Chapter II

ON CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS CONDITIONED ON A PAIR OF NON-ADJACENT RECORDS

1. Introduction
Characterization of distributions through conditional expectation of record values have been considered using $E[h(X_{u(j)})|X_{u(r)} = x] = a h(x) + b$ and $E[h(X_{u(j)})|X_{u(s)} = y] = a_1 h(y) + b_1$ for $1 \leq r < j < s$ among others by Nagaraja (1977, 1988), Franco and Ruiz (1996, 1997), Wesolowski and Ahsanullah (1997), López-Blázquez and Moreno-Rebollo (1997), Ahsanullah and Wesolowski (1998), Dembińska and Wesolowski (2000), Wu and Lee (2001), Raqab (2002), Athar et al. (2003), Ahsanullah and Raqab (2004), Gupta and Ahsanullah (2004), Wu (2004) and Khan and Athar (2008). Bairamov et al. (2005) have characterized the exponential distribution and its monotone transforms, conditioned on a pair of adjacent record values. Here, in this chapter we have extended the results of Bairamov et al. (2005) and have characterized a family of continuous distributions through conditional expectation of record values based on a pair of non-adjacent records. The approach here is entirely different as given in Yanev et al. (2008).
2. Characterization of probability distributions

Conditional pdf of \( X_u(j) \) given \( X_u(r) = x \) and \( X_u(s) = y \), \( 1 \leq r < j < s \) is

\[
f_{X_u(j) \mid X_u(r), X_u(s)}(t \mid x, y) = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!} \times \frac{[-\log \bar{F}(t) + \log \bar{F}(x)]^{j-r-1} [-\log \bar{F}(y) + \log \bar{F}(t)]^{s-j-1} f(t)}{[-\log \bar{F}(y) + \log \bar{F}(x)]^{s-r-1} [\bar{F}(t)]} \quad \alpha < x < t < y < \beta \quad (2.1)
\]

**Theorem 2.1:** Let \( X_u(1), X_u(2), \ldots \) be the upper records from a continuous population with the df \( F(x) \) and the pdf \( f(x) \) over the support \((\alpha, \beta)\) and \( h(t) \) be a monotonic and differentiable function of \( t \). If for two consecutive values \( r \) and \( r+1, \, 1 \leq r < j-1 < s \),

\[g_{r+1} (x, y) = E[h(X_u(j)) \mid X_u(l) = x, X_u(s) = y], \, l = r, r+1, \quad (2.2)\]
exist, where \( g(\cdot) \) is a finite and differentiable function of \( x \),

then

\[
\frac{\log \bar{F}(x)}{\log \bar{F}(y)} = 1 - e^{-I_1}, \quad (2.3)
\]

where

\[
A_1(x, y) = \frac{\partial}{\partial x} g_{r+1} (x, y) \frac{\partial}{\partial x} \left[ \frac{g_{r+1, s}(x, y)}{(s-r-1) \left[ g_{r, s}(x, y) - g_{r+1, s}(x, y) \right]} \right],
\]

and

\[
I_1 = \int_{\alpha}^{x} A_1(t, y) dt
\]
**Proof:** We have,

\[ g^{j \mid r, s}(x, y) = E[h(X_{U(j)}) \mid X_{U(r)} = x, X_{U(s)} = y] \]

Therefore in view of (2.1),

\[ g^{j \mid r, s}(x, y)[B(x, y)]^{s-r-1} = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!} \int_0^y h(t)[B(x,t)]^{j-r-2}[B(t,y)]^{s-j-1} \frac{f(t)}{F(t)} dt \]

where

\[ B(x, y) = [-\log F(y) + \log F(x)] \]

Differentiating both the sides w.r.t. \( x \), we have

\[ \frac{\partial}{\partial x} g^{j \mid r, s}(x, y)[B(x, y)]^{s-r-1} = g^{j \mid r, s}(x, y) (s-r-1) \frac{f(x)}{F(x)} [B(x, y)]^{s-r-2} \]

\[ = -(s-r-1) \frac{f(x)}{F(x)} [B(x, y)]^{s-r-2} g^{j \mid r+1, s}(x, y), \]

implying that,

\[ \frac{f(x)}{[F(x)] B(x, y)} = \frac{\frac{\partial}{\partial x} g^{j \mid r, s}(x, y)}{(s-r-1) [g^{j \mid r, s}(x, y) - g^{j \mid r+1, s}(x, y)]} = A_1(x, y) \]

Integrating both the sides w.r.t. \( x \) over \( (x, y) \), we get

\[ \log B(x, y) \bigg|_\alpha^x = - \int_\alpha^x A_1(t, y) dt \]

and hence the result.
Remark 2.1: From (2.3), we have

\[
\left[ 1 + \frac{\log \overline{F}(x)}{\log \overline{F}(y)} \right]^{s-r-1} = \exp \left[ - \frac{x}{\alpha} \int A(t, y) \, dt \right],
\]

(2.6)

where \( A(x, y) = \frac{A(x, y)}{(s-r-1)} \).

As \( y \to \beta, -\log \overline{F}(y) \to \infty \), therefore in the limiting case as \( s \to \infty \) and \( y \to \beta \), L.H.S. of (2.6) tends to \( \overline{F}(x) \), implying that

\[
\overline{F}(x) = \exp \left[ - \frac{x}{\alpha} \int A(t) \, dt \right],
\]

where,

\[
A(x) = \frac{\partial}{\partial x} [g_{j, r}(x)] - \frac{g_{j, r+1}(x)}{g_{j, r}(x) - g_{j, r+1}(x)},
\]

and \( g_{j, r}(x) = E[h(X_{u(j)}) \mid X_{u(r)} = x] \),

as obtained by Khan and Athar (2009).

Theorem 2.2: Let \( X_{u(1)}, X_{u(2)}, \ldots \) be the upper records from a continuous population with the df \( F(x) \) and the pdf \( f(x) \) over the support \((\alpha, \beta)\) and \( h(t) \) be a monotonic and differentiable function of \( t \). If for two consecutive values \( s-1 \) and \( s, 1 \leq r < j+1 < s \),

\[
g_{j, r, l}(x, y) = E[h(X_{u(j)}) \mid X_{u(r)} = x, X_{u(l)} = y], \quad l = s, s-1,
\]

(2.7)

exist, where \( g(\cdot) \) is a finite and differentiable function of \( y \),

then,

\[
\frac{1 + \log \overline{F}(y)}{1 + \log \overline{F}(x)} = 1 - e^{-l_2}, \quad x < q,
\]

(2.8)
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where \( q \in (\alpha, \beta) \) such that \(- \log F(q) = 1\),

\[
A_2(x, y) = \frac{\partial}{\partial y} g_{j \mid r, s}(x, y) \left( \frac{q}{s-r-1} \frac{g_{j \mid r, s}(x, y) - g_{j \mid r, s-1}(x, y)}{[F(y)]^s} \right),
\]

and \( I_2 = \int_{q}^{y} A_2(x, t) \, dt \).

**Proof:** We have,

\[
g_{j \mid r, s}(x, y)[B(x, y)]^{s-r-1} = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!} \int_{x}^{y} h(t)[B(x, t)]^{j-r-1}[B(t, y)]^{s-j} \frac{f(t)}{[F(t)]^s} \, dt
\]

Differentiating both the sides w.r.t. \( y \), we have

\[
\frac{\partial}{\partial y} g_{j \mid r, s}(x, y)[B(x, y)]^{s-r-1} + g_{j \mid r, s}(x, y)(s-r-1) \frac{f(y)}{[F(y)]^s} [B(x, y)]^{s-r-2}
\]

\[
= \frac{(s-r-1)!}{(j-r-1)!(s-j-2)!} \int_{x}^{y} h(t)[B(x, t)]^{j-r-1}[B(t, y)]^{s-j-2} \frac{f(t)}{[F(t)]^s} \, dt
\]

\[
= (s-r-1) \frac{f(y)}{[F(y)]^s} [B(x, y)]^{s-r-2} g_{j \mid r, s-1}(x, y)
\]

implying that,

\[
\frac{f(y)}{[F(y)] B(x, y)} = - \frac{\partial}{\partial y} g_{j \mid r, s}(x, y) \left( \frac{q}{s-r-1} \frac{g_{j \mid r, s}(x, y) - g_{j \mid r, s-1}(x, y)}{[F(y)]^s} \right) = -A_2(x, y)
\]

Integrating both the sides w.r.t. \( y \) over \((q, y)\), we have

\[
\log B(x, y) \big|_{q}^{y} = - \int_{q}^{y} A_2(x, t) \, dt,
\]

and hence the Theorem.
Remark 2.2: By convention $X_u(0) = \alpha$, therefore we have:

If $g_{j|s}(y) = E[h(X_u(j)) \mid X_u(s) = y]$ then

$$-\log \bar{F}(y) = \exp \left\{ - \int_0^y B(t) \, dt \right\},$$

a result given by Khan and Athar (2009), conditioned on a single record statistic, where

$$B(t) = \frac{g'_{j|s}(t)}{(s-1) [g_{j|s}(t) - g_{j|s-1}(t)]}.$$

In Theorem 2.1 we have assumed that $g(x, y)$ is differentiable w.r.t. $x$, whereas in Theorem 2.2, $g(x, y)$ is assumed to be differentiable w.r.t. $y$. However if $g(x, y)$ is assumed to be differentiable w.r.t. both $x$ and $y$, we can combine Theorem 2.1 and Theorem 2.2 as:

**Theorem 2.3:** Under the conditions of Theorem 2.1 and Theorem 2.2

$$\bar{F}(x) = \exp \left[ - e^{l_1} \left( \frac{1}{e^{l_1} + e^{l_2}} - 1 \right) \right], \quad (2.9)$$

and,

$$\bar{F}(y) = \exp \left[ - e^{l_1} \left( \frac{1}{e^{l_1} + e^{l_2}} - 1 \right) \right] \quad \text{(2.10)}$$

**Proof:** In view of (2.3) and (2.8), we get

$$e^{l_1} = \frac{\log \bar{F}(y)}{\log \bar{F}(y) - \log \bar{F}(x)} \quad \text{and} \quad e^{l_1 + e^{l_2}} - 1 = \frac{1}{\log \bar{F}(y) - \log \bar{F}(x)},$$

and hence the Theorem.
Now we deduce Theorems for specific family of distributions.

**Corollary 2.1:**

\[
g_{j|r,s}(x, y) = E[h(X_{u(j)}) \mid X_{u(r)} = x, X_{u(s)} = y] = \frac{(s - j)h(x) + (j - r)h(y)}{(s - r)}
\]

(2.11)

if and only if the df is

\[
F(x) = 1 - e^{-[ah(x) + b]}, \quad \alpha \leq x \leq \beta,
\]

(2.12)

when \( h(t) \) is a non-decreasing and differentiable function of \( t \) for \( a > 0 \), and the df is

\[
G(x) = e^{-[ah(x) + b]}, \quad \alpha \leq x \leq \beta
\]

(2.13)

when \( h(t) \) is a non-increasing and differentiable function of \( t \) for \( a > 0 \).

**Proof:** First we prove that (2.12) implies (2.11).

Let \( C_{r, j, s} = \frac{(s - r - 1)!}{(j - r - 1)!(s - j - 1)!} \),

then

\[
g_{j|r,s}(x, y) = \frac{C_{r, j, s}}{[-\log F(y) + \log F(x)]}
\]

\[
\times \int_{x}^{y} h(t) \left[ \frac{[-\log F(t) + \log F(x)]^{j-r-1}}{[-\log F(y) + \log F(x)]} \right] \frac{f(t)}{[F(t)]} \, dt
\]

\[
= \frac{C_{r, j, s}}{[h(y) - h(x)]} \int_{x}^{y} h(t) \left[ \frac{\{h(t) - h(x)\}^{j-r-1}}{\{h(y) - h(x)\}} \right] \frac{h'(t)\, dt}{[h(y) - h(x)]}
\]

Set \( u = \frac{h(t) - h(x)}{h(y) - h(x)} \) to get
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\[ g_{j\mid r, s}(x, y) = C_{r, j, s} \int_0^1 [h(x) + u(h(y) - h(x))] u^{j-r-1} (1-u)^{s-j-1} \, du \]

\[ = \frac{(s-j)h(x) + (j-r)h(y)}{(s-r)} \]

To prove (2.11) implies (2.12), note that

\[ A_1(x, y) = \frac{\partial}{\partial x} g_{j\mid r, s}(x, y) \]

\[ = \frac{h'(x)}{(s-r-1) [g_{j\mid r, s}(x, y) - g_{j\mid r+1, s}(x, y)]} \]

Thus in view of the Theorem 2.1,

\[ \frac{\log \bar{F}(x)}{\log \bar{F}(y)} = \frac{a h(x) + b}{a h(y) + b} \]

That is

\[ \log \bar{F}(x) = K [ah(x) + b], \]

where \( K \) is a normalizing constant. If \( h(x) \) is a non-decreasing then \( 0 < -\log \bar{F}(x) < \infty \), and therefore there exist a \( q \) such that \( -\log \bar{F}(q) = 1 \).

Hence

\[ F(x) = 1 - e^{-[ah(x)+b]} \]

Similarly for \( h(x) \) non-increasing, \( -\log G(q) = 1 \), and hence

\[ G(x) = e^{-[ah(x)+b]}. \]

Theorem 2.2 and Theorem 2.3 may also be used to prove the Corollary 2.1.
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Table 2.1: Examples based on the distribution \( F(x) = 1 - e^{-[a h(x)+b]} \)

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( F(x) )</th>
<th>( a )</th>
<th>( b )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power function</td>
<td>( \nu^{-p}x^p ) \quad 0 &lt; x &lt; \nu</td>
<td>1</td>
<td>( p\log\nu )</td>
<td>( -\log(\nu^p - x^p) )</td>
</tr>
<tr>
<td>Pareto</td>
<td>( 1 - \nu^{-p}x^{-p} ) \quad \nu &lt; x &lt; \infty</td>
<td>( p )</td>
<td>( -p\log\nu )</td>
<td>( \log x )</td>
</tr>
<tr>
<td>Beta of the I kind</td>
<td>( 1 - (1 - x)^p ) \quad 0 &lt; x &lt; 1</td>
<td>( p )</td>
<td>0</td>
<td>( -\log(1 - x) )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( 1 - e^{-\theta x} ) \quad 0 &lt; x &lt; \infty</td>
<td>( \theta )</td>
<td>0</td>
<td>( x )</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>( 1 - e^{-\theta x^2} ) \quad 0 &lt; x &lt; \infty</td>
<td>( \theta )</td>
<td>0</td>
<td>( x^2 )</td>
</tr>
<tr>
<td>Weibull</td>
<td>( 1 - e^{-\theta x^p} ) \quad 0 &lt; x &lt; \infty</td>
<td>( \theta )</td>
<td>0</td>
<td>( x^p )</td>
</tr>
<tr>
<td>Extreme value II</td>
<td>( 1 - \exp[-e^{\theta x}] ) \quad -\infty &lt; x &lt; \infty</td>
<td>1</td>
<td>0</td>
<td>( e^{\theta x} )</td>
</tr>
<tr>
<td>Burr Type XII</td>
<td>( 1 - (1 + \theta x^p)^{-m} ) \quad 0 &lt; x &lt; \infty</td>
<td>( m )</td>
<td>0</td>
<td>( \log(1 + \theta x^p) )</td>
</tr>
</tbody>
</table>
Table 2.2: Examples based on the distribution \( G(x) = e^{-[ah(x)+b]} \)

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( G(x) )</th>
<th>( a )</th>
<th>( b )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power function</td>
<td>( x^{-p}p )</td>
<td>( p )</td>
<td>( p \log v )</td>
<td>( -\log x )</td>
</tr>
<tr>
<td></td>
<td>( 0 &lt; x &lt; v )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inverse Weibull</td>
<td>( e^{-\theta x^{-\rho}} )</td>
<td>( \theta )</td>
<td>( 0 )</td>
<td>( x^{-\rho} )</td>
</tr>
<tr>
<td></td>
<td>( 0 &lt; x &lt; \infty )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gumbel</td>
<td>( \exp[-ce^{-x}] )</td>
<td>( c )</td>
<td>( 0 )</td>
<td>( e^{-x} )</td>
</tr>
<tr>
<td></td>
<td>( -\infty &lt; x &lt; \infty )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logistic</td>
<td>( (1+e^{-x})^{-c} )</td>
<td>( c )</td>
<td>( 0 )</td>
<td>( \log(1+e^{-x}) )</td>
</tr>
<tr>
<td></td>
<td>( -\infty &lt; x &lt; \infty )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Remark 2.3: Bairamov et al. (2005) have given three examples when \( h(x) \) is non-decreasing for adjacent records at \( j = n, \ r = n-1 \) and \( s = n+1 \). Their results are essentially as given in the Table 2.1 for Weibull, beta of the first kind and Pareto distributions. Their remaining two examples are for Gumbel and logistic distributions when \( h(x) \) is non-increasing, as given in the Table 2.2.

3. Results for adjacent records

In this section, we have deduced the results for adjacent records from Section 2. Ruiz and Navarro (1996) have also characterized distributions when conditioned records are adjacent but our approach is entirely different.
To this end, we define,

\[ g_{r,s}(x, y) = E[h(X_{u(r)}) \mid X_{u(r)} = x, X_{u(s)} = y] = h(x) \]  \hspace{1cm} (3.1) 

and

\[ g_{r+1,r+2}(x, y) = E[h(X) \mid x \leq X \leq y] = m(x, y) \]  \hspace{1cm} (3.3) 

Therefore at \( j = r + 1 \) and \( s = r + 2 \)

\[ A_1(x, y) = \frac{\partial}{\partial x} \frac{m(x, y)}{[m(x, y) - h(x)]} \]  \hspace{1cm} (3.4) 

and

\[ A_2(x, y) = \frac{\partial}{\partial y} \frac{m(x, y)}{[h(y) - m(x, y)]} \]  \hspace{1cm} (3.5) 

Now, we deduce the characterizing results obtained in Section 2 for adjacent records in Corollary 3.1, 3.2 and 3.3.

**Corollary 3.1:**

\[ m(x, y) = \frac{c}{a(c + 1)} \frac{[ah(y) + b]^{c+1} - [ah(x) + b]^{c+1}}{[ah(y) + b]^c - [ah(x) + b]^c} - \frac{b}{a}, \quad c \neq -1 \]  \hspace{1cm} (3.6) 

if and only if

\[ -\log \bar{F}(x) = [ah(x) + b]^c, \]  \hspace{1cm} (3.7) 

where \( a, b, c \) and \( h(x) \) are so chosen that \( F(x) \) is a df.

**Proof:** For \( -\log \bar{F}(x) = [ah(x) + b]^c \), we have

\[ m(x, y) = \int_x^y \frac{h(t)f(t)}{[\bar{F}(t)][-\log \bar{F}(y) + \log \bar{F}(x)]} dt \]
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\[ \frac{ac}{B(x, y)} \int_{x}^{y} [ah(t) + b]^{c-1} h(t) h'(t) \, dt , \]

where

\[ B(x, y) = [-\log \bar{F}(y) + \log \bar{F}(x)] = [ah(y) + b]^c - [ah(x) + b]^c \]

That is,

\[ m(x, y) = \frac{c}{B(x, y)} \int_{ah(x)+b}^{ah(y)+b} u^{c-1} \left( \frac{u-b}{a} \right) \, du \]

implying that,

\[ m(x, y) = \frac{c}{a(c+1)} \frac{[ah(y) + b]^{c+1} - [ah(x) + b]^{c+1}}{[ah(y) + b]^c - [ah(x) + b]^c} - \frac{b}{a} , \quad c \neq -1 \]

Now to prove (3.6) implies (3.7), we have

\[ A_1(x, y) = \frac{\frac{\partial}{\partial x} m(x, y)}{m(x, y) - h(x)} = \frac{ach'(x)[ah(x) + b]^{c-1}}{[ah(y) + b]^c - [ah(x) + b]^c} \]

Integrating both the sides \( w.r.t. \) \( x \), we get

\[ \frac{x}{a} \int_{t}^{y} A_1(t, y) \, dt = \log \left[ 1 - \frac{[ah(x) + b]^c}{[ah(y) + b]^c} \right] \]

Therefore in view of the Theorem 2.1,

\[ \frac{\log \bar{F}(x)}{\log \bar{F}(y)} = \frac{[ah(x) + b]^c}{[ah(y) + b]^c} \]

Thus

\[ -\log \bar{F}(x) = [ah(x) + b]^c. \]
Corollary 3.2:

\[
m(x, y) = \frac{c}{a(c + 1)} \frac{[ah(x) + b]^{c+1} - [ah(y) + b]^{c+1}}{[ah(x) + b]^c - [ah(y) + b]^c} - \frac{b}{a}, \quad c \neq -1
\]

if and only if

\[
1 - [-\log F(x)] = [ah(x) + b]^c,
\]

where \( a, b, c \) and \( h(x) \) are such that \( F(x) \) is a df.

**Proof:** For \( 1 - [-\log F(x)] = [ah(x) + b]^c \), we have

\[
m(x, y) = \frac{\int_{x}^{y} \frac{h(t) f(t)}{F(t)} \left[ 1 - \log F(y) \right] - \left[ 1 - \log F(x) \right]}{1 - \log F(y) - \left[ 1 - \log F(x) \right]}
\]

\[
= \frac{ac}{B(x, y)} \int_{x}^{y} [ah(t) + b]^{c-1} h(t) h'(t) \, dt
\]

where

\[
B(x, y) = \left[ \left[ 1 - \log F(y) \right] - \left[ 1 - \log F(x) \right] \right] = [ah(y) + b]^c - [ah(x) + b]^c
\]

That is,

\[
m(x, y) = \frac{c}{B(x, y)} \frac{ah(y) + b}{ah(x) + b} \int_{ah(x) + b}^{ah(y) + b} u^{c-1} \left( \frac{u - b}{a} \right) \, du
\]

\[
= \frac{c}{a(c + 1)} \frac{[ah(x) + b]^{c+1} - [ah(y) + b]^{c+1}}{[ah(x) + b]^c - [ah(y) + b]^c} - \frac{b}{a}.
\]

Now to prove (3.8) implies (3.9), proceeding as in Corollary 3.1, we get

\[
A_2(x, y) = \frac{\partial}{\partial y} m(x, y) = \frac{ac h'(y) [ah(y) + b]^{c-1}}{[h(y) - m(x, y)] \left[ [ah(x) + b]^c - [ah(y) + b]^c \right]}
\]

Integrating both the sides w.r.t. \( y \), we get
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\[ 1 + \log \bar{F}(y) = \frac{\{ah(y) + b\}^c}{\{ah(x) + b\}^c} \]

Thus,

\[ 1 - [- \log \bar{F}(y)] = [ah(x) + b]^c \]

**Corollary 3.3:** \( m(x, y) = \frac{c}{(c-1)} \frac{[\{h(x)\}^{c-1} - \{h(y)\}^{c-1}] h(x) h(y)}{[h(x)]^c - [h(y)]^c} \), \( c \neq 1 \) (3.10)

if and only if

\[ - \log \bar{F}(x) = a [h(x)]^{-c} + b, \]

where \( a, b, c \) and \( h(x) \) are such that \( F(x) \) is a df.

**Proof:** For \( - \log \bar{F}(x) = a [h(x)]^{-c} + b \), we have,

\[ m(x, y) = \frac{\int_x^y h(t)f(t)}{\int_x^y \bar{F}(t)[-\log \bar{F}(y) + \log \bar{F}(x)]} \]

\[ = \frac{ac}{B(x, y)} \int_x^y [h(t)]^{-c-1} h(t) h'(t) \, dt \]

\[ = \frac{c}{(c-1)} \frac{[\{h(x)\}^{c-1} - \{h(y)\}^{c-1}] h(x) h(y)}{[h(x)]^c - [h(y)]^c} \]

Now to prove (3.10) implies (3.11), we have

\[ A_1(x, y) = \frac{\frac{\partial}{\partial x} m(x, y)}{m(x, y) - h(x)} = -\frac{ch'(x)\{h(y)\}^c}{h(x)[\{h(x)\}^c - \{h(y)\}^c]} \]
\[
\frac{c h'(x) [h(y)]^c}{\{h(x)\}^{c+1} \left[ 1 - \frac{[h(y)]^c}{h(x)} \right]}
\]

Integrating both the sides w.r.t. \(x\),

\[
- \int_{\alpha}^{x} A_1(t, y) \, dt = \log \left[ 1 - \frac{a [h(x)]^{-c} + b}{a [h(y)]^{-c} + b} \right]
\]

we get,

\[
- \log F(x) = a [h(x)]^{-c} + b.
\]

**Remark 3.1:** For \( F(x) = e^{-a [h(x)]^{-c} - b} \), we have

(a) At \( c = -1 \) and \( F(x) = e^{-a h(x) - b} \),

\[
m(x, y) = \frac{1}{2} \left( \frac{1}{h(x)} + \frac{1}{h(y)} \right) h(x) h(y) = \frac{h(x) + h(y)}{2} = \text{A.M.} \quad (3.12)
\]

For \( h(x) = x \), it is an exponential distribution, as also given in Table 2.1.

(b) At \( c = 2 \) and \( F(x) = e^{-a [h(x)]^{-2} - b} \),

\[
m(x, y) = \frac{2 h(x) h(y)}{h(x) + h(y)} = \frac{1}{2} \left( \frac{1}{h(x)} + \frac{1}{h(y)} \right) = \text{H.M.} \quad (3.13)
\]

(c) At \( c = \frac{1}{2} \) and \( F(x) = e^{-a [h(x)]^{-\frac{1}{2}} - b} \),

\[
m(x, y) = \frac{\left( \frac{1}{\sqrt{h(x)}} - \frac{1}{\sqrt{h(y)}} \right) h(x) h(y)}{\sqrt{h(x)} - \sqrt{h(y)}} = \frac{(\sqrt{h(y)} - \sqrt{h(x)}) \, h(x) h(y)}{\sqrt{h(x)} h(y) (\sqrt{h(y)} - \sqrt{h(x)})} = \sqrt{h(x) h(y)} = \text{G.M.} \quad (3.14)
\]
Where A.M., H.M. and G.M. are arithmetic mean, harmonic mean and geometric mean respectively. That is, these results specify the probability distributions if the conditional expectation is A.M., H.M. and G.M.