Chapter 2

Triple Gauge Boson Production

2.1 Introduction

The di-lepton [49, 50, 51, 52, 53, 54], di-gauge boson [53, 54, 55, 56, 57, 58, 59, 60] and di-jet [61, 62] final states have been extensively studied in the context of extra dimension models. The triple gauge boson final state is also an interesting new physics signal in some of the beyond SM scenarios [63]. In this chapter, we consider the neutral triple gauge boson production at the LHC and study how the ADD model would alter the SM expectation. In the SM, the triple gauge boson final state is an important signal as it depends on the 3-point and 4-point couplings among the gauge bosons which is a test of the electroweak theory. This process in the SM has been studied to LO [64, 65] and its extension to the NLO was on the Les Houches wishlist [63, 66] and has been finally achieved in [67, 68, 69, 70]. The triple gauge boson production processes in the SM are the precise predictions of the electroweak gauge theory and gauge self-couplings. They are also potential backgrounds to many new physics models like SUSY and technicolor. For example, $Z\gamma\gamma$ in SM is a background to signals with di-photons and missing transverse energy in gauge
mediated supersymmetric theories [71] and $\gamma\gamma\gamma$ production in SM is a background to one photon plus techni-pion [72]. Processes with three gauge bosons can also come from the ADD model as gravitons couple directly to gauge bosons of the SM. While mono-jet or di-lepton production is more sensitive to parameters of the model with extra dimensions compared to the triple gauge boson production, all these processes involve same universal coupling of gravity with the SM particles and hence can provide equally important information about the model. Moreover, in discriminating physics beyond the SM namely SUSY or technicolor models using triple gauge boson production, one can not ignore the potential contributions resulting from models with extra dimensions.

In this analysis, we consider the process $PP \rightarrow VVV X$, where we restrict to the neutral gauge bosons $V = \gamma, Z$ and $X$ is some hadronic final state. The following four final states are the subject of this analysis: (i) $\gamma\gamma\gamma$ (ii) $\gamma\gamma Z$ (iii) $\gamma ZZ$ and (iv) $ZZZ$.

### 2.2 Neutral Triple Gauge Boson Production

The neutral gauge boson final state at the hadron collider $PP \rightarrow VVV X$ at LO comes from the following subprocess,

$$ q(p_1) + \bar{q}(p_2) \rightarrow V(p_3) + V(p_4) + V(p_5) \ , \quad (2.1) $$

where $V = \gamma, Z$ and $X$ is any final state hadron. The SM diagram for the above process is shown in Fig. 2.1 with all possible permutations of final states. For the final state with at least two $ZZ$s, Higgs boson could contribute by coupling to the quarks, but this is negligible in the vanishing quark mass limit. In the case of $ZZZ$ final state, there are additional Higgs strahlung diagrams, but their contribution is
also quite small and becomes faded in the present Higgs mass limit. Hence, we have not included the processes with the Higgs boson. Moreover, $gg \to VVV$ subprocess, though it is formally NNLO in QCD, could substantially contribute at $\mathcal{O}(\alpha_s^0)$ in the low invariant mass region of the final state vector bosons due to large gluon densities at small $x$. However, this effect starts diminishing as the invariant mass grows up to higher values wherein the ADD model begins to dominate over the SM contribution and therefore such effect has not been taken care of in our present study. In the ADD

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Typical Feynman diagram for triple gauge boson production in SM.}
\end{figure}

model, the KK modes of the graviton (G) couple to $V$ bosons, quarks, anti-quarks as well as to quark-antiquark-$V$ boson vertex [40]. Four categories of Feynman diagrams that give a $VVV$ final state in ADD model are shown in Fig. 2.2. We have

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{Typical Feynman diagrams for triple gauge boson production in ADD model. Dashed line represents the KK graviton (G) and the other particle lines are same as they are in Fig. 2.1.}
\end{figure}

used unitary gauge ($\xi \to \infty$) for the $Z$ boson and the Feynman gauge ($\xi = 1$) for the photon.

In the SM, the LO process for the production of $\gamma\gamma\gamma$ at hadron colliders results
from the annihilation of a quark and an anti-quark. In the ADD model, the production mechanism is again from the same initial states, but one of the three photons remains attached to either of the $q\bar{q}\gamma$, $\gamma\gamma G$, $q\bar{q}\gamma G$ vertices and the other two photons come from the decay of KK graviton. The typical Feynman diagrams that contribute in the SM and in the ADD model are shown in Fig. 2.1 and 2.2. The Feynman rules for the processes with KK graviton can be found in [40, 41]. All the expressions for the matrix element squared with proper spin, color sums and averages are obtained using a symbolic program based on FORM [73]. The KK graviton propagator $D_{ij}$ and the numerator of the spin-2 propagator [40] of the KK graviton are illustrated in eq. (1.10) and (1.11) respectively. Terms proportional to negative powers of mass of KK mode in $\zeta_{\mu\nu}$ do not contribute as they are proportional to $k_\mu k_\nu$. This provides a useful check on our calculation. The matrix elements have been checked for gauge invariance. We performed similar computation for evaluating the parton level subprocesses for $\gamma\gamma Z$, $\gamma ZZ$ and $ZZZ$ productions. In the following we list few of the important observations.

For the $\gamma\gamma Z$ production, in the limit $m_Z \rightarrow 0$ ($m_Z$ being the mass of Z boson), we reproduce the matrix elements for $\gamma\gamma\gamma$ process with the changes: $(C_V^2 + C_A^2)/4 \rightarrow Q_f^2$, $T_Z \rightarrow e$, where $C_V$, $C_A$ are the vector and axial vector couplings of the weak gauge boson respectively, $T_Z = e/(\sin \theta_w \cos \theta_w)$ and $Q_f$ is the electric charge of the quark flavors. In the case of $\gamma ZZ$ production, we find that the parton level subprocesses in SM and ADD model are similar to those of the $\gamma\gamma Z$ production with the changes $\gamma \leftrightarrow Z$. The squared matrix element for $\gamma ZZ$ production that comes from ADD model alone is not related to the one coming from $\gamma\gamma\gamma$ production. The reason is that some of the terms proportional to $m_Z^2$, that appear in the $GZZ$ vertex, cancel all the inverse power of $m_Z^2$ present in the $Z$ boson polarisation sum, giving contributions that have no analogous ones in the $\gamma\gamma\gamma$ process. However,
the expression for the SM squared matrix element of $\gamma ZZ$ is related to that of $\gamma \gamma \gamma$ process in the SM if we take $m_Z \to 0$, $(C_V^2 + 6C_2^3C_A^3 + C_A^4)/16 \to Q_f^4$ and $T_Z \to e$. For $ZZZ$ production, squared matrix elements involving ADD vertices do not have any relation with those of $\gamma \gamma \gamma$ production for the same reason as described in $\gamma ZZ$ production case. The SM squared matrix element of this process is related to the one for the $\gamma \gamma \gamma$ process in SM with the following replacement in the limit $m_Z \to 0$, 

\[(C_V^6 + 15C_4^3C_A^2 + 15C_2^2C_A^4 + C_A^6)/64 \to Q_f^6, T_Z \to e.\]

In fact, we empirically find that the most general formula for the replacement of $n$ number of $Z$ boson(s) with photon(s) in the SM squared matrix element is,

\[
\frac{(C_V^2 + C_A^3)^n + 2n(n-1)(C_V^2C_A^3(C_V^2 + C_A^3)^{n-2})}{4^n} \to Q_f^{2n},
\]  

(2.2)

which works for all the above three processes with $n = 1, 2, 3$. We have provided the expressions of the squared matrix elements for the $\gamma \gamma \gamma$ production process in Appendix A. For the rest of the processes discussed above, such expressions of the squared matrix elements are too large to be presented in this thesis. Rather, they could be made available upon request.

### 2.3 Numerical Results

In this section, we present different kinematical distributions for the production of neutral triple gauge bosons. The predictions are for the LHC at center of mass energy $\sqrt{S} = 14$ TeV. We have used CTEQ6L parton densities [74]. For the strong coupling constant that appears in CTEQ6L, we use $\Lambda_{QCD} = 0.226$ GeV and $n_f = 5$ flavors. We set the factorisation scale $\mu_F = P_T^V$ for the transverse momentum distribution of $V$ and $\mu_F = Q$ for the invariant mass ($Q$) distribution of the di-boson pair. In
addition we apply the following cuts on $P_T^V$ and the rapidity $y^V$,

$$P_T^γ,Z ≥ 25 \text{ GeV} \quad \text{and} \quad y^γ,Z < 2.7 . \quad (2.3)$$

We also ensure that in general the invariant mass of the di-boson (i.e., any two identical bosons among the three Vs) is less than $M_S$. We use $m_Z = 91.1876 \text{ GeV}$ and $\sin^2 \theta_w = 0.2312$. The fine structure constant is taken as $\alpha = 1/128$.

CMS [75] and ATLAS [76] have already reported searches for signatures of extra dimensions in the diphoton mass spectrum at the LHC for 7 TeV p p collisions. The 95 % lower bound on $M_S$ vary between $2.27 – 3.53 \text{ TeV}$ depending on the number of extra dimensions $d = 3 – 7$ for ATLAS and $M_S$ vary between $2.3 – 3.8 \text{ TeV}$ depending on the number of extra dimensions $d = 2 – 7$ for CMS, both using a fixed K-factor of about 1.6 [56, 57]. We have used the phenomenologically viable ADD

![Total Cross-section](image)
model parameters for our present study.

For the processes involving more than one photon, it is important to isolate photons from each other i.e., they need to be well separated in phase space so that they can be identified as separate objects in the detector. To do this, we consider a cone of radius \( R = \sqrt{(\Delta y)^2 + (\Delta \phi)^2} \) in the rapidity-azimuthal angle plane \((y, \phi)\) and ensure that the minimum separation between any two photons is taken to be \( R_{\gamma \gamma} = 0.4 \). In the following, we describe our findings for the various triple gauge boson production processes.

The total cross sections for various processes involving neutral triple gauge boson final states as a function of \( M_S \) for a fixed value of \( d = 2 \) are given in Fig. 2.3. We set \( d = 2 \) to make the effect of varying \( M_S \) on the SM+ADD cross sections visible for all the processes considered in the present chapter. The SM contributions that
2.3.1 $\gamma\gamma\gamma$ Production

In this case, the three photons in the final state are classified in such a way that $P_{T}^{\gamma_{1}} > P_{T}^{\gamma_{2}} > P_{T}^{\gamma_{3}}$. We have compared our predictions for $P_{T}^{\gamma_{1}}$ distribution in the SM against those given in [70] and found a very good agreement confirming the correct implementation of our analytical results in our numerical code. In the left panel of Fig. 2.4, we present the transverse momentum distribution of $\gamma_{1}$ in SM as well as in SM+ADD (i.e., SM, ADD and the interference between them). We have chosen $M_{S} = 3.5$ TeV and $d = 3$ as representative parameters of the ADD model. In the high $P_{T}^{\gamma_{1}}$ region, the distribution of SM+ADD is fully controlled by processes coming from ADD model and is enhanced due to the dominant contributions of the KK modes. In the right panel of Fig. 2.4, rapidity distribution of the most energetic
photon $\gamma_1$ is shown for $750 < P_T^{\gamma_1} < 1250$ GeV in SM and SM+ADD. It is seen that the SM contribution is extremely small in this range.

Figure 2.6: Invariant mass distribution of the photon pair in $\gamma\gamma Z$ final state (top panel) and $Z$ boson pair in $\gamma ZZ$ final state (bottom panel) for $d = 3$ with different values of $M_S$ (left) and for $M_S = 3.5$ TeV with different values of $d$ (right).

In order to estimate the factorisation scale $\mu_F$ dependence present in our LO
results, in the right panel of Fig. 2.4 we have plotted rapidity distributions for three different choices of $\mu_F$ i.e., $\mu_F = (0.2, 1, 2)P_T^{\gamma_1}$. In the central rapidity region, the variation of the rapidity distribution with respect to the factorisation scale is the largest. With respect to the central choice of $\mu_F = P_T^{\gamma_1}$, the variation is about 23.6 % and 8.2 % for the choice of $\mu_F = 0.2 P_T^{\gamma_1}$ and $\mu_F = 2 P_T^{\gamma_1}$ respectively.

The $P_T$ distribution of $\gamma_2$ is found to be similar to that of $\gamma_1$, but it is different for $\gamma_3$ (the least energetic photon among the three) as shown in Fig. 2.5 (left panel). Similarly its rapidity distribution, which is shown in Fig. 2.5 (right panel), is also different from the most energetic photon.

### 2.3.2 $\gamma\gamma Z$ Production

Here, the invariant mass distribution of the photon pair is a useful observable because in the ADD model, the photon pair is one of the clean decay modes of the KK graviton and in the region of interest, this could give an enhancement of the tail of the distribution. In Fig. 2.6 (top left panel) we have presented the invariant mass distributions of the photon pair for different choices of $M_S = (3.5, 4, 4.5)$ TeV fixing $d = 3$, while in the top right panel the same distribution is plotted for different choices of $d = 3, 4, 6$, but for a fixed value of $M_S = 3.5$ TeV. We find that the KK modes dominate over the SM contribution for larger values of invariant masses (around 400 GeV or above, for a given set of $M_S$ and $d$ values) of photon pairs leading to a significant enhancement of the signal over the background. We plot the factorisation scale dependence of invariant mass distributions of photon pairs in Fig. 2.7 (left panel) for different choices of $\mu_F$, i.e., $\mu_F = (0.2, 2)Q$. 

26
Figure 2.7: Dependence of invariant mass distribution of the photon pair in $\gamma\gamma Z$ final state (left panel) and Z boson pair in $\gamma ZZ$ final state (right panel) on the factorisation scale for $d = 3$ and $M_S = 3.5$ TeV.

2.3.3 $\gamma ZZ$ Production

Invariant mass of Z boson pair is again a useful observable. We have done a similar analysis as we did for $\gamma\gamma Z$ and use the same choice of factorisation scale and ADD model parameters. The invariant mass distributions are shown in the lower panels of Fig. 2.6 for different choices of $M_S$ and $d$. We find that the invariant mass distributions of photon pairs in $\gamma\gamma Z$ production and Z boson pairs here have similar qualitative behavior. In order to investigate the uncertainty resulting from the factorisation scale $\mu_F$, in Fig. 2.7 (right panel), we have plotted the invariant mass distributions of the Z boson pair for different choices of $\mu_F$, i.e., $\mu_F = (0.2, 2)Q$. 
Figure 2.8: Transverse momentum distribution of $Z_1$ (left panel) and $Z_3$ (right panel) for $M_S = 3.5$ TeV and $d = 3$.

Figure 2.9: Rapidity distribution of $Z_1$ for $M_S = 3.5$ TeV and $d = 3$ in the region where $P_T^{Z_1} \in (900, 1400)$ GeV.
2.3.4 ZZZ Production

We have classified triple Z bosons in such a way that $P_T^{Z_1} > P_T^{Z_2} > P_T^{Z_3}$ and for the $P_T^{Z_i}$ distribution, we make the choice of factorisation scale as $\mu_F = P_T^{Z_i}$, where $i = 1, 2, 3$. In Fig. 2.8, we have presented the transverse momentum distributions of $Z_1$ (left panel) and $Z_3$ (right panel) for SM and SM+ADD with $M_S = 3.5$ TeV and $d = 3$. Also, rapidity distribution of $Z_1$ for SM and SM+ADD with the same model parameters is shown in Fig. 2.9. For the rapidity distribution, we have put the constrain: $900 < P_T^{Z_1} < 1400$ GeV. As in the case of $\gamma\gamma\gamma$, the $P_T^{Z_2}$ distribution is similar to that of $P_T^{Z_1}$ distribution. We have also shown the sensitivity of rapidity distribution to the factorisation scale $\mu_F$ by varying it between $\mu_F = 0.2P_T^{Z_1}$ and $\mu_F = 2P_T^{Z_1}$. In the central rapidity region, we estimate the variation of the rapidity distribution with the factorisation scale and find that for $\mu_F = 0.2P_T^{Z_1}$ and $\mu_F = 2P_T^{Z_1}$, such variations are about 27.5 % and 8.9 % respectively with respect to those at $\mu_F = P_T^{Z_1}$. The rapidity distribution for $Z_2$ is similar to that of $Z_1$ while $Z_3$ is different.

So far, in our numerical analysis, we have put the UV cutoff $\Lambda = M_S$ which is the conventional choice to do the phenomenology as mentioned earlier. The sensitivity of the choice of UV cutoff is presented in Fig. 2.10 for $P_T^{Z_1}$ distribution of $\gamma\gamma\gamma$ final state and also for the invariant mass distribution of $\gamma\gamma$ pair of $\gamma\gamma Z$ process by varying $\Lambda = (0.9, 0.95, 1)M_S$. The cross section at $P_T^{Z_1} = 1200$ GeV varies between 10 - 24 % as we vary $\Lambda = (0.9, 0.95)M_S$ as compared to $\Lambda = M_S$ for the $\gamma\gamma\gamma$ process. Similarly, for the cross section of $\gamma\gamma Z$ process at $Q = 2000$ GeV, the variation stands between 7 - 15 % in the same range of $\Lambda$. 

29
Figure 2.10: $P_T^{\gamma \gamma}$ distribution of $\gamma \gamma \gamma$ final state (left) and invariant mass distribution of the photon pair in $\gamma \gamma Z$ final state (right) using the cutoff scale $\Lambda = (0.9, 0.95, 1)M_S$ for $M_S = 3.5$ TeV and $d = 3$.

2.4 Pentagon Reduction

To make the pavement towards NLO corrections of these processes involving tensor couplings, reduction of 5-point tensor integrals will inevitably be required. Therefore, in this section, we deal with the way of reducing the one loop 5-point tensor integrals (up to rank-4) using the Passarino-Veltman reduction technique [77, 78, 79]. In fact, numerous activities have been performed in reducing one loop tensor integrals and calculating the scalar ones (see for example [80, 81, 82, 83, 84]). The following work is basically an extension of what was done in [85], where reduction of 4-point tensor integrals (up to rank-3) was taken care of. Here, we describe the usage of projective momenta technique and define new projective momenta to perform a complete study of reducing 4-rank 4-point and the full 5-point tensor integrals up to rank-4. All the analytical results are given in detail so that they can easily be
coded in any analytical or numerical programme.

2.4.1 Notation & Convention

We define up to 5-point integrals in the following way,

\[ A_0(M_1) = (2\pi \mu)^{4-n} \int \frac{d^n l}{i\pi^2} \frac{1}{D_1} , \]

\[ B_{\{0,\mu,\nu\}}(p_1, M_1, M_2) = (2\pi \mu)^{4-n} \int \frac{d^n l}{i\pi^2} \frac{\{1, l_\mu, l_\nu l_\nu\}}{D_1 D_2} , \]

\[ C_{\{0,\mu,\nu,\mu\nu\}}(p_1, p_2, M_1, M_2, M_3) = (2\pi \mu)^{4-n} \int \frac{d^n l}{i\pi^2} \frac{\{1, l_\mu, l_\nu l_\nu, l_\mu l_\nu l_\nu l_\rho\}}{D_1 D_2 D_3} , \]

\[ D_{\{0,\mu,\nu,\mu\nu,\mu\nu\nu\}}(p_1, p_2, p_3, M_1, M_2, M_3, M_4) \]
\[ = (2\pi \mu)^{4-n} \int \frac{d^n l}{i\pi^2} \frac{\{1, l_\mu, l_\nu l_\nu, l_\mu l_\nu l_\nu l_\rho, l_\mu l_\nu l_\nu l_\rho l_\lambda\}}{D_1 D_2 D_3 D_4} , \]

\[ E_{\{0,\mu,\nu,\mu\nu,\mu\nu\nu\nu\}}(p_1, p_2, p_3, p_4, M_1, M_2, M_3, M_4, M_5) \]
\[ = (2\pi \mu)^{4-n} \int \frac{d^n l}{i\pi^2} \frac{\{1, l_\mu, l_\nu l_\nu, l_\mu l_\nu l_\nu l_\rho, l_\mu l_\nu l_\nu l_\rho l_\lambda\}}{D_1 D_2 D_3 D_4 D_5} . \] (2.4)

where \( M_i \)'s are the masses of off-shell internal lines and \( p_i \)'s are the on-shell 4-momentum of external particles and \( D_i \)'s are given here under:

\[ D_1 = l^2 - M_1^2 + i\epsilon \]
\[ D_2 = (l + p_1)^2 - M_2^2 + i\epsilon, \]
\[ D_3 = (l + p_1 + p_2)^2 - M_3^2 + i\epsilon, \]
\[ D_4 = (l + p_1 + p_2 + p_3)^2 - M_4^2 + i\epsilon, \]
\[ D_5 = (l + p_1 + p_2 + p_3 + p_4)^2 - M_5^2 + i\epsilon. \]  

Note that, for the sake of simplicity, we keep ourselves confined to present analytical results involving massless internal lines in the loop. However, it is straightforward to extend such calculation for massive internal lines with nominal changes in few selective variables. It is evident that the above integrals listed in eq. (2.4) are symmetric in their Lorentz indices and they can be easily demonstrated in Lorentz covariant way as follows,

\[ B_\mu = p_1 \mu B_1, \]
\[ B_{\mu\nu} = p_1 \mu p_1 \nu B_{21} + g_{\mu\nu} B_{22}, \]
\[ C_\mu = p_1 \mu C_{11} + p_2 \mu C_{12}, \]
\[ C_{\mu\nu} = p_1 \mu p_1 \nu C_{21} + p_2 \mu p_2 \nu C_{22} + \{p_1 p_2\}_{\mu\nu} C_{23} + g_{\mu\nu} C_{24}, \]
\[ C_{\mu\nu\rho} = p_1 \mu p_1 \nu p_1 \rho C_{31} + p_2 \mu p_2 \nu p_2 \rho C_{32} + \{p_1 p_1 p_2\}_{\mu\nu\rho} C_{33} + \{p_1 p_2 p_2\}_{\mu\nu\rho} C_{34}. \]
\[ D_{\mu} = p_{1\mu} D_{11} + p_{2\mu} D_{12} + p_{3\mu} D_{13} \]
\[ D_{\mu\nu} = p_{1\mu} p_{1\nu} D_{21} + p_{2\mu} p_{2\nu} D_{22} + p_{3\mu} p_{3\nu} D_{23} \]
\[ + p_{1p2} \mu \nu D_{24} + \{ p_{1p3} \mu \nu D_{25} + \{ p_{2p3} \mu \nu D_{26} + g_{\mu \nu} D_{27} \} \}
\]
\[ D_{\mu\nu\rho} = p_{1\mu} p_{1\nu} p_{1\rho} D_{31} + p_{2\mu} p_{2\nu} p_{2\rho} D_{32} + p_{3\mu} p_{3\nu} p_{3\rho} D_{33} \]
\[ + \{ p_{1p2} \mu \nu \rho D_{34} + \{ p_{1p3} \mu \nu \rho D_{35} + \{ p_{2p3} \mu \nu \rho D_{36} \} \} \}
\]
\[ D_{\mu\nu\rho\lambda} = p_{1\mu} p_{1\nu} p_{1\rho} p_{1\lambda} D_{41} + p_{2\mu} p_{2\nu} p_{2\rho} p_{2\lambda} D_{42} + p_{3\mu} p_{3\nu} p_{3\rho} p_{3\lambda} D_{43} \]
\[ + \{ p_{1p2} \mu \nu \rho \lambda D_{44} + \{ p_{1p3} \mu \nu \rho \lambda D_{45} + \{ p_{2p3} \mu \nu \rho \lambda D_{46} \} \} \}
\]
\[ E_{\mu} = p_{1\mu} E_{11} + p_{2\mu} E_{12} + p_{3\mu} E_{13} + p_{4\mu} E_{14} \]
\[ E_{\mu\nu} = p_{1\mu} p_{1\nu} E_{21} + p_{2\mu} p_{2\nu} E_{22} + p_{3\mu} p_{3\nu} E_{23} + p_{4\mu} p_{4\nu} E_{24} \]
\[ + \{ p_{1p2} \mu \nu E_{25} + \{ p_{1p3} \mu \nu E_{26} + \{ p_{2p3} \mu \nu E_{27} \} \} \}
\]
\[ + \{ p_{2p3} \mu \nu E_{28} + \{ p_{2p4} \mu \nu E_{29} + \{ p_{3p4} \mu \nu E_{210} + g_{\mu \nu} E_{211} \} \} \}, \]
\[ (2.7) \]
\[ (2.8) \]
\[ E_{\mu\nu\rho} = p_{1\mu}p_{1\nu}p_{1\rho}E_{31} + p_{2\mu}p_{2\nu}p_{2\rho}E_{32} + p_{3\mu}p_{3\nu}p_{3\rho}E_{33} + p_{4\mu}p_{4\nu}p_{4\rho}E_{34} + \{p_{1\mu}p_{1\nu}\}_{\mu\nu\rho}E_{35} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho}E_{36} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho}E_{37} + \{p_{1\mu}p_{2\nu}\}_{\mu\nu\rho}E_{38} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho}E_{39} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho}E_{310} + \{p_{1\mu}p_{2\nu}\}_{\mu\nu\rho}E_{311} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho}E_{312} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho}E_{313} + \{p_{2\mu}p_{2\nu}\}_{\mu\nu\rho}E_{314} + \{p_{2\mu}p_{4\nu}\}_{\mu\nu\rho}E_{315} + \{p_{2\mu}p_{3\nu}\}_{\mu\nu\rho}E_{316} + \{p_{2\mu}p_{4\nu}\}_{\mu\nu\rho}E_{317} + \{p_{2\mu}p_{3\nu}\}_{\mu\nu\rho}E_{318} + \{p_{3\mu}p_{3\nu}\}_{\mu\nu\rho}E_{319} + \{p_{3\mu}p_{4\nu}\}_{\mu\nu\rho}E_{320} + \{p_{4\mu}g\}_{\mu\nu\rho}E_{321} + \{p_{2}g\}_{\mu\nu\rho}E_{322} + \{p_{3}g\}_{\mu\nu\rho}E_{323} + \{p_{4}g\}_{\mu\nu\rho}E_{324} , \]
\[ E_{\mu\nu\rho\lambda} = p_{1\mu}p_{1\nu}p_{1\rho}p_{1\lambda}E_{41} + p_{2\mu}p_{2\nu}p_{2\rho}p_{2\lambda}E_{42} + p_{3\mu}p_{3\nu}p_{3\rho}p_{3\lambda}E_{43} + p_{4\mu}p_{4\nu}p_{4\rho}p_{4\lambda}E_{44} + \{p_{1\mu}p_{1\nu}\}_{\mu\nu\rho\lambda}E_{45} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{46} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{47} + \{p_{1\mu}p_{2\nu}\}_{\mu\nu\rho\lambda}E_{48} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{49} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{410} + \{p_{1\mu}p_{2\nu}\}_{\mu\nu\rho\lambda}E_{411} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{412} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{413} + \{p_{1\mu}p_{2\nu}\}_{\mu\nu\rho\lambda}E_{414} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{415} + \{p_{1\mu}p_{2\nu}\}_{\mu\nu\rho\lambda}E_{416} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{417} + \{p_{1\mu}p_{2\nu}\}_{\mu\nu\rho\lambda}E_{418} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{419} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{420} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{421} + \{p_{1\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{422} + \{p_{1\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{423} + \{p_{2\mu}p_{2\nu}\}_{\mu\nu\rho\lambda}E_{424} + \{p_{2\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{425} + \{p_{2\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{426} + \{p_{2\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{427} + \{p_{2\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{428} + \{p_{2\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{429} + \{p_{2\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{430} + \{p_{2\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{431} + \{p_{2\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{432} + \{p_{3\mu}p_{3\nu}\}_{\mu\nu\rho\lambda}E_{433} + \{p_{3\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{434} + \{p_{3\mu}p_{4\nu}\}_{\mu\nu\rho\lambda}E_{435} + \{p_{1\mu}g\}_{\mu\nu\rho\lambda}E_{436} + \{p_{1\mu}g\}_{\mu\nu\rho\lambda}E_{437} + \{p_{1\mu}g\}_{\mu\nu\rho\lambda}E_{438} + \{p_{1\mu}g\}_{\mu\nu\rho\lambda}E_{439} + \{p_{2\mu}g\}_{\mu\nu\rho\lambda}E_{440} + \{p_{2\mu}g\}_{\mu\nu\rho\lambda}E_{441} + \{p_{2\mu}g\}_{\mu\nu\rho\lambda}E_{442} + \{p_{3\mu}g\}_{\mu\nu\rho\lambda}E_{443} + \{p_{3\mu}g\}_{\mu\nu\rho\lambda}E_{444} + \{p_{4\mu}g\}_{\mu\nu\rho\lambda}E_{445} + \{g\}_{\mu\nu\rho\lambda}E_{446} . \]
In the above equations (eq. (2.6)-(2.9)), we have adopted some short-hand notations which are given here under:

\[
\{p_ip_jp_kp_l\}_{\mu\nu\rho\lambda} = \sum_{\sigma(i,j,k,l)} p_{\sigma(i)\mu} p_{\sigma(j)\nu} p_{\sigma(k)\rho} p_{\sigma(l)\lambda},
\]

with \(\sigma(i, j, k, l)\) denoting all different permutations of \((i, j, k, l)\). Similar is the case for \(\{p_ip_jp_k\}_{\mu\nu\rho}\) and \(\{p_ip_j\}_{\mu\nu}\) expansions.

\[
\{p_ip_jg\}_{\mu\nu\rho\lambda} = \{p_ip_j\}_{\mu\nu\rho\lambda} + \{p_ip_j\}_{\mu\nu\rho\lambda} + \{p_ip_j\}_{\mu\nu\rho\lambda} + \{p_ip_j\}_{\mu\nu\rho\lambda} + \{p_ip_j\}_{\mu\nu\rho\lambda} + \{p_ip_j\}_{\mu\nu\rho\lambda},
\]

(2.11)

\[
\{pi\}_{\mu\nu\rho} = p_{i\mu} g_{\nu\rho} + p_{i\nu} g_{\mu\rho} + p_{i\rho} g_{\mu\nu},
\]

(2.12)

\[
\{gg\}_{\mu\nu\rho\lambda} = g_{\mu\nu} g_{\rho\lambda} + g_{\mu\rho} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\rho}.
\]

(2.13)

At this stage, our aim is to find all the co-efficients of \(D_{\mu\nu\rho\lambda}\) (in eq. (2.8)) and for all others represented in eq. (2.9). Rest of the co-efficients of eq. (2.6),(2.7),(2.8) have already been calculated and they are listed in [85].

### 2.4.2 Reduction of 4-point 4-rank Tensor

Apparently, it seems that, if we want to find out the co-efficients of \(D_{\mu\nu\rho\lambda}\), we have to deal with a \(22 \times 22\) matrix. But, this can be reduced to a \(3 \times 3\) matrix problem by introducing three projective momenta \(P_i\)'s which would have the following properties:

\[
P^\mu_i p_{j\mu} = \delta_{ij} \quad \forall \quad i, j = 1, 2, 3.
\]

(2.14)
The existence of such projective momenta is directly related to the existence of $X^{-1}$ matrix, where the $X$ matrix is defined as follows,

$$X(p_1, p_2, p_3) \equiv X_{[1,2,3]} = \begin{pmatrix}
  p_1^2 & p_1.p_2 & p_1.p_3 \\
  p_1.p_2 & p_2^2 & p_2.p_3 \\
  p_1.p_3 & p_2.p_3 & p_3^2 
\end{pmatrix}. \quad (2.15)$$

In other words, if these three 4-momenta $p_1, p_2, p_3$ form an independent set resulting $\det[X] \neq 0$, then only construction of such $P_i$s would be possible. Construction of another projective tensor $P^{\mu\nu}$ is inevitable in order to find out the co-efficients of $D_{\mu\nu}$ (eq. (2.8)) and its form and properties are given below:

$$P^{\mu\nu} = \frac{1}{(n-3)} \left\{ g^{\mu\nu} - \sum_{i=1}^{3} P^\mu_i P^\nu_i \right\}, \quad (2.16)$$

$$p_{\mu} P^{\mu\nu} = 0 \quad \text{and} \quad g_{\mu\nu} P^{\mu\nu} = 1. \quad (2.17)$$

With the correct combination of these two types of projective tensors mentioned above, we can now define a new projective tensor which is essential to be able to find some of the co-efficients of $D_{\mu\nu\rho\lambda}$ and it is of the following form,

$$P^{\mu\nu\rho}_{i,j,k} = P^{\mu}_{i} P^{\nu}_{j} P^{\rho}_{k} - (P_{i}, P_{j}) P^{\mu\nu} P^{\rho}_{k} - (P_{j}, P_{k}) P^{\mu\nu} P^{\rho}_{i} - (P_{k}, P_{i}) P^{\mu\nu} P^{\rho}_{j}. \quad (2.18)$$

For example, by applying $P^{\mu\nu\rho}_{1,1,1}$ on $D_{\mu\nu\rho\lambda}$, we get the following matrix identity which is indeed a $3 \times 3$ matrix relation:

$$P^{\mu\nu\rho}_{1,1,1} D_{\mu\nu\rho\lambda} \begin{pmatrix}
  p_1^\lambda \\
  p_2^\lambda \\
  p_3^\lambda 
\end{pmatrix} = X_{[1,2,3]} \begin{pmatrix}
  D_{41} \\
  D_{44} \\
  D_{45} 
\end{pmatrix} + \begin{pmatrix}
  3D_{416} \\
  0 \\
  0 
\end{pmatrix} = \begin{pmatrix}
  R_{441} \\
  R_{442} \\
  R_{443} 
\end{pmatrix}. \quad (2.19)$$
In a like manner, we can easily get similar kind of matrix equations (see Appendix B.1), which in fact provide the solution for the co-efficients ranging from $D_{41}$ to $D_{415}$ with the proper choice of $P_{i,j,k}$, provided we need to know the exact solution for the rest of the unknown variables (e.g. $D_{416}$ and $R_{441-443}$ in eq. (2.19)) beforehand.

In order to find such relations involving the co-efficients $D_{416}$ to $D_{421}$, we need to operate $P^\mu_\nu P^\rho_\iota D_{\mu\nu\rho\lambda}$ where $i = 1, 2, 3$. Following is just one of these relations:

$$P^\mu_\nu P^\rho_1 D_{\mu\nu\rho\lambda} \begin{pmatrix} p^\lambda_1 \\ p^\lambda_2 \\ p^\lambda_3 \end{pmatrix} = X_{[1,2,3]} \begin{pmatrix} D_{416} \\ D_{419} \\ D_{420} \end{pmatrix} + \begin{pmatrix} D_{422} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_{442} \\ R_{443} \\ R_{444} \end{pmatrix}. \quad (2.20)$$

Rest of them are listed in Appendix B.1. Now, the only co-efficient left to evaluate is $D_{422}$, which demands invocation of another new projection operator ($P^{\mu\nu\rho\lambda}$), that obeys the following relation:

$$P^{\mu\nu\rho\lambda} = \left( \frac{n-3}{n-1} \right) P^{\mu\nu} P^{\rho\lambda}, \quad (2.21)$$

and applying this projective tensor on $D_{\mu\nu\rho\lambda}$, we finally get,

$$P^{\mu\nu\rho\lambda} D_{\mu\nu\rho\lambda} = D_{422}. \quad (2.22)$$

At this point, complete solutions for these co-efficients are one step away, as we are to derive the solutions for the R-functions right away. The calculation is straightforward and it will be more vivid with the following explicit derivation of $R_{441}$ given here under:

$$R_{441} = P^{\mu\nu}_{1,1,1} D_{\mu\nu\rho\lambda} p^\lambda_1$$
(2\pi\mu)^{4-n} \int \frac{d^n l}{i\pi^2} P_{i1}^{\mu
u} D_1 D_2 D_3 D_4 P_1^\lambda

(2\pi\mu)^{4-n} \int \frac{d^n l}{i\pi^2} [P_{i1}^{\mu
u} P_1^\rho - 3(P_1 P_1) P_{i1}^{\mu\nu} P_1^\rho] \frac{l_\mu l_\nu l_\rho l_\lambda}{D_1 D_2 D_3 D_4} (l.p_1)

(2\pi\mu)^{4-n} \int \frac{d^n l}{i\pi^2} [P_{i1}^{\mu\nu} P_1^\rho - 3(P_1 P_1) P_{i1}^{\mu\nu} P_1^\rho] \frac{l_\mu l_\nu l_\rho}{D_1 D_2 D_3 D_4}

\times \frac{1}{2} [(l + p_1)^2 - l^2 - p_1^2]

\frac{1}{2} \left\{ (2\pi\mu)^{4-n} \int \frac{d^n l}{i\pi^2} [P_{i1}^{\mu\nu} P_1^\rho - 3(P_1 P_1) P_{i1}^{\mu\nu} P_1^\rho] \frac{l_\mu l_\nu l_\rho}{D_1 D_2 D_3 D_4}

- (2\pi\mu)^{4-n} \int \frac{d^n l}{i\pi^2} [P_{i1}^{\mu\nu} P_1^\rho - 3(P_1 P_1) P_{i1}^{\mu\nu} P_1^\rho] \frac{(l - p_1)_\mu (l - p_1)_\nu (l - p_1)_\rho}{l^2(l + p_2)^2(l + p_2 + p_3)^2}

- (2\pi\mu)^{4-n} \int \frac{d^n l}{i\pi^2} [P_{i1}^{\mu\nu} P_1^\rho - 3(P_1 P_1) P_{i1}^{\mu\nu} P_1^\rho] \frac{l_\mu l_\nu l_\rho p_1^2}{D_1 D_2 D_3 D_4} \right\}

\frac{1}{2} [P_{i1}^{\mu\nu} P_1^\rho - 3(P_1 P_1) P_{i1}^{\mu\nu} P_1^\rho] [C_{\mu\nu\rho}(p_1 + p_2, p_3) - C_{\mu\nu\rho}(p_2, p_3)

+ \{l(p_1)\}_{\mu\nu\rho} - \{l(p_1)\}_{\mu\nu\rho} + p_1^\nu p_1^\rho C_0(p_2, p_3) - p_1^2 D_{\mu\nu\rho}(p_1, p_2, p_3)]

\frac{1}{2} [C_{31}(p_1 + p_2, p_3) + C_0(p_2, p_3) - p_1^2 D_{31}(p_1, p_2, p_3)] \quad . \quad (2.23)

Rest of the R-functions can be derived in the similar way and all of them are listed in Appendix B.1.

### 2.4.3 Reduction of 5-point Tensor

The main thing to remember at the time of reducing 5-point tensor integrals is that, here the number of independent external 4-momenta is four (i.e., p_1, p_2, p_3, p_4) and one has to define all the projective momenta and projective tensors consistently.

So, to keep pace with the above statement, it is obvious that we would require four projective momenta with the following properties,

\[ P^\mu_i p_{j\mu} = \delta_{ij} \quad \forall \quad i, j = 1, 2, 3, 4 \quad . \quad (2.24) \]
In this case, X-matrix has to be redefined and the projective tensor $P^{\mu\nu}$ has to be modified maintaining the same properties as described in eq. (2.17), in the following way,

$$X(p_1, p_2, p_3, p_4) \equiv X_{[1,2,3,4]} = \begin{pmatrix} p_1^2 & p_1.p_2 & p_1.p_3 & p_1.p_4 \\ p_1.p_2 & p_2^2 & p_2.p_3 & p_2.p_4 \\ p_1.p_3 & p_2.p_3 & p_3^2 & p_3.p_4 \\ p_1.p_4 & p_2.p_4 & p_3.p_4 & p_4^2 \end{pmatrix} , \quad (2.25)$$

$$P^{\mu\nu} = \frac{1}{(n-4)} \left\{ g^{\mu\nu} - \sum_{i=1}^{4} P_i^{\mu} P_i^{\nu} \right\} . \quad (2.26)$$

With the help of the above two projection operators, we can easily reduce $E_{\mu}$ and $E_{\mu\nu}$ and their expressions are provided in detail in Appendix B.2. In order to reduce $E_{\mu\nu\rho}$ and $E_{\mu\nu\rho\lambda}$, projective tensors similar to $P^{\mu\nu}_{i,j}$ and $P^{\mu\nu\rho}_{i,j,k}$ would work with the only modification therein that the latin indices will now run from 1 to 4, unlike the 4-point reduction case, where they are running from 1 to 3, i.e.,

$$P^{\mu\nu}_{i,j} = P_i^{\mu} P_j^{\nu} - (P_i.P_j) P^{\mu\nu} , \quad (2.27)$$

$$P^{\mu\nu\rho}_{i,j,k} = P_i^{\mu} P_j^{\nu} P_k^{\rho} - (P_i.P_j) P^{\mu\nu} P_k^{\rho} - (P_j.P_k) P^{\mu\nu} P_i^{\rho} - (P_k.P_i) P^{\mu\nu} P_j^{\rho} , \quad (2.28)$$

where $i, j, k = 1, 2, 3, 4$. In addition, to find out the solution for the co-efficient $E_{446}$ in eq. (2.9), one has to consider the following relation,

$$\mathcal{P}^{\mu\nu\rho\lambda} E_{\mu\nu\rho\lambda} = E_{446} \quad , \quad (2.29)$$

where

$$\mathcal{P}^{\mu\nu\rho\lambda} = \left( \frac{1}{2n-7} \right) P^{\mu\nu} P^{\rho\lambda} \quad . \quad (2.30)$$
All the $(4 \times 4)$ matrix relations along with the $R$-functions for all reduced 5-point integrals are systematically jotted down in Appendix B.2.

### 2.5 Conclusion

In this chapter, we have studied the neutral triple gauge boson production processes at the LHC in theories with large extra dimensions which are produced via the exchange of a tower of KK graviton, taking into account the SM contributions altogether. All the final state photons and $Z$ bosons are taken to be real. We have performed various checks on our analytical results and the numerical predictions are obtained using a Monte Carlo code which allows us to implement various experimental cuts. For the case in which the gauge bosons in the final state are identical we have presented the transverse momentum distribution by ordering the transverse momentum as $P_{V1} > P_{V2} > P_{V3}$. We find that $P_{V1}$ and $P_{V2}$ distributions are similar but the one for $P_{V3}$ is different. The rapidity distributions are also presented. For the case where one of the gauge bosons in the final state is different, we choose to use the invariant mass distribution of the identical di-bosons, as it would be a better discriminator in the region of interest. We have also studied their dependencies on the ADD model parameter $M_S$ and the number of extra dimensions $d$, keeping the UV scale $\Lambda = M_S$. In addition, we have reported the sensitivity of the choice of $\Lambda$ by varying it from $\Lambda = 0.9M_S$ to $0.95M_S$. We have also studied the dependence of our LO predictions on the factorisation scale. Nevertheless, a detailed calculation of 5-point tensor integral reduction using Passarino-Veltman technique has been presented in order to reveal its analytical results in a ready-to-use format. Howsoever, we have not yet dealt with complete calculation of any process to the NLO accuracy. In the next chapter, we will present NLO QCD correction to the associated
production of the vector gauge boson \((Z/W^\pm)\) and the graviton in the LED model at the LHC and discuss its effect on various kinematical observables.