).

However, generic warped product submanifolds of the type (a), namely submanifolds of the type \( M_T \times f M_0 \) do exist in nearly Kaehler manifolds. We quote here an example of CR-warped product submanifold of \( S^6 \).
Example 2.2.1 [57]. Let \( \{e_0, e_i (1 \leq i \leq 7)\} \) be the canonical basis of the Cayley division algebra on \( \mathbb{R}^8 \) over \( \mathbb{R} \) and \( \mathbb{R}' \) be the subspace of \( \mathbb{R}^8 \) generated by the purely imaginary Cayley numbers \( e_i (1 \leq i \leq 7) \). Then

\[
S^6 = \{y_1 e_1 + y_2 e_2 + \ldots + y_7 e_7 : y_1^2 + y_2^2 + \ldots + y_7^2 = 1\} \subset \mathbb{R}^7,
\]
is a unit 6-sphere admitting a nearly Kaehler structure \((J, g, \nabla)\). Now suppose that

\[
S^2 = \{y = (y_2, y_4, y_6) \in \mathbb{R}^3 : y_2^2 + y_4^2 + y_6^2 = 1\},
\]
is a unit 2-sphere and

\[
S^1 = \{z = e^{it}, t \in \mathbb{R}\},
\]
is a unit circle. Considering the mapping

\[
\psi : S^2 \times S^1 \to S^6
\]
defined by

\[
\psi(y, z) = \psi((y_2, y_4, y_6), e^{it}) = (y_2 \cos t)e_2 - (y_2 \sin t)e_3 + (y_4 \cos 2t)e_4 + (y_4 \sin 2t)e_5 + (y_6 \cos t)e_6 + (y_6 \sin t)e_7,
\]
for \( y = (y_2, y_4, y_6) \in S^2 \) and \( z = e^{it} \in S^1, t \in \mathbb{R} \). Then the tangent bundle of submanifold is given by

\[
Z_1 = -y_2 \sin t e_2 - y_2 \cos t e_3 - 2y_4 \sin 2t e_4 + 2y_4 \cos 2t e_5 - y_6 \sin t e_6 - y_6 \cos t e_7,
\]

\[
Z_2 = y_6 \cos 2t e_4 + y_6 \sin 2t e_5 - y_4 \cos t e_6 - y_4 \sin t e_7,
\]

\[
Z_3 = y_6 \cos 2t e_4 + y_6 \sin 2t e_5 - y_4 \cos t e_6 - y_4 \sin t e_7.
\]

Where \( D = \text{span}\{Z_2, Z_3\} \) and \( D^\perp = \text{span}\{Z_1\} \). Moreover, we can drive that \( D \) is integrable. Denoting the integral manifolds of \( D \) and \( D^\perp \) by \( M_T \) and \( M_\perp \) respectively, then the induced metric tensor is

\[
ds^2 = (y_6^2 + y_4^2)dy_2^2 + y_2y_4dy_2dy_4 + (y_6^2 + y_4^2)dy_4^2 + (1 + 3y_4^2)dt^2 + g_{M_T} + (1 + 3y_4^2)g_{M_\perp}.
\]

Thus it follows that \( S^2 \times_f S^1 \) is a warped product submanifold of \( S^6 \) with warping
function $f = \sqrt{(1 + 3y_4^2)}$.

In the last Theorem of this section, we will study the existence of proper doubly twisted product generic submanifolds in nearly Kaehler manifolds.

**Theorem 2.2.4.** Let $\bar{M}$ be a nearly Kaehler manifold. Then there do not exist doubly twisted product generic submanifolds of $\bar{M}$ which are not (single) twisted product generic submanifolds in the form $f_1M_T \times f_1M_0$ such that $M_T$ is holomorphic and $M_0$ is an arbitrary submanifold of $\bar{M}$.

**Proof.** Let $M = f_2M_T \times f_1M_0$ be a doubly twisted product submanifold of $\bar{M}$. Then for any $X, Y \in D$ and $Z \in D^0$, using formula 2.2.1, we obtain

$$g(\nabla JX, JY) = (Zlnf_2)g(X, Y).$$  \hfill (2.2.6)

Using (1.1.7), (1.4.4) and (1.2.2), the left hand side is simplified as

$$g(\nabla JX, JY) = -g(Z, \nabla JX JY)$$

$$= -g(Z, (\nabla YJ)JX + J\nabla JX Y)$$

$$= g(Z, (\nabla YJ)JX) - g(Z, J\nabla JX Y)$$

$$= g(JZ, (\nabla YJ)X) + g(PZ, \nabla JX Y)$$

$$+ g(FZ, h(JX, Y))$$

$$= g(JZ, (\nabla YJ)X) - (PZlnf_2)g(JX, Y)$$

$$+ g(FZ, h(JX, Y)).$$

Thus, we have

$$(Zlnf_2)g(X, Y) = g(JZ, (\nabla YJ)X) - (PZlnf_2)g(JX, Y) + g(h(JX, Y), FZ).$$

Interchanging $X$ and $Y$ and adding the resulting equation into the above while making use of Theorem 1.3.4 and the fact that $\bar{M}$ is nearly Kaehler, we obtain

$$g(h(JX, Y), FZ) = (Zlnf_2)g(X, Y).$$  \hfill (2.2.7)
Now, again using the fact that $\tilde{M}$ is nearly Kaehler and formula (1.4.4.), we get

$$g(\Box_X JX, JZ) = -(Zlnf_2)\|X\|^2.$$  \hspace{1cm} (2.2.8)

On the other hand taking account of (1.2.2) and (1.3.1), the left hand side of the above equation reduces to

$$g(\nabla_X JX, PZ) + g(h(X, JX), FZ).$$

The first term in the above expression vanishes by (1.4.4), whereas the second term on making use of (2.2.7) reduces to $Zlnf_2\|X\|^2$, i.e.,

$$g(\Box_X JX, JZ) = Zlnf_2\|X\|^2.$$  \hspace{1cm} (2.2.9)

Now, by (2.2.8) and (2.2.9) we get

$$Zlnf_2\|X\|^2 = 0$$

for all $Z \in D^0$. That is, $f_2$ depends only on the points of $M_T$. Hence $M$ is a twisted generic product of the form $M_T \times_f M_0$.

Therefore, we finally see that doubly twisted product generic submanifolds are (single) generic twisted product submanifolds in nearly Kaehler manifolds.