CHAPTER 4

ON HOMOTYPICAL IDENTITIES

§ 4.1. INTRODUCTION

In [37], Khan, jointly with Shah obtained some partial results by establishing some sufficient conditions for homotypical identities whose both sides contain repeated variables and are preserved under epis in conjunction with a seminormal identity. In the present chapter, we generalize these results [37, Theorems 2.5, 2.7 and 3.1]. However, a full determination of all semigroup identities that are preserved under epis in conjunction with a seminormal identity remains an open problem.

In Section 4.2, we consider balanced identities, while Section 4.3 deals with non-balanced homotypical identities.

§ 4.2. BALANCED IDENTITIES

An identity \( u = v \) is said to be balanced if \( \lvert x \rvert_u = \lvert x \rvert_v, \forall x \in C(u)(= C(v)) \).

In this section, we establish some sufficient conditions for balanced identities to be preserved under epis in conjunction with a seminormal identity.

Lemma 4.2.1: Let \( U \) be a permutative subsemigroup satisfying a seminormal permutation identity of a semigroup \( S \) such that \( \text{Dom}(U, S) = S \). If \( U \) satisfies

\[
x_1^{p_1} \cdots x_r^{p_r} u(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s}
\]

(63)

where \( u \) and \( v \) are any words in \( z_1, z_2, \ldots, z_\ell \), then the identity (63) is also satisfied for all \( x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s \in S \) and \( z_1, z_2, \ldots, z_\ell \in U \), where \( p_1, \ldots, p_r, q_1, \ldots, q_s \) are any positive integers such that \( p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r ; q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1 (r, s \geq 1) \).

Proof. Take any semigroups \( U \) and \( S \) with \( U \) a subsemigroup of \( S \) such that \( \text{Dom}(U, S) = S \). Since \( U \) satisfies (1), by Result 1.5.5, \( S \) also satisfies (1). Now we
shall show that the identity (63) satisfied by \( U \) is also satisfied when \( x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s \in S \) and \( z_1, z_2, \ldots, z_\ell \in U \).

**Case (i):** First, take any \( x_1, x_2, \ldots, x_r \in S \) and \( y_1, y_2, \ldots, y_s, z_1, z_2, \ldots, z_\ell \in U \). If \( x_1, x_2, \ldots, x_r \in \U \), then (63) holds trivially. So assume without loss of generality that \( x_1 \in S \setminus U \). Let (2) be a zigzag of minimal length \( m \) over \( U \) with value \( x_1 \).

Letting \( y = y_1^{q_1} y_2^{q_2} \ldots y_s^{q_s} \), we have

\[
x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}
\]

\[
= y_m^{p_1} a_{2m} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y
\]

(by the zigzag equations and Result 1.5.13)

\[
= y_m^{p_1} a_{2m} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y
\]

(as \( U \) satisfies (63))

\[
= x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}
\]

(by the zigzag equations and Result 1.5.13 and as \( y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \))

as required.

Next, we assume inductively that the result is true for all \( x_1, \ldots, x_{k-1} \in S \) and \( x_1, \ldots, x_r \in \U \). We shall prove that the result is also true for all \( x_1, \ldots, x_k \in S \) and \( x_{k+1}, \ldots, x_r \in \U \). Again if \( x_k \in \U \), then the result follows by inductive hypothesis. So assume that \( x_k \in S \setminus \U \). Let (2) be a zigzag of minimal length \( m \) over \( U \) with value \( x_k \).

Now, we have

\[
x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y
\]

\[
= x_1^{p_1} x_2^{p_2} \cdots x_{k-1}^{p_{k-1}} y_m^{p_k} a_{2m} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y
\]

(by Result 1.5.13 and zigzag equations)

\[
= w y_m^{(m)} b_1^{(m)} \cdots b_{k-1}^{(m)} p_k \cdots p_{k+1} \cdots p_r a_{2m} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y
\]

(by Results 1.5.10 and 1.5.11 for some \( b_1^{(m)}, \ldots, b_{k-1}^{(m)} \in U \) and \( y_m^{(m)} \in S \setminus \U \) as \( y_m \in S \setminus \U \) and where \( w = x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} \) and where \( a_{2m} = a_{2m-1} t_m \) with \( t_m \in S \setminus \U \)

\[
= w y_m^{(m)} v^{(m)} b_1^{(m)} \cdots b_{k-1}^{(m)} a_{2m} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_\ell) y
\]

(by Result 1.5.11 as \( y_m^{(m)}, t_m \in S \setminus \U \) and where \( v^{(m)} = b_1^{(m)} p_1 \cdots b_{k-1}^{(m)} p_{k-1} \))
\[\begin{align*}
&= w y_m^{(m)p_k} v^{(m)} b_1^{(m)p_1} \ldots b_{k-1}^{(m)p_{k-1}} a_m^{p_k} x_{k+1}^{p_k} \ldots x_r^{p_r} v(z_1, z_2, \ldots, z_t) y \\
&\quad \text{(as } U \text{ satisfies (63))} \\
&= w y_m^{(m)p_k} b_1^{(m)p_k} \ldots b_{k-1}^{(m)p_{k-1}} a_m^{p_k} x_{k+1}^{p_k} \ldots x_r^{p_r} v(z_1, z_2, \ldots, z_t) y \\
&\quad \text{(by Result 1.5.11 and the definition of } v^{(m)}) \\
&= x_1^{p_1} x_2^{p_2} \ldots x_{k-1}^{p_{k-1}} x_k^{p_k} x_{k+1}^{p_k} \ldots x_r^{p_r} v(z_1, z_2, \ldots, z_t) y \\
&\quad \text{(as } y = y_1^{y_1} y_2^{y_2} \ldots y_s^{y_s}) \\
&\quad \text{(by Result 1.5.13 and zigzag equations)} \\
&\quad \text{as required.}
\end{align*}\]

**Case (ii):** Now, we show that (63) is satisfied for all \(x_1, \ldots, x_r, y_1, \ldots, y_s \in S\) and \(z_1, \ldots, z_t \in U\). Again, we can assume without loss of generality that \(y_1 \in S \setminus U\). Let (2) be a zigzag of minimal length \(m\) over \(U\) with value \(y_1\). Now, as the equalities (64) and (65) below follow by Results 1.5.10 and 1.5.11 for some \(b_2^{(1)}, \ldots, b_s^{(1)} \in U\) and \(t_1^{(1)}\) in \(S \setminus U\) as \(y_1, t_1 \in S \setminus U\) and where \(w^{(1)} = b_2^{(1)n-q_2} \ldots b_s^{(1)n-q_s}\) respectively. Letting \(x = x_1^{p_1} x_2^{p_2} \ldots x_r^{p_r}\), we have

\[\begin{align*}
& x_1^{p_1} x_2^{p_2} \ldots x_r^{p_r} u(z_1, z_2, \ldots, z_t) y_1^{y_1} y_2^{y_2} \ldots y_s^{y_s} \\
&= x u(z_1, z_2, \ldots, z_t) a_0^{a_0} t_1^{t_1} y_1^{y_1} y_2^{y_2} \ldots y_s^{y_s} \\
&\quad \text{(by the zigzag equations and Result 1.5.13)} \\
&= x u(z_1, z_2, \ldots, z_t) a_0^{a_0} b_2^{(1)q_2} \ldots b_s^{(1)q_s} t_1^{(1)} y_2^{y_2} \ldots y_s^{y_s} \quad \text{(64)} \\
&= x u(z_1, z_2, \ldots, z_t) a_0^{a_0} b_2^{(1)q_2} \ldots b_s^{(1)q_s} w^{(1)} t_1^{(1)q_1} y_2^{y_2} \ldots y_s^{y_s} \quad \text{(65)} \\
&= x v(z_1, z_2, \ldots, z_t) a_0^{a_0} b_2^{(1)q_2} \ldots b_s^{(1)q_s} w^{(1)} t_1^{(1)q_1} y_2^{y_2} \ldots y_s^{y_s} \\
&\quad \text{(as } U \text{ satisfies (63))} \\
&= x v(z_1, z_2, \ldots, z_t) a_0^{a_0} b_2^{(1)q_2} \ldots b_s^{(1)q_s} t_1^{(1)q_1} y_2^{y_2} \ldots y_s^{y_s} \\
&\quad \text{(by Result 1.5.11 and definition of } w^{(1)}) \\
\end{align*}\]
Next, we assume inductively that the result is true for all \(y_1, \ldots, y_{k-1} \in S\) and \(y_k, \ldots, y_s \in U\). We shall prove that the result is also true for all \(y_1, \ldots, y_{k-1}, y_k\) in \(S\) and \(y_{k+1}, \ldots, y_s \in U\). Again if \(y_k \in U\), then the result follows by inductive hypothesis. So assume that \(y_k \in S \setminus U\). Let (2) be a zigzag of minimal length \(m\) over \(U\) with value \(y_k\). Now as the equalities (66) and (67) below follow by Results 1.5.10 and 1.5.11 for some \(b_k^{(1)}, \ldots, b_s^{(1)} \in U\) and \(t_1^{(1)} \in S \setminus U\) and where \(v = y_k^{q_k+1} \cdots y_s^{q_s}\), and by Result 1.5.11 as \(a_0 = y_1 a_1, y_1, t_1^{(1)} \in S \setminus U\) and where \(w^{(1)} = b_k^{(1)} a_{q_k+1} \cdots b_s^{(1)} a_{q_s}\), respectively, we have

\[
x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(z_1, z_2, \ldots, z_r) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}
\]

(by inductive hypothesis)

\[
x_1(x_1, x_2, \ldots, x_r) y_1^{q_1} \cdots y_k^{q_k-1} a_0^{q_k} b_k^{q_k+1} \cdots b_s^{q_s} t_1^{q_1} v
\]

(by Result 1.5.13 and zigzag equations)
Corollary 4.2.2: Let $U$ be a permutative subsemigroup satisfying a seminormal permutation identity of a semigroup $S$ such that $\text{Dom}(U, S) = S$. Let $t_1, t_2, \ldots, t_\ell$ be any positive integers. If $U$ satisfies

$$x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} z_1^{t_1} z_2^{t_2} \cdots z_j^{t_j} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} z_1^{t_1} z_2^{t_2} \cdots z_j^{t_j} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \quad (68)$$

where $j$ is any permutation of the set $\{1, 2, \ldots, \ell\}$ and $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$, are any positive integers such that $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$, $q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1 (r, s \geq 1)$, then the identity (68) is also satisfied for all $x_1, x_2, \ldots, x_r, y_1, \ldots, y_s \in S$ and $z_1, z_2, \ldots, z_j \in U$.

Proposition 4.2.3: Let $U$ be a permutative subsemigroup satisfying a seminormal permutation identity of a semigroup $S$ such that $\text{Dom}(U, S) = S$. Let $u$ and $v$ be any words in $w_1, \ldots, w_\ell$. If the identity

$$x_1^{p_1} \cdots x_r^{p_r} u(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s}$$

holds for all $x_1, \ldots, x_r, y_1, \ldots, y_s \in S$ and $w_1, \ldots, w_\ell \in U$. Then the identity

$$x_1^{p_1} \cdots x_r^{p_r} u(w_1, \ldots, w_\ell) y^q = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) y^q$$

$$[x^p u(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s} = x^q v(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s}]$$

also holds for all $y \in S \setminus U$, $x_1, \ldots, x_r \in S$, $w_1, \ldots, w_\ell \in U$ and positive integer $q \geq q_1$ [for all $x \in S \setminus U$, $y_1, \ldots, y_s \in S$, $w_1, \ldots, w_\ell \in U$ and positive integer $p \geq p_1$].

Proof. We have

$$x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) y^q$$

$$= x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) y^q y^{q-q_1}$$

$$= x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_s} y^{q-q_1}$$

(by Results 1.5.10 and Corollary 1.5.12 for some $a_1, \ldots, a_s \in U$ and $y' \in S \setminus U$ as $a_1 = z_1 b_1$ for some $z_1' \in S \setminus U$, $b_1 \in U$)

$$= x_1^{p_1} \cdots x_r^{p_r} u(w_1, w_2, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_s} w y^{q-q_1}$$

(by Corollary 1.5.12 as $a_1 = z_1 b_1$ for some $z_1' \in S \setminus U$, $b_1 \in U$ and where $w = a_2^{q_2-q_1} \cdots a_s^{q_s-q_1}$)
\[ x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_s} w y^{q_1} y^{q_s - q_1} \]
\[ = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) a_1^{q_1} \cdots a_s^{q_s} y^{q_1} y^{q_s - q_1} \]  
(by definition of \( w \))
\[ = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) y^{q_1} y^{q_s - q_1} \]  
(by Result 1.5.10 and Corollary 1.5.12 as \( y^{q_1} = a_1^{q_1} \cdots a_s^{q_s} y^{q_1} \))
\[ = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) y^q \]
as required. Dual statement may be proved on the similar lines. \( \square \)

**Theorem 4.2.4:** Let \((1)\) be a seminormal identity and let \( p_1, p_2, \ldots, p_r, q_1, \ldots, q_s \) be any positive integers such that \( p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r; q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1 (r, s \geq 1) \) with \( p_1 + \cdots + p_r \geq q_0 - 2 \) and \( q_1 + \cdots + q_s \geq h_0 - 2 \) respectively. Then for any integer \( p \geq \max\{p_r, q_1\} \), all semigroup identities of the form
\[ x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_\ell^{p_1} y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_\ell^{p_1} y_1^{q_1} \cdots y_s^{q_s} \]  
(69)
where \( \ell \geq 3 \) and \( j \) is any permutation of the set \( \{1, 2, \ldots, \ell\} \), are preserved under epis in conjunction with \((1)\).

**Proof.** Take any semigroups \( U \) and \( S \) with \( U \) dense in \( S \). Assume \( U \) and hence \( S \) by Result 1.5.5, satisfy the identity \((1)\). We shall show that if \( U \) satisfies \((69)\), then so does \( S \). So let \( z_1, z_2, \ldots, z_\ell \in S \). If \( z_1, z_2, \ldots, z_\ell \in U \), then the result holds by Corollary 4.2.2.

For ease of writing, we introduce some notation:

\[ w_1(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_\ell}, y_1, \ldots, y_s) \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_1} \cdots z_{j_\ell}^{p_1} y_1^{q_1} \cdots y_s^{q_s} \]
\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, z_\ell, y_1, \ldots, y_s) \]
and

\[ w_2(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_\ell}, y_1, \ldots, y_s) \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_1} \cdots z_{j_\ell}^{p_1} y_1^{q_1} \cdots y_s^{q_s} \]
\[ = u_2(x_1, \ldots, x_r, z_1, \ldots, z_\ell, y_1, \ldots, y_s) \]
Using these definitions, the theorem asserts that

\[ w_1(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_q}, y_1, \ldots, y_s) = w_2(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_q}, y_1, \ldots, y_s) \]

or, equivalently, that

\[ u_1(x_1, \ldots, x_r, z_1, \ldots, z_{\ell}, y_1, \ldots, y_s) = u_2(x_1, \ldots, x_r, z_1, \ldots, z_{\ell}, y_1, \ldots, y_s) \]

We will prove the theorem by induction on \( q \), where the elements \( x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s, z_{j_1}, \ldots, z_{j_q} \) lie in \( S \) (\( q \geq 2 \)) and the remaining elements \( z_{j_1}, \ldots, z_{j_\ell} \) lie in \( U \).

First for \( q = 2 \), that is, when \( x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s, z_{j_1} \in S \) and \( z_{j_1}, \ldots, z_{j_2} \in U \), we wish to show that (69) holds. When \( z_{j_1} \in U \), (69) holds by Corollary 4.2.2. If, on the other hand \( z_{j_1} \in S \setminus U \), let (2) be a zigzag of minimal length \( m \) over \( U \) with value \( z_{j_1} \).

**Case (i).** \( j_1 = 1 \). Now

\[
\begin{align*}
x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_{j_1}} & \cdots z_{j_\ell}^{p_{j_\ell}} y_1^{q_1} \cdots y_s^{q_s} \\
= x_1^{p_1} \cdots x_r^{p_r} y_m a_{2m} z_{j_2}^{p_{j_2}} \cdots z_{j_\ell}^{p_{j_\ell}} y_1^{q_1} \cdots y_s^{q_s} & \text{(by the zigzag equations and Result 1.5.13)} \\
= x_1^{p_1} \cdots x_r^{p_r} y_m a_{2m} y_1^{q_1} y_1^{q_1} \cdots y_s^{q_s} & \text{by Result 1.5.11 for some } a_1^{(m)} \ldots a_r^{(m)} \in U \text{ and } y_m^{(m)} \in S \setminus U \text{ as } y_m \in S \setminus U \text{ and } a_{2m} = a_{2m-1} l_m \text{ with } l_m \in S \setminus U \\
= x_1^{p_1} \cdots x_r^{p_r} y_m a_{2m} y_1^{q_1} y_1^{q_1} \cdots y_s^{q_s} & \text{(by Result 1.5.11 as } y_m^{(m)} \text{, } t_m \in S \setminus U \text{ and where } w^{(m)} = a_{1}^{(m)pr-p_r-1} a_r^{(m)pr}} \\
= x_1^{p_1} \cdots x_r^{p_r} y_m a_{2m} y_1^{q_1} y_1^{q_1} \cdots y_s^{q_s} & \text{(by Corollary 4.2.2)} \\
= x_1^{p_1} \cdots x_r^{p_r} y_m a_{2m} y_1^{q_1} y_1^{q_1} \cdots y_s^{q_s} & \text{(by definition of } w^{(m)}) \\
= x_1^{p_1} \cdots x_r^{p_r} y_m a_{2m} y_1^{q_1} y_1^{q_1} \cdots y_s^{q_s} & \text{(by Results 1.5.10 and 1.5.11 as } y_m = y_m^{(m)pr-p_r} a_1^{(m)pr} \cdots a_r^{(m)pr}} \\
= x_1^{p_1} \cdots x_r^{p_r} y_m a_{2m} y_1^{q_1} y_1^{q_1} \cdots y_s^{q_s} & \text{(by the zigzag equations and Result 1.5.13)}
\end{align*}
\]

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as required.

**Case (ii).** \(1 < j_1 < \ell\). Now letting \(k = j_1\), we have

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} \cdots x_{j_\ell}^{(m)} y_1 \cdots y_s
\]

(by the zigzag equations and Result 1.5.13)

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} a_m a_{2m} \cdots y_1 y_1^{q_1} \cdots y_s
\]

(by Results 1.5.10 and 1.5.11 for some \(a_1^{(m)}, \ldots, a_r^{(m)} \in U\) and \(y_m^{(m)} \in S \setminus U\) as \(y_m \in S \setminus U\) and \(a_{2m} = a_{2m-1} t_m\) with \(t_m \in S \setminus U\))

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

(by Result 1.5.11 as \(y_m^{(m)}\), \(t_m \in S \setminus U\) and where \(w^{(m)} = a_1^{(m) - p_{j_1} - p_r - 1}\))

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} w_1(a_1^{(m)}, \ldots, a_r^{(m)}, a_{2m}, \cdots, y_1, y_1^{q_1}, \cdots, y_s)
\]

(by Corollary 4.2.2)

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

(by Results 1.5.10 and 1.5.11 for some \(a_1^{(m)}, \ldots, a_r^{(m)} \in U\) and \(y_m^{(m)} \in S \setminus U\) as \(y_m^{(m)}\) and \(t_m \in S \setminus U\))

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} a_1^{(m)} a_2^{(m)} \cdots y_1 y_1^{q_1} \cdots y_s
\]

where the last equality holds by the zigzag equations and Result 1.5.11 as \(y_m^{(m)}\), \(t_m \in S \setminus U\). Now, setting \(u^{(m)} = x_1^{(m)} \cdots x_{j_1}^{(m)} y_m^{p_{j_1} - p_r} y_m^{p_r} w^{(m)} u^{(m)}\), we have

\[
x_1^{(m)} \cdots x_{j_1}^{(m)} y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} z_{k} z_{k}^{p_{j_1} - p_r} \cdots z_{k} z_{k}^{p_{j_1} - p_r} \cdots y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} y_1 y_1^{q_1} \cdots y_s
\]

\[
u^{(m)} a_1^{(m)} \cdots a_r^{(m)} z_1 z_1^{p_{j_1} - p_r} z_{k-1} z_{k-1}^{p_{j_1} - p_r} \cdots z_{k-1} z_{k-1}^{p_{j_1} - p_r} y_1 y_1^{q_1} \cdots y_s
\]

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\[ u^{(m)} a_1^{(m)} p_1 \cdots a_r^{(m)} p_r z_1^{(p)} \cdots z_{k-1}^{(p)} a_{2m-1}^{(p)} b_{k+1}^{(p)} \cdots b_{\ell}^{(p)} c_1^{(q_1)} \cdots c_s^{(q_s)} e^{(m)} t_m^{(m)} p \]

(by Result 1.5.11 as \( y_{m}^{(m)} t_m^{(m)} \in S \cup \)) and where \( e^{(m)} = c_1^{(m)p-q_1} \cdots c_s^{(m)p-q_s} \)

\[ u^{(m)} a_1^{(m)} p_1 \cdots a_r^{(m)} p_r z_1^{(p)} \cdots z_{k-1}^{(p)} a_{2m-1}^{(p)} b_{k+1}^{(p)} \cdots b_{\ell}^{(p)} c_1^{(q_1)} \cdots c_s^{(q_s)} \nu^{(m)} \]

(where \( \nu^{(m)} = e^{(m)} t_m^{(m)} p d \))

\[ u^{(m)} u_2(a_1^{(m)}, \ldots, a_r^{(m)}, z_1^{(m)}, \ldots, z_{k-1}^{(m)} a_{2m-1}^{(m)}, b_{k+1}^{(m)}, \ldots, b_{\ell}^{(m)}, c_1^{(m)}, \ldots, c_s^{(m)}) \nu^{(m)} \]

\[ u^{(m)} u_1(a_1^{(m)}, \ldots, a_r^{(m)}, z_1^{(m)}, \ldots, z_{k-1}^{(m)} a_{2m-1}^{(m)}, b_{k+1}^{(m)}, \ldots, b_{\ell}^{(m)}, c_1^{(m)}, \ldots, c_s^{(m)}) \nu^{(m)} \]

(by Corollary 4.2.2)

Now the word \( u_1(\xi_1, \ldots, \xi_r, z_1, z_2, \ldots, z_l, \xi'_1, \ldots, \xi'_s) \) begins with \( \xi_1^{p_1} \cdots \xi_r^{p_r} z_1^{p} \) which equals \( \xi_1^{p_1} \cdots \xi_r^{p_r} z_1^{p} \). So, the above product in \( S \) contains \( y_{m}^{(m)p} y_{m}^{(m)p} z_1^{p} \). Thus, using Result 1.5.11 and the fact that \( y_{m}^{(m)p} y_{m}^{(m)p} z_1^{p} \) from the zigzag equations, we have

\[ x_1^{p_1} \cdots x_r^{p_r} z_1^{p} y_1^{q_1} \cdots y_s^{q_s} \]

\[ = u^{(m)} u_1(a_1^{(m)}, \ldots, a_r^{(m)}, z_1^{(m)}, \ldots, z_{k-1}^{(m)} a_{2m-1}^{(m)}, b_{k+1}^{(m)}, \ldots, b_{\ell}^{(m)}, c_1^{(m)}, \ldots, c_s^{(m)}) \nu^{(m)} \]

\[ = u^{(m-1)} u_1(a_1^{(m-1)}, \ldots, a_r^{(m-1)}, z_1^{(m-1)}, \ldots, z_{k-1}^{(m-1)} a_{2m-1}^{(m-1)}, b_{k+1}^{(m)}, \ldots, b_{\ell}^{(m)}, c_1^{(m)}, \ldots, c_s^{(m)}) \nu^{(m)} \]

(where \( u^{(m-1)} = x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{(m-1)p} z_1^{(m-1)} a_{2m-1}^{(m-1)} b_{k+1}^{(m-1)} \cdots b_{\ell}^{(m-1)}, c_1^{(m-1)}, \ldots, c_s^{(m-1)} ) \nu^{(m)} \) and by Results 1.5.10 and 1.5.11 for some \( a_1^{(m-1)}, \ldots, a_r^{(m-1)} \in U \) and \( y_{m-1}^{(m-1)} \in S \cup \) as \( y_{m-1} \) and \( t_m^{(m-1)} \in S \cup \) (by Corollary 4.2.2)

\[ = x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{(m-1)p} z_1^{(m-1)} a_{2m-1}^{(m-1)} b_{k+1}^{(m-1)} \cdots b_{\ell}^{(m-1)}, c_1^{(m-1)}, \ldots, c_s^{(m-1)}) \nu^{(m)} \]

(by definition of \( u^{(m-1)} \) and \( v^{(m)} \))

\[ = x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{(m-1)p} z_1^{(m-1)} a_{2m-1}^{(m-1)} b_{k+1}^{(m-1)} \cdots b_{\ell}^{(m-1)}, c_1^{(m)}, \ldots, c_s^{(m-1)p} \]

(by definition of \( e^{(m)} \))

\[ = x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{(m-1)p} z_1^{(m-1)} a_{2m-1}^{(m-1)} b_{k+1}^{(m-1)} \cdots b_{\ell}^{(m-1)}, c_1^{(m)}, \ldots, c_s^{(m)} \]

(by Result 1.5.11 as \( t_m^{(m)} = c_1^{(m)p} \cdots c_s^{(m)} p \))

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Continuing this way, we obtain

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} y_1^{p_1} \cdots y_s^{p_s}
\]

(by Result 1.5.11 as \( t_m = b_{k+1} \cdots b_{(m)p} t_{(m)p} \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} d
\]

(by the zigzag equations and Result 1.5.11 as \( y_{m-1}, t_{m-1} \in S \backslash U \))

Continuing this way, we obtain

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} y_1^{p_1} \cdots y_s^{p_s}
\]

(by Results 1.5.10 and 1.5.11 for some \( a_1^{(1)}, \ldots, a_r^{(1)} \in U \) and \( y_1^{(1)} \in S \backslash U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} a_1^{(1)p_1} d
\]

(by Result 1.5.11 as \( y_{m-1} \in S \backslash U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} w^{(1)} a_1^{(1)p_1} \cdots a_r^{(1)p_r} d
\]

(by Results 1.5.10 and 1.5.11 for some \( a_1^{(1)p_1}, \ldots, a_r^{(1)p_r} \in U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} a_1^{(1)p_1} \cdots a_r^{(1)p_r} d
\]

(by Results 1.5.10 and 1.5.11 for some \( a_1^{(1)p_1}, \ldots, a_r^{(1)p_r} \in U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} a_1^{(1)p_1} \cdots a_r^{(1)p_r} d
\]

(by Results 1.5.10 and 1.5.11 for some \( a_1^{(1)p_1}, \ldots, a_r^{(1)p_r} \in U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} b^{(1)p_1} c_1^{(1)p_1} \cdots c_s^{(1)p_1} d
\]

(by Results 1.5.10 and 1.5.11 for some \( b^{(1)p_1}, \ldots, c_s^{(1)p_1} \in S \backslash U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} b^{(1)p_1} c_1^{(1)p_1} \cdots c_s^{(1)p_1} d
\]

(by Results 1.5.10 and 1.5.11 for some \( b^{(1)p_1}, \ldots, c_s^{(1)p_1} \in S \backslash U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} b^{(1)p_1} c_1^{(1)p_1} \cdots c_s^{(1)p_1} d
\]

(by Results 1.5.10 and 1.5.11 for some \( b^{(1)p_1}, \ldots, c_s^{(1)p_1} \in S \backslash U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} b^{(1)p_1} c_1^{(1)p_1} \cdots c_s^{(1)p_1} d
\]

(by Results 1.5.10 and 1.5.11 for some \( b^{(1)p_1}, \ldots, c_s^{(1)p_1} \in S \backslash U \))

\[
x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_{m-1}} z_{j_1}^{p_{j_1}} \cdots z_{j_t}^{p_{j_t}} b^{(1)p_1} c_1^{(1)p_1} \cdots c_s^{(1)p_1} d
\]

(by Results 1.5.10 and 1.5.11 for some \( b^{(1)p_1}, \ldots, c_s^{(1)p_1} \in S \backslash U \))
As before, the above product contains the subword $y_1^{p_p}y_1^{p_p}a_p$. Thus, using Result 1.5.11 and the fact that $y_1a_1 = a_0$ from the zigzag equations, we have

$$x_1^{p_1}x_2^{p_2} \cdots x_r^{p_r}z_1^{p_1}z_2^{p_2} \cdots z_s^{p_s}y_1^{q_1}y_2^{q_2} \cdots y_s^{q_s}$$

$$= u^{(1)}u_1(a_1^{(1)}, \ldots, a_r^{(1)}, b_1^{(1)}, \ldots, b_{k+1}^{(1)}, \ldots, b_{t}^{(1)}, c_1^{(1)}, \ldots, c_s^{(1)})v^{(1)}$$

(by Corollary 4.2.2)

$$= u^{(1)}u_1(a_1^{(1)}, \ldots, a_r^{(1)}, b_1^{(1)}, \ldots, b_{k+1}^{(1)}, \ldots, b_{t}^{(1)}, c_1^{(1)}, \ldots, c_s^{(1)})v^{(1)}$$

(by definition of $e^{(1)}$).

$$= u^{(1)}u_1(a_1^{(1)}, \ldots, a_r^{(1)}, b_1^{(1)}, \ldots, b_{k+1}^{(1)}, \ldots, b_{t}^{(1)}, c_1^{(1)}, \ldots, c_s^{(1)})v^{(1)}$$

(by Corollary 4.2.2)

$$= u^{(1)}u_1(a_1^{(1)}, \ldots, a_r^{(1)}, b_1^{(1)}, \ldots, b_{k+1}^{(1)}, \ldots, b_{t}^{(1)}, c_1^{(1)}, \ldots, c_s^{(1)})v^{(1)}$$

(by definition of $e^{(1)}$).

$$= x_1^{p_1}x_2^{p_2} \cdots x_r^{p_r}z_1^{p_1}z_2^{p_2} \cdots z_s^{p_s}y_1^{q_1}y_2^{q_2} \cdots y_s^{q_s}$$

(as $v^{(1)} = e^{(1)}t_1^{(1)}p$).

This completes the proof in Case(ii).

**Case(iii).** $j_1 = \ell$. Now

$$x_1^{p_1} \cdots x_r^{p_r}z_1^{p_1} \cdots z_{j-1}^{p_1} \cdots z_{j-1}^{p_1}y_1^{q_1} \cdots y_s^{q_s}$$

$$= x_1^{p_1} \cdots x_r^{p_r}z_1^{p_1} \cdots z_{j-1}^{p_1} \cdots z_{j-1}^{p_1}y_1^{q_1} \cdots y_s^{q_s}$$

(by the zigzag equations and Result 1.5.13)
\[ x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r-p_1} y_m^{p_r-a_2m-2} \cdots z_j^{p_1} y_1^{q_1} \cdots y_s^{q_s} \]

(by Results 1.5.10 and 1.5.11 for some \( a_1^{(m)}, \ldots, a_r^{(m)} \in U \) and \( y_m^{(m)} \in S \setminus U \) as \( y_m \in S \setminus U \) and \( a_2m = a_2m-1 \) with \( t_m \in S \setminus U \))

\[ x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r-p_1} y_m^{p_r-a_2m-2} \cdots z_j^{p_1} y_1^{q_1} \cdots y_s^{q_s} \] (by Result 1.5.11 as \( y_m^{(m)}, t_m \in S \setminus U \) and where \( w^{(m)} = a_1^{(m)p_r-p_1} \cdots a_r^{(m)p_r-p_1} \))

\[ u^{(m)} w_1^{(m)}(a_1^{(m)}, \ldots, a_r^{(m)}, a_2m, z_j, 1, \ldots, y_s) \]

(where \( u^{(m)} = x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r-p_1} y_m^{p_r-a_2m-2} w(m) \))

\[ u^{(m)} w_2^{(m)}(a_1^{(m)}, \ldots, a_r^{(m)}, a_2m, z_j, 1, \ldots, y_s) \] (by Corollary 4.2.2)

\[ u^{(m)} a_1^{(m)p_1} \cdots a_r^{(m)p_r} z_1^{p_1} z_2^{p_2} \cdots z_t^{p_t} y_1^{q_1} \cdots y_s^{q_s} \]

(by the zigzag equations and Result 1.5.11)

\[ u^{(m)} a_1^{(m)p_1} \cdots a_r^{(m)p_r} z_1^{p_1} z_2^{p_2} \cdots z_t^{p_t} y_1^{q_1} \cdots y_s^{q_s} \]

(by Results 1.5.10 and 1.5.11 for some \( c_1^{(m)}, \ldots, c_r^{(m)} \in U \) and \( t_m^{(m)} \in S \setminus U \) as \( y_m \) and \( t_m \in S \setminus U \))

\[ e^{(m)} = c_1^{(m)p_r-q_1} \cdots c_r^{(m)p_r-q_1} \]

(by Result 1.5.11 as \( y_m^{(m)}, t_m^{(m)} \in S \setminus U \) and where \( e^{(m)} = c_1^{(m)p_r-q_1} \cdots c_r^{(m)p_r-q_1} \))

\[ u^{(m)} a_1^{(m)p_1} \cdots a_r^{(m)p_r} z_1^{p_1} z_2^{p_2} \cdots z_t^{p_t} y_1^{q_1} \cdots y_s^{q_s} \]

(where \( u^{(m)} = e^{(m)} t_m^{(m)} y_1^{q_1} \cdots y_s^{q_s} \))

\[ u^{(m)} u_2^{(m)}(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_t, a_2m-1, c_1^{(m)}, \ldots, c_r^{(m)}) y^{(m)} \]

\[ u^{(m)} u_1^{(m)}(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_t, a_2m-1, c_1^{(m)}, \ldots, c_r^{(m)}) y^{(m)} \] (by Corollary 4.2.2).

Now the word \( w_j^{(m)}(\xi_1, \cdots, \xi_r, z_1, \cdots, z_t, \xi_1', \cdots, \xi_r') \) begins with \( \xi_1^{p_1} \cdots \xi_r^{p_r} z_j^{p_1} \), which equals \( \xi_1^{p_1} \cdots \xi_r^{p_r} z_j^{p_1} \). So, the above product in \( S \) contains \( y_m^{p_r-p_1} y_m^{p_r-a_2m-1} \). Thus, using Result 1.5.11 and the fact that \( y_m a_2m = y_m a_2m-2 \) from the zigzag equations,
we have,
\[ x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{q_1} \cdots z_{j_s}^{q_s} \]
\[ = u^{(m)} u_1^{(m)}(a_1^{(m)}, \ldots, a_r^{(m)}, z_1, \ldots, z_{l-1}, a_{2m-1}, c_1^{(m)}, \ldots, c_s^{(m)}) u^{(m)} \]
\[ = u^{(m-1)} u_1^{(m-1)}(a_1^{(m-1)}, \ldots, a_r^{(m-1)}, z_1, \ldots, z_{l-1}, a_{2m-1}, c_1^{(m)}, \ldots, c_s^{(m)}) u^{(m)} \]

(Where \( u^{(m-1)} = x_1^{p_1} \cdots x_r^{p_r} y_{m-1}^{p_r} y_{m-1}^{q_1} \cdots y_{m-1}^{q_s} a_1^{(m-1)} a_2^{(m-1)} \cdots a_{m-1}^{(m-1)} \).

And by Results 1.5.10 and 1.5.11 for some \( a_1^{(m-1)}, \ldots, a_r^{(m-1)} \in U \) and \( y_{m-1}^{(m-1)} \in S \setminus U \) as \( y_{m-1}^{(m-1)} \in S \setminus U \) and \( t_m^{(m)} \in S \setminus U \)

\[ = u^{(m-1)} u_2^{(m-1)}(a_1^{(m-1)}, \ldots, a_r^{(m-1)}, z_1, \ldots, z_{l-1}, a_{2m-2}, c_1^{(m)}, \ldots, c_s^{(m)}) u^{(m)} \]

(by Corollary 4.2.2)

\[ = x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r} y_m^{p_r} \cdots y_m^{p_r} a_1^{(m)} a_2^{(m)} \cdots a_{m-2}^{(m)} c_1^{(m)} \cdots c_s^{(m)} e^{(m)} t_m^{(m)} y_1^{q_1} \cdots y_s^{q_s} \]

(by definitions of \( u^{(m)} \) and \( u^{(m)} \))

\[ = x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r} y_m^{p_r} \cdots y_m^{p_r} a_1^{(m)} a_2^{(m)} \cdots a_{m-2}^{(m)} c_1^{(m)} \cdots c_s^{(m)} t_m^{(m)} y_1^{q_1} \cdots y_s^{q_s} \]

(by definition of \( e^{(m)} \))

\[ = x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r} y_m^{p_r} \cdots y_m^{p_r} a_1^{(m)} a_2^{(m)} \cdots a_{m-2}^{(m)} y_1^{q_1} \cdots y_s^{q_s} \]

(by Result 1.5.11 as \( t_m^{(m)} = c_1^{(m)} \cdots c_s^{(m)} t_m^{(m)} \))

\[ = x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r} y_m^{p_r} \cdots y_m^{p_r} a_1^{(m)} a_2^{(m)} y_1^{q_1} \cdots y_s^{q_s} \]

where the last equality follows by Result 1.5.11 and the zigzag equations.

Continuing this way and letting \( y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \), we obtain

\[ x_1^{p_1} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \]

\[ = x_1^{p_1} \cdots x_r^{p_r} y_m^{p_r} y_m^{p_r} \cdots y_m^{p_r} y_1^{q_1} \cdots y_s^{q_s} \]

\[ : \]

\[ = x_1^{p_1} \cdots x_r^{p_r} y_1^{q_1} y_1^{q_1} y_1^{q_1} \cdots y_1^{q_1} t_1^{(1)} \]

(by Results 1.5.10 and 1.5.11 for some \( a_1^{(1)}, \ldots, a_r^{(1)} \in U \) and \( y_1^{(1)} \in S \setminus U \) as \( y_1, t_1 \in S \setminus U \))

\[ = x_1^{p_1} \cdots x_r^{p_r} y_1^{q_1} y_1^{q_1} w_1^{(1)} a_1^{(1)} a_2^{(1)} y_1^{q_1} \cdots y_1^{q_1} a_1^{(1)} t_1^{(1)} \]

(by Result 1.5.11 as \( y_1^{(1)} \), \( t_1 \in S \setminus U \) and where \( w_1^{(1)} = a_1^{(1)} a_2^{(1)} \cdots a_r^{(1)} \)).
\[ u^{(1)} a_1^{(p_1)} \ldots a_r^{(p_r)} z_1^{(p_1)} \ldots z_{\ell-1}^{(p_1)} a_1^{(q_1)} \ldots c_s^{(q_s)} t_1^{(p_1)} y \]

(where \( u^{(1)} = x^{(p_1)}_1 \ldots x^{(p_r)}_r y_1^{(p_r)} y_1^{(1)} \))

\[ = u^{(1)} a_1^{(p_1)} \ldots a_r^{(p_r)} z_1^{(p_1)} \ldots z_{\ell-1}^{(p_1)} a_1^{(q_1)} \ldots c_s^{(q_s)} t_1^{(p_1)} y \]

(by Results 1.5.10 and 1.5.11 for some \( c_1^{(1)}, \ldots, c_s^{(1)} \in U \) and \( t_1^{(1)} \in S \backslash U \) as \( y_1^{(1)} \) and \( t_1 \in S \backslash U \))

\[ = u^{(1)} a_1^{(p_1)} \ldots a_r^{(p_r)} z_1^{(p_1)} \ldots z_{\ell-1}^{(p_1)} a_1^{(q_1)} \ldots c_s^{(q_s)} e^{(1)} t_1^{(p_1)} y \]

(by Result 1.5.11 as \( y_1^{(1)}, t_1^{(1)} \in S \backslash U \) and where \( e^{(1)} = c_1^{(p_1-q_1)} \ldots c_s^{(p_1-q_s)} \))

\[ = u^{(1)} u_2(a_1^{(1)}, \ldots, a_r^{(1)}, z_1, \ldots, z_{\ell-1}, a_1^{(1)}, \ldots, c_s^{(1)}) v^{(1)} \]

(where \( v^{(1)} = e^{(1)} t_1^{(p_1)} y \))

\[ = u^{(1)} u_1(a_1^{(1)}, \ldots, a_r^{(1)}, z_1, \ldots, z_{\ell-1}, a_1^{(1)}, \ldots, c_s^{(1)}) v^{(1)} \]

(by Corollary 4.2.2.)

As before, the above product in \( S \) contains the subword \( y_1^{(p_1-q_1)} y_1^{(p_1)} a_1^{(p_1)} \). Thus, using Result 1.5.11 and the fact that \( y_1 a_1 = a_0 \) from the zigzag equations, we have

\[ = u^{(1)} u_1(a_1^{(1)}, \ldots, a_r^{(1)}, z_1, \ldots, z_{\ell-1}, a_1^{(1)}, \ldots, c_s^{(1)}) v^{(1)} \]

\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, z_{\ell-1}, a_0, c_1^{(1)}, \ldots, c_s^{(1)}) v^{(1)} \]

(by Corollary 4.2.2)

\[ = x_1^{p_1} x_2^{p_2} \ldots x_r^{p_r} z_1^{p_1} z_2^{p_2} \ldots z_{\ell-1}^{p_1} a_0^{p_1} c_1^{(1)} \ldots c_s^{(1)} e^{(1)} t_1^{(p_1)} y \]

(by definition of \( v^{(1)} \))

\[ = x_1^{p_1} x_2^{p_2} \ldots x_r^{p_r} z_1^{p_1} z_2^{p_2} \ldots z_{\ell-1}^{p_1} a_0^{p_1} c_1^{(1)} \ldots c_s^{(1)} t_1^{(p_1)} y \]

(by definition of \( e^{(1)} \))

\[ = x_1^{p_1} x_2^{p_2} \ldots x_r^{p_r} z_1^{p_1} z_2^{p_2} \ldots z_{\ell-1}^{p_1} a_0^{p_1} y \]

(by Results 1.5.10 and 1.5.11 as \( c_1^{(1)} \ldots c_s^{(1)} t_1^{(p_1)} = t_1^{(1)} \))

\[ = x_1^{p_1} \ldots x_r^{p_r} z_1^{p_1} z_2^{p_2} \ldots z_{\ell-1}^{p_1} y_1^{q_1} \ldots y_1^{q_s} \]

where the last equality follows by zigzag equations, Result 1.5.13; and as \( j_1 = \ell \) and \( y = y_1^{q_1} \ldots y_1^{q_s} \).

This is the end of the proof in case (iii) and, thus, of the base \( q = 2 \) of the induction.
Next, assume inductively that the result holds when \( x_1, \ldots, x_r, y_1, \ldots, y_s, z_{j_1}, \ldots, z_{j_{q-1}} \) are in \( S \) (\( q > 2 \)) and \( z_{j_q}, \ldots, z_{j_e} \in U \). From this assumption, we shall prove that the result also holds when \( x_1, \ldots, x_r, y_1, \ldots, y_s, z_{j_1}, \ldots, z_{j_{q-1}}, z_{j_q} \in S \) and \( z_{j_{q+1}}, \ldots, z_{j_e} \in U \). So take any \( x_1, \ldots, x_r, y_1, \ldots, y_s, z_{j_1}, \ldots, z_{j_{q-1}}, z_{j_q}, z_{j_{q+1}}, \ldots, z_{j_e} \in S \) and \( z_{j_{q+1}}, \ldots, z_{j_e} \in U \). Assume that \( z_{j_q} \in S \setminus U \). Let (2) be a zigzag of minimal length \( m \) over \( U \) with value \( z_{j_q} \). Put \( k = j_q \) and \( t = j_{q-1} \). As equalities (70) and (71) below hold by Result 1.5.15 as \( y_m \in S \setminus U \) and \( p_1 + \ldots + p_r + 1 \geq g_0 - 1 \), we have

**Case (1).** \( t = k - 1 \). Now

\[
\begin{align*}
& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q \\
& = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots z_{j_{q-1}}^{p_j} y_m^{p_j} a_{2m}^{p_j} z_{j_{q+1}}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q \\
& \quad \text{(by the zigzag equations and Result 1.5.13)} \\
& = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots (z_{j_{q-1}}^{p_j} y_m^{p_j} a_{2m}^{p_j} z_{j_{q+1}}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q) \\
& = w_1(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_{q-2}}, z_{j_{q-1}} y_m, a_{2m}, z_{j_{q+1}}, \ldots, z_{j_e}, y_1, \ldots, y_s) \\
& = w_2(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_{q-2}}, z_{j_{q-1}} y_m, a_{2m}, z_{j_{q+1}}, \ldots, z_{j_e}, y_1, \ldots, y_s) \\
& \quad \text{(by the inductive hypothesis)} \\
& = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots z_{k-2}^{p_j} (z_{j_{q-1}}^{p_j} y_m^{p_j} a_{2m}^{p_j} z_{k}^{p_j} z_{k+1}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q) \quad \text{(as \( t = k - 1 \))} \\
& = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots z_{k-2}^{p_j} z_{j_{q-1}}^{p_j} y_m^{p_j} a_{2m}^{p_j} z_{k+1}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q \\
& \quad \text{(by the zigzag equations and Result 1.5.13)} \\
& = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots z_{k-2}^{p_j} z_{k-1}^{p_j} z_{k}^{p_j} z_{k+1}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q \quad \text{(as \( j_{q-1} = k - 1 \))} 
\end{align*}
\]

as required.

**Case (2).** \( t < k - 1 \) and \( k \leq \ell \). Now, as above, we have

\[
\begin{align*}
& x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q \\
& = x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_j} \cdots z_{j_{q-1}}^{p_j} y_m^{p_j} a_{2m}^{p_j} z_{j_{q+1}}^{p_j} \cdots z_{j_e}^{p_j} y_1^q \cdots y_s^q \\
& \quad \text{(by the zigzag equations and Result 1.5.13)} 
\end{align*}
\]
\[ x_1^{p_1} \cdots x_r^{p_r} z_j^p \cdots (z_{m-1} y_m)^p a_{2m}^{p_2} z_{j+1}^{p_1} \cdots z_j^{p_1} y_1^{p_1} \cdots y_s^{p_1} \]

(by Result 1.5.15 as \( y_m \in S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \))

\[ = w_1(x_1, \ldots, x_r, z_j, \ldots, z_{j-k}, z_{j-k+1} y_m, a_{2m}, z_{j+1}, \ldots, z_j, y_1, \ldots, y_s) \]

\[ = w_2(x_1, \ldots, x_r, z_j, \ldots, z_{j-k}, z_{j-k+1} y_m, a_{2m}, z_{j+1}, \ldots, z_j, y_1, \ldots, y_s) \]

(by the inductive hypothesis)

\[ = x_1^{p_1} \cdots x_r^{p_r} z_j^p \cdots (z_{j-k} y_m)^p z_{t+1}^p \cdots z_{t-k}^p a_{2m}^p z_{k+1}^p \cdots z_{t}^p y_1^{p_1} \cdots y_s^{p_1} \]

(72)

where the last equality holds by the zigzag equations and Result 1.5.13, and the equality (72) above follows by Result 1.5.15 as \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \) and \( a_{2m} = a_{2m-1} t_m \) with \( t_m \in S \setminus U \), as required.

**Case (3).** \( k + 1 = t \). As the equality (73) below holds by the inductive hypothesis, and equalities (74) and (75) below follow by the zigzag equations and by Result 1.5.15 as \( y_m \in S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \) respectively, we have

\[ x_1^{p_1} \cdots x_r^{p_r} z_j^p \cdots z_j^{p_1} y_1 \cdots y_s^{p_1} \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_j^p \cdots (z_{j-k} y_m)^p a_{2m}^p z_{j+1}^p \cdots z_j^{p_1} y_1^{p_1} \cdots y_s^{p_1} \]

(by the zigzag equations and Result 1.5.13)

\[ = x_1^{p_1} \cdots x_r^{p_r} z_j^p \cdots (z_{j-k} y_m)^p a_{2m}^p z_{j+1}^p \cdots z_j^{p_1} y_1^{p_1} \cdots y_s^{p_1} \]

(by Result 1.5.15 as \( y_m \in S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \))

\[ = w_1(x_1, \ldots, x_r, z_j, \ldots, z_{j-k+1}, z_{j+1} y_m, a_{2m}, z_{j+1}, \ldots, z_j, y_1, \ldots, y_s) \]

\[ = w_2(x_1, \ldots, x_r, z_j, \ldots, z_{j-k+1}, z_{j+1} y_m, a_{2m}, z_{j+1}, \ldots, z_j, y_1, \ldots, y_s) \]

(73)

\[ = w_2(x_1, \ldots, x_r, z_j, \ldots, z_{j-k+1}, z_{j+1} y_m, a_{2m-1} t_m, z_{j+1}, \ldots, z_j, y_1, \ldots, y_s) \]

(74)

\[ = x_1^{p_1} \cdots x_r^{p_r} z_j^p \cdots z_{k-1}^p (a_{2m-1} t_m)^p (z_{j-k} y_m)^p z_{t+1}^p \cdots z_{t-k}^p y_1^p \cdots y_s^p \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_j^p \cdots z_{k-1}^p (a_{2m-1} t_m)^p (z_{j-k} y_m)^p z_{t+1}^p \cdots z_{t-k}^p y_1^p \cdots y_s^p \]

(75)
\[ u_2(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_{2m-1}, t_m z_{j_t-1} y_m, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s) \]

\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_{2m-1}, t_m z_{j_t-1} y_m, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s), \]

where the last equality holds by the inductive hypothesis. Since \( z_{j_t-1} z_{j_q} \) is a subword of the word \( u_1(\xi_1, \ldots, \xi_r, z_1, \ldots, z_t, \xi_1', \ldots, \xi_s') \), the above product in \( S \) contains \( (t_m z_{j_t-1} y_m)^p a_{2m-1}^p \). Thus, using dual of Result 1.5.15 as \( t_m \in S \setminus U \) and \( y_1^q \cdots y_s^q \in S^q \), and the fact that \( y_m a_{2m-1} = y_{m-1} a_{2m-2} \) from the zigzag equations, we have

\[ \sum_{i=1}^{p_1} x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} y_1^q \cdots y_s^q \]

\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_{2m-1}, t_m z_{j_t-1} y_m, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s) \]

\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_{2m-2}, t_m z_{j_t-1} y_m, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s) \]

\[ = u_2(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_{2m-2}, t_m z_{j_t-1} y_m, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s) \quad (76) \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} (t_m z_{j_t-1} y_m)^p z_{t+1} \cdots z_t y_1^q \cdots y_s^q \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} (t_m z_{j_t-1} y_m)^p z_{t+1} \cdots z_t y_1^q \cdots y_s^q, \]

where the last equality and the equality (76) above follow by Result 1.5.15, as \( y_{m-1} \in S \setminus U \) and \( p_1 + \ldots + p_r + 1 \geq g_0 - 1 \), and the inductive hypothesis respectively.

As equalities (77) and (78) below follow by zigzag equations and Result 1.5.15, as \( y_{m-1} \in S \setminus U \) and \( p_1 + \ldots + p_r + 1 \geq g_0 - 1 \), we have

\[ x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} y_1^q \cdots y_s^q \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} (t_m z_{j_t-1} y_m)^p z_{t+1} \cdots z_t y_1^q \cdots y_s^q \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} (t_{m-1} z_{j_t-1} y_m)^p z_{t+1} \cdots z_t y_1^q \cdots y_s^q \quad (77) \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} (t_{m-1} z_{j_t-1} y_m)^p z_{t+1} \cdots z_t y_1^q \cdots y_s^q. \quad (78) \]

Thus, continuing this way, we obtain

\[ x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} y_1^q \cdots y_s^q \]

\[ = x_1^{p_1} \cdots x_r^{p_r} z_{j_{i-1}}^{p_i} \cdots z_j^{p_r} (t_{m-1} z_{j_t-1} y_m)^p z_{t+1} \cdots z_t y_1^q \cdots y_s^q. \]

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As before, the word \( u_1(\xi_1, \ldots, \xi_r, z_1, \ldots, z_t, \xi'_1, \ldots, \xi'_s) \) contains \( z_{j_{q-1}} z_j \) as a subword, so the above product in \( S \) contains \( (t_1 z_{j_{q-1}} y_1)^p a_1^p \). Thus, using Result 1.5.15 and the fact that \( y_t a_t = a_0 \) from the zigzag equations, we have

\[
x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{j_{q-1}}^{p_1} y_1^{q_1} \cdots y_s^{q_s}
\]

\[
= u_1(x_1, \ldots, x_r, z_1, \ldots, z_t, a_0, t_1 z_{j_{q-1}} y_1, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s)
\]

\[
= u_1(x_1, \ldots, x_r, z_1, \ldots, z_{t+1}, a_0, t_1 z_{j_{q-1}}, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s)
\]

\[
= u_2(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_0, t_1 z_{j_{q-1}}, z_{t+1}, \ldots, z_t, y_1, \ldots, y_s)
\]

(by the inductive hypothesis)

\[
= x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} y_1^{q_1} \cdots y_s^{q_s}
\]

\[
= x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} y_1^{q_1} \cdots y_s^{q_s}
\]

(by Result 1.5.15 as \( t_1 \in S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \))

\[
= x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} z_{j_{q-1}}^{p_1} z_{t+1}^{p_1} \cdots y_1^{q_1} \cdots y_s^{q_s}
\]

(by Result 1.5.13 and the zigzag equations)

\[
= x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} z_{j_{q-1}}^{p_1} z_{t+1}^{p_1} \cdots z_1^{p_1} \cdots y_s^{q_s}
\]

as required.

**Case (4).** \( k + 1 < t \) and \( t < \ell \). As the equality (79) below follows by Result 1.5.15 because \( y_m \in S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \), we have

\[
x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_1} \cdots z_{j_{q-1}}^{p_1} y_1^{q_1} \cdots y_s^{q_s}
\]

\[
= x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_1} \cdots z_{j_{q-1}}^{p_1} y_m \cdots y_1^{q_1} \cdots y_s^{q_s}
\]

(by the zigzag equations and Result 1.5.13)

\[
= x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_1} \cdots z_{j_{q-1}}^{p_1} (z_{j_{q-1}} y_m)^p a_{2m}^{p} z_{j_{q+1}}^{p_1} \cdots y_1^{q_1} \cdots y_s^{q_s}
\]

(79)
\[ w_1(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_{q-1}}, z_{j_{q-1}}, y_1, \ldots, y_s) \]
\[ = w_2(x_1, \ldots, x_r, z_{j_1}, \ldots, z_{j_{q-1}}, z_{j_{q-1}}, y_1, \ldots, y_s) \]
(by the inductive hypothesis)
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} z_{k+1}^{p_1} \cdots z_{t-1}^{p_1}(z_{j_{q-1}}, y_{m})^{p_1} z_{t+1}^{p_1} \cdots z_{s}^{p_1} y_1^{q_1} \cdots y_s^{q_s} \]
where the last equality holds by the zigzag equations and Result 1.5.15 as \( t_m \) is in \( S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \). As the equality (80) below holds by Results 1.5.10 and 1.5.15 for some \( b_{k+1}^{(m)}, \ldots, b_{k+(t-k)}^{(m)} \in U \) and \( t_m^{(m)} \in S \setminus U \), as \( t_m \in S \setminus U \) and letting \( z = z_{t+1}^{p_1} \cdots z_{t}^{p_1} y_1^{q_1} \cdots y_s^{q_s} \) and \( w_i^{(k)} = t_i^{(k)} z_{k+1} \cdots z_{t-1}(z_{j_{q-1}}, y_k) \) where \( i, k \in \{1, 2, \ldots, m\} \), we have
\[ x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} z_{k+1}^{p_1} \cdots z_{t-1}^{p_1}(z_{j_{q-1}}, y_{m})^{p_1} z \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} z_{k+1}^{p_1} \cdots z_{t-1}^{p_1}(z_{j_{q-1}}, y_{m})^{p_1} z \]
\[ = u_2(x_1, \ldots, x_r, z_1, \ldots, a_{2m-1}, b_{k+1}^{(m)}, \ldots, b_{k+(t-k)}^{(m)}, w_m^{(m)}, \ldots, z_{t}, y_1, \ldots, y_s) \]
\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, a_{2m-1}, b_{k+1}^{(m)}, \ldots, b_{k+(t-k)}^{(m)}, w_m^{(m)}, \ldots, z_{t}, y_1, \ldots, y_s), \]
where the last equality holds by the inductive hypothesis and the equality (81) holds by Result 1.5.15, as \( y_{m} \in S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \). Also equalities (82), (83) and (84) follow respectively by the inductive hypothesis, by Result 1.5.15 as \( y_{m-1} \in S \setminus U \) and \( p_1 + \cdots + p_r + 1 \geq g_0 - 1 \), and by Results 1.5.10 and 1.5.15 as \( b_{k}^{(m) p} \cdots b_{k+(t-k)}^{(m) p} = t_m^{p} \). Since the word \( u_1(\xi_1, \ldots, \xi_r, z_1, \ldots, z_\ell, \xi'_{1}, \ldots, \xi'_{s}) \) contains \( z_{j_{q-1}}, z_{j_{q-1}} \) as a subword, \( (t_m^{(m)} \cdots z_{t-1}(z_{j_{q-1}}, y_{m}))^{p} a_{2m-1}^{(m)} \) is contained in the above product in \( S \). Thus, using Result 1.5.15 and the fact that \( y_m a_{2m-1} = y_{m-1} a_{2m-2} \) from the zigzag equations, we have
\[ x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{j_{1}}^{p_1} z_{j_{1}}^{p_1} \cdots z_{j_{s}}^{p_1} y_1^{q_1} \cdots y_s^{q_s} \]
\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, a_{2m-1}, b_{k+1}^{(m)}, \ldots, b_{k+(t-k)}^{(m)}, w_m^{(m)}, \ldots, z_{t}, y_1, \ldots, y_s) \]
\[ u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, w_{m-1}, \ldots, z_t, y_1, \ldots, y_s) \]
\[ = u_2(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, w_{m-1}, \ldots, z_t, y_1, \ldots, y_s) \quad \text{(82)} \]
\[ x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{r-1}^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} w_{m-1}^{(m-1)p} \cdots z_t^{p} y \]
\[(\text{where } y = y_1 \cdots y_s)\]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} \cdots z_{t-1}^{p_1} (z_{j_1}, y_{m-1}) \cdots z_t^{p} y \quad \text{(83)} \]
\[(\text{by definition of } w_{m-1}^{(m-1)p}) \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} \cdots z_{t-1}^{p_1} (z_{j_1}, y_{m-1}) \cdots z_t^{p} y \quad \text{(84)} \]
where the last equality holds by the zigzag equations and Result 1.5.15.

Continuing this way, we obtain

\[ x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{r-1}^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} \cdots z_{t-1}^{p_1} (z_{j_1}, y_{m-1}) \cdots z_t^{p} y \]
\[ \cdots \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} \cdots z_{t-1}^{p_1} (z_{j_1}, y_{m-1}) \cdots z_t^{p} y \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} \cdots z_{t-1}^{p_1} (z_{j_1}, y_{m-1}) \cdots z_t^{p} y \quad \text{(85)} \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} (w_1^{(1)p}) \cdots z_t^{p} y, \]
where the last equality follows by Result 1.5.15 as \( y_1 \in S \setminus U \) and \( p_1 + \ldots + p_r + 1 \geq g_0 - 1 \) and the equality (85) above follows by Results 1.5.10 and 1.5.15 for some \( b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)} \) in \( U \) and \( t_1^{(1)} \in S \setminus U \), as \( t_1 \in S \setminus U \) and \( p_1 + \ldots + p_r + 1 \geq g_0 - 1 \).

Therefore,

\[ x_1^{p_1} \cdots x_r^{p_r} z_{j_1}^{p_1} \cdots z_{j_t}^{p_1} \cdots z_{t-1}^{p_1} (w_1^{(1)p}) \cdots z_t^{p} y \]
\[ \cdots \]
\[ = x_1^{p_1} \cdots x_r^{p_r} z_1^{p_1} \cdots z_{k-1}^{p_1} \cdots z_{k+(t-k-1)}^{p_1} (w_1^{(1)p}) \cdots z_t^{p} y \]
\[ = u_2(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_1, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}; w_1^{(1)}, \ldots, z_t, y_1, \ldots, y_s) \]
\[(\text{by definition of } y)\]
\[ = u_1(x_1, \ldots, x_r, z_1, \ldots, z_{k-1}, a_1, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}; w_1^{(1)}, \ldots, z_t, y_1, \ldots, y_s), \]
87
where the last equality holds by the inductive hypothesis, and the equality (86) below holds by the inductive hypothesis and Result 1.5.15 as, $t_{1}^{(1)} \in S \setminus U$ and $q_{1} + \cdots + q_{s} + 1 \geq h_{0} - 1$ respectively. Also equalities (87), (88) and (89) below follow respectively by the inductive hypothesis, Corollary 1.5.12 as, $t_{1}^{(1)} \in S \setminus U$ and $q_{1} + \cdots + q_{s} + 1 \geq h_{0} - 1$; and as $t_{1}^{(1)} \in S \setminus U$ and $y_{k+1}^{(1)} = y_{k+1}^{(1)} c_{k+1}'$ with $y_{k+1}^{(1)} \in S \setminus U$ and $c_{k+1}' \in U$, and by the zigzag equations and Result 1.5.13 respectively. As before, the word $u_{1}(\xi_{1}, \ldots, \xi_{r}, z_{1}, \ldots, z_{t}, \xi_{1}', \ldots, \xi_{s}')$ contains $z_{j_{k-1}} z_{j_{k}}$ as a subword, the above product in $S$ contains $(z_{j_{k-1}}, y_{1})^{p} a_{1}^{p}$. Thus, using Result 1.5.15 as $t_{1}^{(1)} \in S \setminus U$ and $q_{1} + \cdots + q_{s} + 1 \geq h_{0} - 1$ and the fact that $y_{a_{1}} = a_{0}$ from the zigzag equations, we have

\[
x_{1}^{p_{1}} x_{2}^{p_{2}} \cdots x_{r}^{p_{r}} z_{j_{1}}^{p_{1}} z_{j_{2}}^{p_{2}} \cdots z_{j_{s}}^{p_{s}} y_{1}^{q_{1}} y_{2}^{q_{2}} \cdots y_{s}^{q_{s}}
\]

\[
= u_{1}(x_{1}, \ldots, x_{r}, z_{1}, \ldots, z_{k-1}, a_{1}, b_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}, u_{1}^{(1)}, \ldots, z_{t}, y_{1}, \ldots, y_{s})
\]

\[
= u_{1}(x_{1}, \ldots, x_{r}, z_{1}, \ldots, z_{k-1}, a_{0}, c_{k+1}^{(1)}, \ldots, b_{k+(t-k-1)}^{(1)}, f, \ldots, z_{t}, y_{1}, \ldots, y_{s})
\]

(by definition of $u_{1}^{(1)}$ and where $f = t_{1}^{(1)} z_{k+1} \cdots z_{k-1} z_{j_{k-1}}$) (86)

\[
x_{1}^{p_{1}} \cdots x_{r}^{p_{r}} z_{j_{1}}^{p_{1}} \cdots z_{j_{s}}^{p_{s}} y_{1}^{q_{1}} \cdots y_{s}^{q_{s}}
\]

(by definition of $f$)

\[
x_{1}^{p_{1}} \cdots x_{r}^{p_{r}} z_{j_{1}}^{p_{1}} \cdots z_{j_{s}}^{p_{s}} a_{0}^{p_{1}^{(1)}} b_{k+1}^{(1)} \cdots b_{k+(t-k-1)}^{(1)} (t_{1}^{(1)} z_{k+1} \cdots z_{t-1} z_{j_{k-1}})^{p} \cdots z_{t}^{p_{t}} y_{1}^{q_{1}} \cdots y_{s}^{q_{s}}
\]

(87)

\[
x_{1}^{p_{1}} \cdots x_{r}^{p_{r}} z_{j_{1}}^{p_{1}} \cdots z_{j_{s}}^{p_{s}} a_{0}^{p_{1}^{(1)}} b_{k+1}^{(1)} \cdots b_{k+(t-k-1)}^{(1)} (t_{1}^{(1)} z_{k+1} \cdots z_{t-1} z_{j_{k-1}})^{p} \cdots z_{t}^{p_{t}} y_{1}^{q_{1}} \cdots y_{s}^{q_{s}}
\]

(88)

\[
x_{1}^{p_{1}} \cdots x_{r}^{p_{r}} z_{j_{1}}^{p_{1}} \cdots z_{j_{s}}^{p_{s}} a_{0}^{p_{1}^{(1)}} b_{k+1}^{(1)} \cdots b_{k+(t-k-1)}^{(1)} (t_{1}^{(1)} z_{k+1} \cdots z_{t-1} z_{j_{k-1}})^{p} \cdots z_{t}^{p_{t}} y_{1}^{q_{1}} \cdots y_{s}^{q_{s}}
\]

(89)

\[
x_{1}^{p_{1}} \cdots x_{r}^{p_{r}} z_{j_{1}}^{p_{1}} \cdots z_{j_{s}}^{p_{s}} a_{0}^{p_{1}^{(1)}} b_{k+1}^{(1)} \cdots b_{k+(t-k-1)}^{(1)} (t_{1}^{(1)} z_{k+1} \cdots z_{t-1} z_{j_{k-1}})^{p} \cdots z_{t}^{p_{t}} y_{1}^{q_{1}} \cdots y_{s}^{q_{s}}
\]

(89)

as required.

**Case (5).** $k + 1 < t$ and $t = \ell$.

The proof in this case may be obtained by modifying the proof of Case (4) in the following way:

Letting $a = x_{1}, \ldots, x_{r}$ and $b = y_{1}, \ldots, y_{s}$.
(a) Replace the word \( z_{t+1}^p \cdots z_{t}^p \) by 1;

(b) Replace
\[
(a, z_1, \ldots, z_{k-1}, a_{2c-1}, b_{k+1}, \ldots, b_{k+(t-k-1)}, t_{e}^{(c)} z_{k+1} \cdots z_{t-1}(z_{j_u-1}, y_c), z_{t+1}, \ldots, z_{t}, b)
\]
by
\[
(a, z_1, z_2, \ldots, z_{k-1}, a_{2c-1}, b_{k+1}, \ldots, b_{k+(t-k-1)}, t_{e}^{(c)} z_{k+1} \cdots z_{t-1}(z_{j_u-1}, y_c), b)
\]
for all \( c = 1, 2, \ldots, m; \)

(c) Replace
\[
(a, z_1, \ldots, z_{k-1}, a_{2c-2}, b_{k+1}, \ldots, b_{k+(t-k-1)}, t_{e}^{(c)} z_{k+1} \cdots z_{t-1}(z_{j_u-1}, y_c), z_{t}, b)
\]
by
\[
(a, z_1, z_2, \ldots, z_{k-1}, a_{2c-2}, b_{k+1}, \ldots, b_{k+(t-k-1)}, t_{e}^{(c)} z_{k+1} \cdots z_{t-1}(z_{j_u-1}, y_c), b)
\]
for all \( c = 1, 2, \ldots, m; \) and \( y_0 \) by 1 when \( c = 1. \)

Thus, the proof of the theorem is completed. \( \square \)

The following theorem extends the class of homotypical identities that are preserved under epis in conjunction with a seminormal identity.

**Theorem 4.2.5:** Let \( u \) and \( v \) be any words in the variables \( z_1, z_2, \ldots, z_{\ell} \) such that \(|z_i|_u = |z_i|_v\) for all \( i = 1, 2, \ldots, \ell. \) Then all semigroup identities of the form
\[
x_1^{p_1} \cdots x_{\ell}^{p_{\ell}} u(z_1, \ldots, z_{\ell}) y_1^{q_1} \cdots y_{\ell}^{q_{\ell}} = x_1^{p_1} \cdots x_{\ell}^{p_{\ell}} v(z_1, \ldots, z_{\ell}) y_1^{q_1} \cdots y_{\ell}^{q_{\ell}} (\ell \geq 3)
\]
(90)
are preserved under epis in conjunction with (1).

**Proof.** Take any semigroups \( U \) and \( S \) with \( U \) as a subsemigroup of \( S \) such that \( U \) is dense in \( S. \) Let \( U \) satisfy (1) and (90). As \( U \) satisfies (1), by Result 1.5.5, \( S \) also satisfies (1). Now we shall prove that the identity (90) satisfied by \( U \) is also satisfied by \( S. \) Suppose that \( x_1, x_2, \ldots, x_{\ell}, y_1, y_2, \ldots, y_{\ell}, z_1, z_2, \ldots, z_{\ell} \in S. \) If all of \( z_1, z_2, \ldots, z_{\ell} \) are in \( U, \) then, (90) is satisfied by Lemma 4.2.1. So we assume that not all of \( z_1, z_2, \ldots, z_{\ell} \) are from \( U. \) Suppose that \( z_j \in S \setminus U, \) for some \( j \in \{1, 2, \ldots, \ell\}. \)

By Result 1.5.1, \( z_j = x'b = x'b'y' \) for some \( x', y' \in S \setminus U \) and \( b, b' \in U \) with \( b = b'y'. \)
Now, as equalities (91) and (92) below hold respectively by Result 1.5.15 as \( y' \in S \setminus U \) and \( p_1 + \cdots + p_r \geq g_0 - 1 \), and by Result 1.5.15 as \( x' \in S \setminus U \), \( q_1 + \cdots + q_s \geq h_0 - 1 \) together with the fact that \( |z_i|_u = |z_i|_v \) for all \( i = 1, 2, \ldots, \ell \), we have

\[
x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r} u(z_1, z_2, \ldots, z_r) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = x_1^{q_1} x_2^{q_2} \cdots x_r^{q_r} (x')^j u(z_1, z_2, \ldots, z_{j-1}, b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s},
\]

(91)

\[
x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_{j-1}, b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} (x')^j v(z_1, z_2, \ldots, z_{j-1}, b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s},
\]

(92)

\[
x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_{j-1}, b'y', z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(z_1, z_2, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s},
\]

(93)

where the equality (93) above follows by Result 1.5.15 as \( y' \in S \setminus U \) and \( p_1 + \cdots + p_r \geq g_0 - 1 \), as required.

Corollary 4.2.6: Let (1) be any seminormal identity. Then all semigroup identities of the form

\[
x_1^{p_1} \cdots x_r^{p_r} z_1^{t_1} \cdots z_\ell^{t_\ell} y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} z_1^{t_1} \cdots z_\ell^{t_\ell} y_1^{q_1} \cdots y_s^{q_s}
\]

\( (\ell \geq 3) \),

where \( j \) is any permutation of the set \( \{1, 2, \ldots, \ell\} \) and \( t_1, t_2, \ldots, t_\ell \geq 1 \), are preserved under epis in conjunction with (1).

§ 4.3. NON-BALANCED IDENTITIES

A homotypical identity \( u = v \) is said to be non-balanced if \( |x|_u \neq |x|_v \), for some \( x \in C(u) (= C(v)) \). In this section, we establish some sufficient conditions for non-balanced identities to be preserved under epis in conjunction with a seminormal identity.

Theorem 4.3.1: All semigroup identities of the form

\[
x_1^{p_1} \cdots x_r^{p_r} u(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(z_1, \ldots, z_\ell) y_1^{q_1} \cdots y_s^{q_s} \quad (\ell \geq 3),
\]

(94)

where \( u \) and \( v \) be any words in the variables \( z_1, z_2, \ldots, z_\ell \) such that \( \min \{|z_i|_u, |z_i|_v\} \geq \min \{p_r, q_i\} \) for all \( i \) in \( \{1, 2, \ldots, \ell\} \), are preserved under epis in conjunction with a
seminormal identity.

The following lemmas, where \( U \) and \( S \) be any semigroups such that \( U \) is dense in \( S \), will be required to complete the proof of Theorem 4.3.1; bracketed clauses yield the dual statements.

In the following, \( S^1 \) will denote the semigroup obtained from the semigroup \( S \) by adjoining an identity, if necessary; while the length of a word \( u \), denoted by \( \ell(u) \), is defined as the sum of the occurrences of all the variables appearing in \( u \).

**Lemma 4.3.2**([37, Lemma 2.7.1]): Let (1) be any seminormal identity, and let \( u, v \) and \( w \) be any words in the variables \( x_1, x_2, \ldots, x_k \) \((k \geq 2)\) such that \( \ell(u) \geq g_0 - 1 \) and \( \ell(v) \geq h_0 - 1 \). Take any \( a_1, a_2, \ldots, a_k \in U \) and \( t_1, t_2, \ldots, t_k \in S^1 \). If for each \( i \) such that \( t_i \in S \), \( a_i = y_i b_i \) \([a_i = b_i y_i]\) for some \( y_i \in S \setminus U \) and \( b_i \in S \) \((i = 1, 2, \ldots, k)\), then for any choice \( d_1, d_2, \ldots, d_k \) for the variables \( x_1, x_2, \ldots, x_k \) in \( S^1 \) respectively

\[
u(\tilde{d}) w(a_1 t_1, a_2 t_2, \ldots, a_k t_k)v(\tilde{d}) = u(\tilde{d}) w(a_1, a_2, \ldots, a_k)w(t_1, t_2, \ldots, t_k)v(\tilde{d})
\]

\[
[u(\tilde{d}) w(t_1 a_1, t_2 a_2, \ldots, t_k a_k)v(\tilde{d}) = u(\tilde{d}) w(t_1, t_2, \ldots, t_k)w(a_1, a_2, \ldots, a_k)v(\tilde{d})],
\]

where \( \tilde{d} = (d_1, d_2, \ldots, d_k) \).

**Lemma 4.3.3**([37, Lemma 2.7.2]): Let (1) be any seminormal identity and let \( u, v, w \) and \( w' \) be any words in the variables \( x_1, x_2, \ldots, x_k \) such that \( \ell(w) \geq g_0 - 1 \), \( \ell(w') \geq h_0 - 1 \). Take any \( d_1, d_2, \ldots, d_k \) in \( S \) for the variables \( x_1, x_2, \ldots, x_k \) respectively. Let \( x_j \in C(v) \) \([x_j \in C(u)]\) be such that \( d_j \in S \setminus U \) for some \( 1 \leq j \leq k \). Then

\[
w(\tilde{d}) u(\tilde{d}) v(\tilde{d}) w'(\tilde{d}) = w(\tilde{d}) v(\tilde{d}) u(\tilde{d}) w'(\tilde{d}),
\]

where \( \tilde{d} = (d_1, d_2, \ldots, d_k) \).

**Lemma 4.3.4**([37, Lemma 2.7.3]): Let (1) be any seminormal identity and let \( u, v \) and \( w \) be any words in the variables \( x_1, x_2, \ldots, x_k \) \((k \geq 2)\) such that \( \ell(u) \geq g_0 - 1 \) and \( \ell(w) \geq h_0 - 1 \). Take any \( d_1, d_2, \ldots, d_k \) in \( S \) for the variables \( x_1, x_2, \ldots, x_k \) respectively. If \( x_j \in C(v) \), for some \( 1 \leq j \leq k \), be such that \( d_j \in S \setminus U \), then

\[
u(\tilde{d}) w(\tilde{d}) = u(\tilde{d}) (d_j)^{|x_j|} v(\tilde{d}) w(\tilde{d})
\]

91
in $S^1$ (in fact the two products are equal in $S$), where $\vec{d} = (d_1, d_2, \ldots, d_k)$ and $\vec{d'} = (d_1, d_2, \ldots, d_{j-1}, 1, d_{j+1}, \ldots, d_k)$, for all $d_1, d_2, \ldots, d_k \in S$ (thus the product $v(\vec{d'})$ is obtained from the product $v(\vec{d})$ by omitting all the occurrences of the element $d_j$).

**Proof of Theorem 4.3.1.** Take any semigroups $U$ and $S$ with $U$ dense in $S$, and assume that $U$ satisfies (1) and (94). As $U$ satisfies (1), by Result 1.5.5, $S$ also satisfies (1). Now, we show that the identity (94) is also satisfied by $S$.

We shall prove the theorem in the case when $p_i \geq q_1$, so $|z_i|^u \geq q_1$ and $|z_i|^v \geq q_1$ hold for all $i$; the proof when $q_1 \geq p_i$ follows by dual arguments on similar lines.

So, take any $d_1, d_2, \ldots, d_l \in S$. If some $d_i \in U$, there is a zigzag in $S^1$ over $U$ with value $d_i$, namely, $d_i = d_i \cdot 1 = 1 \cdot d_i = 1 \cdot d_i$. And if $d_i \in S \setminus U$, then there is a zigzag over $U$ in $S$, hence in $S^1$. Thus $d_1, d_2, \ldots, d_l$ all have zigzags over $U$ in $S^1$ of some common length [33, Lemma 4.2], say

\[
\begin{align*}
  d_i &= a_0(i)^i t_1^{(i)}, & a_0(i)^i &= y_1^{(i)} a_1^{(i)}; \\
  y_k^{(i)} a_{2k}^{(i)} &= y_{k+1}^{(i)} a_{2k+1}^{(i)}, & a_{2k-1}^{(i)} t_k^{(i)} &= a_{2k}^{(i)} t_{k+1}^{(i)} & (1 \leq i \leq \ell, 1 \leq k \leq m - 1); \\
  d_{2m-1}^{(i)} t_m^{(i)} &= d_m^{(i)}, & y_m^{(i)} a_{2m}^{(i)} &= d_i;
\end{align*}
\]

where $a_j^{(i)} \in U$ ($i = 1, 2, \ldots, \ell; j = 0, 1, \ldots, 2m$) and $t_q^{(i)}, y_q^{(i)} \in S^1$ for $i = 1, 2, \ldots, \ell$ and $q = 1, 2, \ldots, m$. Further, for each $d_i \in S \setminus U$, we may assume that $t_q^{(i)}$ and $y_q^{(i)}$ are in $S \setminus U$ from the proof of [33, Lemma 4.2].

Let $\vec{z} = (z_1, z_2, \ldots, z_\ell)$. With this notation, as in [33], identity (94) becomes

\[
x_1^{p_1} x_2^{p_2} \cdots x_\ell^{p_\ell} u(\vec{z}) y_1^{q_1} y_2^{q_2} \cdots y_\ell^{q_\ell} = x_1^{p_1} x_2^{p_2} \cdots x_\ell^{p_\ell} v(\vec{z}) y_1^{q_1} y_2^{q_2} \cdots y_\ell^{q_\ell}.
\]

Also, let

\[
\begin{align*}
  \vec{d} &= (d_1, d_2, \ldots, d_l); \\
  \vec{a}_k &= (a_k(1), a_k(2), \ldots, a_k(\ell)) & (k = 0, 1, \ldots, 2m); \\
  \vec{t}_q &= (t_q(1), t_q(2), \ldots, t_q(\ell)) & (q = 1, 2, \ldots, m); \\
  \vec{y}_q &= (y_q(1), y_q(2), \ldots, y_q(\ell)) & (q = 1, 2, \ldots, m).
\end{align*}
\]
We now wish to show that

\[
x^{p_i} \cdots x^{p_r} u(\tilde{d}) y^{q_1} \cdots y^{q_s} = x^{p_i} \cdots x^{p_r} v(\tilde{d}) y^{q_1} \cdots y^{q_s}.
\]

By [33, Lemma 4.3], \( \tilde{d} \in S^{[\ell]} \) is in the dominion of \( U^{[\ell]} \) in \( (S^1)^{[\ell]} \), where \( T^{[\ell]} \), for any semigroup \( T \) and for any integer \( \gamma \geq 2 \), denotes the cartesian product of the \( \gamma \)-copies of \( T \). In fact, \( \tilde{d} \) has a zigzag over \( U^{[\ell]} \) in \( (S^1)^{[\ell]} \) of length \( m \) as follows:

\[
\tilde{d} = \tilde{a}_0 \tilde{t}_1, \quad \tilde{a}_0 = \tilde{y}_1 \tilde{a}_1;
\]

\[
\tilde{y}_k \tilde{a}_{2k} = \tilde{y}_{k+1} \tilde{a}_{2k+1}, \quad \tilde{a}_{2k-1} \tilde{t}_k = \tilde{a}_{2k} \tilde{t}_{k+1} \quad (1 \leq k \leq m - 1, 1 \leq i \leq m - 1);
\]

\[
\tilde{a}_{2m-1} \tilde{t}_m = \tilde{a}_{2m}, \quad \tilde{y}_m \tilde{a}_{2m} = \tilde{d};
\]

(96)

where \( \tilde{a}_t \in U^{[\ell]} \) (\( t = 0, 1, \ldots, 2m \)) and \( \tilde{y}_q, \tilde{t}_q \in (S^1)^{[\ell]} \) (\( q = 1, 2, \ldots, m \)).

So suppose that \( x_1, \ldots, x_r, y_1, \ldots, y_s, d_1, \ldots, d_\ell \in S \). If all \( d_j \) that occur in the word \( u \) are in \( U \), then, by Lemma 4.2.1,

\[
x^{p_i} \cdots x^{p_r} u(\tilde{d}) y^{q_1} \cdots y^{q_s} = x^{p_i} \cdots x^{p_r} v(\tilde{d}) y^{q_1} \cdots y^{q_s},
\]
as required. Hence, we may assume that there exists at least one \( d_j \) (\( 1 \leq j \leq \ell \)), say, such that \( d_j \in S \setminus U \). Then \( t^{(j)}_i, y^{(j)}_i \in S \setminus U \) for all \( i = 1,2,\ldots,m \). Letting \( x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} \) and \( y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \), we have

\[
x^{p_i} \cdots x^{p_r} u(\tilde{d}) y^{q_1} \cdots y^{q_s} = xu(a_0(\tilde{t}_1))y
\]

(from equations (96))

\[
xu(a_0)u(\tilde{t}_1)y \quad \text{(by Lemma 4.3.2)}
\]

\[
xu(a_0)(t^{(j)}_1)^{|z_j|_u}u(\tilde{t}_1)y \quad \text{(by Lemma 4.3.4 as } t^{(j)}_1 \in S \setminus U)
\]

\[
xv(a_0)(t^{(j)}_1)^{|z_j|_u}u(\tilde{t}_1)y \quad \text{(by Proposition 4.2.3 as } |z_j|_u \geq q_1 \text{ and } t^{(j)}_1 \in S \setminus U)
\]

\[
xv(\tilde{y}_1 \tilde{a}_1)u(\tilde{t}_1)y
\]

(by equations (96) and Result 1.5.14 as \( t^{(j)}_1 \in S \setminus U \))

\[
xv(\tilde{y}_1)v(\tilde{a}_1)u(\tilde{t}_1)y \quad \text{(by dual of Lemma 4.3.2 as } y^{(j)}_1 \in S \setminus U)
\]

\[
xv(\tilde{a}_1)v(\tilde{y}_1)u(\tilde{t}_1)y \quad \text{(by Lemma 4.3.3 as } y^{(j)}_1 \in S \setminus U)
\]

\[
xv(\tilde{a}_1)(y^{(j)}_1)^{|z_j|_v}v(\tilde{y}_1)u(\tilde{t}_1)y \quad \text{(by Lemma 4.3.4 as } y^{(j)}_1 \in S \setminus U)
\]

\[
xu(\tilde{a}_1)(y^{(j)}_1)^{|z_j|_v}v(\tilde{y}_1)u(\tilde{t}_1)y
\]

(by Proposition 4.2.3 as \( |z_j|_v \geq q_1 \) and \( y^{(j)}_1 \in S \setminus U \))

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Continuing in this way, we obtain
\begin{align*}
x_1^{p_1} \cdots x_n^{p_n} u(d) y_1^{q_1} \cdots y_s^{q_s} \\
= x v(\tilde{y}_1) u(\tilde{a}_1) y \\
= x v(\tilde{y}_2) u(\tilde{a}_2) y \\
= x v(\tilde{y}_{m-1}) u(\tilde{a}_{m-2}) y \\
= x v(\tilde{y}_m-1) u(\tilde{a}_{m-2}) y \\
(\text{by Lemma 4.3.2}) \\
= x v(\tilde{y}_m) u(\tilde{a}_{m-2}) y \\
(\text{from equations (96)).}
\end{align*}
\[ \begin{align*}
&= x v(\tilde{y}_m) u(\tilde{a}_{2m-1} \cdots \tilde{a}_1) y \quad \text{(by Lemmas 4.3.2 and 4.3.3 as } y_m^{(i)} \in S \setminus U) \\
&= x u(\tilde{a}_{2m}) v(\tilde{y}_m) y \quad \text{(from equations (96))} \\
&= x u(\tilde{a}_{2m}) v(\tilde{y}_m) y \quad \text{(by Lemma 4.3.3 as } y_m^{(i)} \in S \setminus U) \\
&= x u(\tilde{a}_{2m}) (y_m^{(i)})^{z_j v} v(\tilde{y}_m) y \quad \text{(by Lemma 4.3.4 as } y_m^{(i)} \in S \setminus U) \\
&= x u(\tilde{a}_{2m}) v(\tilde{y}_m) y \quad \text{(by Proposition 4.2.3 as } |z_j|_v \geq q_1 \text{ and } y_m^{(i)} \in S \setminus U) \\
&= x u(\tilde{a}_{2m}) v(\tilde{y}_m) y \quad \text{(by Lemma 4.3.4)} \\
&= x u(\tilde{a}_{2m}) v(\tilde{a}_{2m}) y \quad \text{(by Lemma 4.3.3 as } y_m^{(i)} \in S \setminus U) \\
&= x u(\tilde{a}_{2m} \tilde{a}_{2m}) y \quad \text{(by the dual of Lemma 4.3.2)} \\
&= x_1^{p_1} \cdots x_r^{p_r} v(d) y_1^{q_1} \cdots y_s^{q_s}
\end{align*} \]

where the last inequality follows from equations (96), and as \( x = x_1^{p_1} \cdots x_r^{p_r} \) and \( y = y_1^{q_1} \cdots y_s^{q_s} \). This completes the proof of the theorem. \( \square \)