PREFACE

The theory of submanifolds of an almost Hermitian manifold has been one of the most interesting topics in differential geometry. The almost complex structure on an almost Hermitian manifold gives rise to two classes of submanifolds, the holomorphic submanifolds (or, complex submanifolds) and the totally real submanifolds. The study of complex submanifolds of a Kaehler manifold with a differential geometric point of view (that is, with emphasis on the Riemannian metric) was initiated by E.Clabi and others (cf [10], [64]). On the other hand, the study of totally real submanifolds from differential geometric point of view was initiated in the early 1970's (cf [12], [13], [44], [60], [65] and [66] etc). Since then many differential geometers have contributed many interesting results in the subject. A. Bejancu [2], [3] provided a single setting to study these submanifolds by introducing CR-submanifolds of Kaehler manifolds. In fact, a CR-submanifold is endowed with two orthogonal complementary distributions such that one is holomorphic and the other is totally real. Later, various generalizations of the notion were obtained namely generic submanifolds, slant and semi-slants submanifolds in Kaehler as well as in contact settings (cf. [14], [17] and [52] etc.). A special case of CR-submanifold is a CR-product, i.e., a Riemannian product of a holomorphic and a totally real submanifold of the ambient manifold.

Warped product spaces form a more general class of manifolds than the Riemannian product of manifolds and have got many applications in physics. In fact warped product manifolds provide excellent setting to model space-time near black holes or bodies with large gravitational force (cf.[1], [23], [30], [31] and [58]). For instance, the best relativistic model of the Schwarzschild space-time that describes the out space around a massive star or a black hole is given as a warped product. The idea of warped product manifolds emerged when R.L.Bishop and B.O'Neill [6] explored manifolds of negative curvature. Recently, the warped product spaces are being investigated for their physical applications as well as with differential geometric point of view. Some other generalizations of warped products, namely multiply warped product and twisted products are recently defined and have found lot of applications too (cf. [18], [47], [50], [54] and [59] etc.) The conditions of space like
boundaries in the multiply warped product space-times are studied by Relativist. From a physical point of view these space-times are interesting, first because they include classical examples of space-time: when \( n=1 \), they are generalized Robertson-Walker space-times, standard model of cosmology; when \( n=2 \), the intermediate zone of Reissner-Nordstrom space-time and interior of Schwarzschild space-time appear as particular case.

In view of the applications of warped product spaces, the geometric study of these spaces assumes significance. The aim of the thesis is to study warped product spaces with extrinsic geometric point of view i.e., as submanifold of some known manifolds. Our target spaces are Kaehler, nearly Kaehler and locally conformal Kaehler manifolds. In fact, this approach has proved effective ever since S.Nolker [51] gave an explicit description of the warped product representation of Euclidean spaces. However, more recently, the studies of warped product manifolds with extrinsic geometric point of view are intensified after the impulse given by B.Y.Chen [19], [20] when he initiated the study of CR-submanifolds as warped products in a Kaehler manifold.

The thesis comprises of five chapters. Each chapter is divided into various sections. The mathematical relations obtained in the text have been labeled with double decimal numbering. The first figure denotes the chapter, seconds represents the section and the third represents the number of definition, equation, lemma or the Theorem as the case may be. For example Theorem 4.3.5 refers to the fifth Theorem of section three in the fourth chapter.

The first chapter is introductory and serves the purpose of fixing the notations and developing the basic concepts keeping in view of the pre-requisites of the subsequent chapters and also to make the thesis self contained.

The first investigative result on warped product manifolds as submanifolds of a known space was obtained by B.Y.Chen [19] when he proved that there does not exist a non-trivial warped product CR-submanifold, i.e., a submanifold of the type \( M_1 \times_f M_T \) in a Kaehler manifold is simply a CR-product. However, non-trivial
warped product submanifold of the type $MT \times f M_\perp$ (known as CR-warped product submanifold) do exist in Kaehler manifolds. Subsequently, B.Sahin [48] explored semi-slant warped products, i.e., submanifolds of the types $M_\theta \times f MT & M_T \times f M_\theta$ in a Kaehler manifold and proved that these warped products are also non-existent for $\theta \in (0, \pi/2)$. To extend the study, in the first section of chapter 2, we have considered non-trivial warped product submanifolds of the type $MT \times f M_0$ and $M_0 \times f M_T$, where $M_0$ is any non-totally real submanifold of the underlying Kaehler manifold and established the non-existence of these warped product submanifolds, thereby generalize the non-existence results of B.Y.Chen and B.Sahin. The study of generic warped products is further extended in section 2 by taking a more general setting of nearly Kaehler manifolds. K. Sekigawa [57], while studying CR-submanifolds of $S^6$ proved that $S^6$ does not admit CR-product submanifolds. This paved way to the investigations of more general product submanifolds of $S^6$. The most natural candidate seems to be a warped product submanifold. He constructed an example of a CR-warped product of $S^6$ to strengthen the existence of such submanifolds in nearly Kaehler manifolds. With regard to the existence of warped product submanifolds of $S^6$, N.Ejiri [24] proved that there exist countably many immersions of $S^1 \times S^{n-1}$ such that the induced metric on it is a warped product metric of constant scalar curvature $n(n-1)$. In section 2, we have analyzed a few generalized warped product submanifolds in nearly Kaehler manifolds to see exactly what warped product submanifolds, one can obtain in nearly Kaehler manifolds. The contents of this chapter are published in Filomat Vol.22(1), 139-144, (2008) and Toyama Math. Journal, Vol.32 (2009).

It is observed at the end of section 1 of chapter 2 that CR-warped products are the only non-trivial warped product submanifolds in the class of generic warped product submanifolds of a Kaehler manifold. Since extrinsic geometry heavily depends on the second fundamental form of the immersion of the given manifold into an ambient manifold, B.Y.Chen [19] obtained an inequality for the squared norm of the second fundamental form of CR-warped product submanifolds in a Kaehler manifold as well as in a real and complex space forms. V.A.Khan and K.A.Khan [39] obtained a similar inequality for the squared norm of the second fundamental form of a semi-slant warped product submanifold in a nearly Kaehler manifold. In chapter 3, we have considered semi-slant warped product submanifolds of a generalized
complex space form and obtained an inequality which generalizes the inequalities of Chen [19] and Khan et.al [39]. In establishing the required inequality, several important formulas are derived for a semi-slant submanifolds as well as semi-slant warped product submanifolds of a nearly Kaehler manifold. The contents of the chapter are accepted for publication in Balkan Journal of Geometry and its Applications.

K.Matsumoto [47], V.Bonanzinga [9] and M.I.Munteanu [50] extended the study of CR-warped product submanifolds to the setting of locally conformal Kaehler manifolds. Various important formulae and geometric properties are obtained by them. Since warped products have got applications in Physics, it is important to see under what conditions a CR-submanifold is isometric to a CR-warped product submanifold. In chapter 4, we have worked out mathematical conditions in terms of the canonical structures $P$ and $F$ as well as in terms of the shape operator under which a CR-submanifold of an l.c.K manifold becomes isometric to a CR-warped product submanifold. In particular, when the ambient manifold is a Kaehler manifold, our conditions reduce to the conditions obtained by Chen [15], [19] for the setting of CR-submanifolds of Kaehler manifolds. The contents of the chapter are accepted for publication in Bulletin Korean Mathematical Society.

As a step forward, in chapter 5, we studied generic warped product submanifolds of l.c.K manifolds. A Theorem is proved that provides a mechanism of constructing non-trivial proper generic warped product submanifolds in l.c.K.manifolds. An example is constructed based on the described technique. Some important geometric consequences are drawn. Moreover, an inequality for the squared norm of the second fundamental form is also obtained. It can be noted that the results obtained in chapter 5 can be specialized for CR-warped product submanifolds in l.c.K as well as in Kaehler manifolds and they agree with already existing results of these settings. The contents of the chapter are published in Acta Mathematica Sciential, 30B(5) (2010).

In the end of the thesis, references have been given which by no means are comprehensive but mention only the papers and books referred to in the main body of the thesis.