CHAPTER 1
PRELIMINARIES

§ 1.1. INTRODUCTION

In this chapter we introduce some basic concepts and results of the semigroup theory, that we shall be using throughout the thesis. Most of the material included in this chapter occurs in the standard literature, namely Cohn [5], Clifford and Preston [6], Higgins [23], Howie [25, 26] and Petrich [43-45].

§ 1.2. BASIC DEFINITIONS

In this section, we give a brief exposition of some basic definitions and terminology of semigroup theory.

Definition 1.2.1. A groupoid \((S, \mu)\) is defined as a non empty set \(S\) on which a binary operation, by which we mean a map \(\mu : S \times S \rightarrow S\), is defined. We say that \((S, \mu)\) is a semigroup if the operation is also associative i.e., for all \(x, y, z \in S\)

\[
((x, y)\mu, z)\mu = (x, (y, z)\mu)\mu
\]

(here and thereafter throughout the thesis, we write symbols of mapping on the right). For convenience we shall follow the usual algebraic practice of writing the binary operation as multiplication. Thus \((x, y)\mu\) will be written as \(x \cdot y\) or (more usually) \(xy\). Thus above formula takes the simple form \((xy)z = x(yz)\)

Definition 1.2.2. A semigroup \(S\) is said to be commutative if

\[xy = yx, \quad \forall \ x, \ y \in S.\]

Definition 1.2.3. Let \(S\) be a semigroup. An element \(a\) of \(S\) is said to be regular if there exists \(x \in S\) such that \(a = axa\). A semigroup whose all elements are regular is called a regular semigroup.
For example, the semigroup of all mappings of a non-empty set into itself, with respect to the operation of composition of maps, is a regular semigroup.

**Definition 1.2.4.** If a semigroup $S$ contains an element $e$ such that

$$ex = xe = x, \quad (\forall x \in S),$$

we say that $e$ is an identity element (or just an identity) of $S$, and $S$ is said to be a semigroup with identity or (more usually) a monoid.

Like groups, an identity element of a semigroup, if it exists, is also unique.

**Definition 1.2.5.** If a semigroup $S$ has no identity element, then we can easily adjoin an extra element $1$ to $S$ to form a monoid, by defining

$$1s = s1 = s, \quad \forall s \in S, \quad \text{and} \quad 1.1 = 1.$$ 

Thus $S \cup \{1\}$ becomes a monoid. We now define

$$S^1 = \begin{cases} S, & \text{if } S \text{ has an identity element} \\ S \cup \{1\}, & \text{otherwise.} \end{cases}$$

We refer to $S^1$ as the monoid obtained from $S$ by adjoining an identity, if necessary.

**Definition 1.2.6.** If a semigroup $S$ with at least two elements contains an element $0$ such that

$$0x = x0 = 0, \quad (\forall x \in S),$$

then we say that $0$ is a zero element (or just zero) of $S$ and $S$ is said to be a semigroup with zero.

A zero element of a semigroup, if exists, is also unique.

**Definition 1.2.7.** If a semigroup $S$ has no zero element, then we can adjoin an extra element $0$, and define

$$0s = s0 = 0, \quad \forall s \in S, \quad \text{and} \quad 00 = 0.$$
It is a routine matter to check that $S \cup \{0\}$ is a semigroup with zero. By analogy with $S^1$, we define

$$S^0 = \begin{cases} S, & \text{if } S \text{ has a zero element} \\ S \cup \{0\}, & \text{otherwise.} \end{cases}$$

Again we refer to $S^0$ as the semigroup obtained from $S$ by adjoining a zero, if necessary.

We assume that here and elsewhere in the text, a bracketed statement will mean dual to the other statement.

**Definition 1.2.8.** A non-empty subset $T$ of a semigroup $S$ is called a *subsemigroup* of $S$ if

$$xy \in T, \ \forall \ x, \ y \in T.$$  

This condition can be expressed more compactly as $T^2 \subseteq T$. The associativity that holds throughout $S$ certainly holds throughout $T$, and so, $T$ itself is a semigroup.

**Definition 1.2.9.** A subsemigroup of $S$ which is a group with respect to the operation inherited from $S$ is called a *subgroup* of $S$.

It is easy to see that a non-empty subset $T$ of a semigroup $S$ is a subgroup of $S$ if and only if

$$(\forall \ a \in T), \ aT = T \text{ and } Ta = T.$$  

**Definition 1.2.10.** A non-empty subset $A$ of a semigroup $S$ is called a *left ideal* of $S$ if $SA \subseteq A$, a *right ideal* of $S$ if $AS \subseteq A$, and a *(two-sided)* ideal of $S$ if it is both a left ideal and a right ideal of $S$.

Evidently, every ideal of $S$ (whether left, right or two-sided) is a subsemigroup of $S$, but the converse is not true in general.

**Definition 1.2.11.** If $S$ and $T$ are semigroups, then the cartesian product

$$S \times T = \{(s,t) : s \in S, \ t \in T\}$$
of $S$ and $T$ becomes a semigroup if we define

$$(s, t)(s', t') = (ss', tt'),$$

where $s, s' \in S$ and $t, t' \in T$. We refer to this semigroup as the direct product of $S$ and $T$.

**Definition 1.2.12.** A semigroup $S$ is said to be a **band** if every element of $S$ is an idempotent, i.e., $a^2 = a$, $\forall a \in S$.

**Definition 1.2.13.** A semigroup $S$ is said to be a left zero semigroup if $ab = a$, $\forall a, b \in S$. Right zero semigroups are defined dually.

**Definition 1.2.14.** If $I$ and $\Lambda$ are non-empty sets, then we may define an associative binary operation “$\circ$” on $I \times \Lambda$ as:

$$(i_1, \lambda_1) \circ (i_2, \lambda_2) = (i_1, \lambda_2), \ \forall i_1, i_2 \in I; \ \lambda_1, \lambda_2 \in \Lambda.$$ Then $(I \times \Lambda, \circ)$ is a semigroup which is called a **rectangular band**.

If $| \Lambda | = 1$ or $| I | = 1$, then the rectangular band $I \times \Lambda$ is a left [right] zero semigroup.

**Definition 1.2.15.** A band $S$ is said to be a left [right] normal band if

$$abc = acb \ [abc = bac], \ \forall a, b, c \in S.$$ **Definition 1.2.16.** A semigroup $S$ is said to be an inverse semigroup if each $a$ in $S$ has a unique inverse, i.e., if there exists a unique element $a^{-1}$ in $S$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$, $\forall a \in S$.

Such a semigroup is certainly regular, but not every regular semigroup is an inverse semigroup. A rectangular band is an obvious example in which every element is an inverse of every other element.

**Definition 1.2.17.** A relation $R$ on a set $X$ is called an **equivalence relation** if it is reflexive, symmetric and transitive.
Definition 1.2.18. Let $S$ be a semigroup. A relation $R$ on $S$ is called left compatible (with the operation on $S$) if

$$\forall s, t, a \in S \quad (s, t) \in R \text{ implies } (as, at) \in R,$$

and right compatible if

$$\forall s, t, a \in S \quad (s, t) \in R \text{ implies } (sa, ta) \in R.$$

It is called compatible if

$$\forall s, t, s', t' \in S \quad [(s, t) \in R \text{ and } (s', t') \in R] \text{ implies } (ss', tt') \in R.$$

A left [right] compatible equivalence relation on a semigroup $S$ is called a left [right] congruence on $S$ and a compatible equivalence relation on $S$ is called a congruence on $S$.

**Result 1.2.19** ([26, Prop. 1.5.1]). An equivalence relation $\rho$ on a semigroup $S$ is a congruence if and only if it is both left and right compatible.

§ 1.3. VARIETIES AND PERMUTATION IDENTITIES

**Definition 1.3.1.** Let $X$ be any set, and let $F_X$ consists of all finite sequences of elements of $X$. If $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ be any two elements of $F_X$ $(x_i, y_j \in X)$, where $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, then we define their product by simple juxtaposition:

$$(x_1, x_2, \ldots, x_m)(y_1, y_2, \ldots, y_n) = (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n).$$

$F_X$, thereby, becomes a semigroup which we call as the free semigroup on $X$. An element of $F_X$ will be called as a word in the alphabet set $X$.

**Definition 1.3.2.** A semigroup identity $u = v$, is the formal equality of two words $u$ and $v$ formed by letters over an alphabet set $X$.

**Definition 1.3.3.** A semigroup $S$ is said to satisfy an identity if for every substitution of elements from $S$ for the letters forming the words of the identity, the resulting words are equal in $S$. 

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Equivalently, one can also define:

**Definition 1.3.4.** Let $X$ be a countably infinite set and let $F_X$ be the free semigroup on $X$. Let $S$ be any semigroup. If $u, v \in F_X$, then we shall say that the identical relation (or identity) $u = v$ is satisfied in $S$ if $u\phi = v\phi$ for every homomorphism $\phi : F_X \to S$.

**Definition 1.3.5.** The class of semigroups, in which a finite or an infinite collection $u_1 = v_1, u_2 = v_2, \ldots$ of identical relations is satisfied, is called the *variety of semigroups determined by these identical relations*, and the list of identical relations is called a *presentation of the variety*, denoted by $[u_1 = v_1, u_2 = v_2, \ldots]$.

We shall take Birkhoff’s Theorem for (semigroup) varieties for granted.

**Result 1.3.6** ([10, Ch. 1 section 26, Theorem 3]). A non-empty class $\mathcal{V}$ of semigroups is a variety if and only if

(a) every subsemigroup of a semigroup in $\mathcal{V}$ is in $\mathcal{V}$;

(b) every homomorphic image of a semigroup in $\mathcal{V}$ is in $\mathcal{V}$;

(c) the direct product of a family of semigroups in $\mathcal{V}$ is in $\mathcal{V}$.

**Definition 1.3.7.** Let $u$ be any word. The *content* of $u$ is the (necessarily finite) set of all the variables appearing in $u$, and will be denoted by $C(u)$. Further, for any variable $x$ in $u$, $|x|_u$ will denote the number of occurrences of the variable $x$ in the word $u$.

**Definition 1.3.8.** An identity

$$u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n),$$

in the variables $x_1, x_2, \ldots, x_n$ is called *homotypical* if $C(u) = C(v)$ and *heterotypical* otherwise.
**Definition 1.3.9.** An identity of the form

\[ x_1 x_2 \cdots x_n = x_{i_1} x_{i_2} \cdots x_{i_n} \quad (n \geq 2), \]  

is called a permutation identity, where \( i \) is any permutation of the set \( \{1, 2, 3, \ldots, n\} \) and \( i_k \), for each \( k \) \((1 \leq k \leq n)\), is the image of \( k \) under the permutation \( i \). A permutation identity of the form (1) is said to be nontrivial if the permutation \( i \) is different from the identity permutation. Further, a nontrivial permutation identity of the form (1) is said to be *left semicommutative* if \( i_1 \neq 1 \), *right semicommutative* if \( i_n \neq n \), and *seminormal* if \( i_1 = 1 \) and \( i_n = n \). For example, some of the better known permutation identities are:

\[
\begin{align*}
x_1 x_2 &= x_2 x_1 \quad \text{[commutativity];} \\
x_1 x_2 x_3 &= x_1 x_3 x_2 \quad \text{[left normality];} \\
x_1 x_2 x_3 &= x_2 x_1 x_3 \quad \text{[right normality];} \\
x_1 x_2 x_3 x_4 &= x_1 x_3 x_2 x_4 \quad \text{[normality].}
\end{align*}
\]

**Remark 1.3.10.** Every nontrivial permutation identity is either *left semicommutative*, *right semicommutative*, or *seminormal*.

**Definition 1.3.11.** A semigroup \( S \) satisfying a nontrivial permutation identity is said to be permutative, and a variety \( V \) of semigroups is said to be permutative if it admits a nontrivial permutation identity.

For further details and other related results on varieties and identities of semigroups, one may refer to Aizenstat [1], christlock [4], Higgins [15] and Khan [30, 34-36].

**§ 1.4. EPI-MORPHISMS AND DOMINIONS**

**Definition 1.4.1.** A morphism \( \alpha : A \rightarrow B \) in the category \( C \) of semigroups is said to be an *epimorphism* (epi for short) if for all morphisms \( \beta, \gamma : B \rightarrow C \), \( \alpha \beta = \alpha \gamma \) implies \( \beta = \gamma \).

**Remark 1.4.2.** One can easily see that any onto morphism is an epimorphism. Whether or not the converse is true, depends on the category under consideration.
It is true in the categories of Sets, Abelian Groups and Groups for instance.

In general, epimorphisms are not onto in the categories of semigroups and rings. Here epimorphisms can be characterized in term of so called “zigzags”, a special sequence of factorizations of elements in the epimorphic image.

In the next section, we include a full proof of the Zigzag Theorem for semigroups, a result due to Isbell [28].

The following example of a semigroup epimorphism which is not onto appears in Drbohlav [7].

**Example 1.4.3.** Take the embedding $i$ of the real interval $(0, 1]$ into $(0, \infty]$, where both are considered as multiplicative semigroups. To see that $i: (0, 1] \rightarrow (0, \infty]$ is epi, take any pair of homomorphisms $\alpha, \beta$ from $(0, \infty]$ such that $i\alpha = i\beta$; that is, $\alpha$ and $\beta$ agree on $(0, 1]$. We shall show that for any $x > 1$, $x\alpha = x\beta$. Let $x > 1$. Then

$$[(x)\alpha(1/x)\alpha](x)\beta = (1)\alpha(x)\beta = (1)\beta(x)\beta = (x)\beta.$$

Equally though, since $1/x < 1$,

$$[(x)\alpha(1/x)\alpha](x)\beta = (x)\alpha[(1/x)\alpha(x)\beta] = (x)\alpha[(1/x)\beta(x)\beta]$$

$$= (x)\alpha(1)\beta = (x)\alpha(1)\alpha = (x)\alpha.$$

Therefore, $\alpha = \beta$ and so $i$ is epi.

Moreover, the embedding of an infinite monogenic semigroup into an infinite cyclic group, and the embedding (under multiplication) of the natural numbers into the positive rational numbers are other examples of this kind. However, Hall [11] has unified all these examples by showing that if $U$ is a full subsemigroup (a subsemigroup that contains all the idempotent elements of the containing semigroup) of an inverse semigroup $S$, which generates $S$ as an inverse semigroup, then the embedding of $U$ in $S$ is an epimorphism.
Definition 1.4.4. Let $U$ be a subsemigroup of a semigroup $S$. Following Isbell [28], we say that $U$ dominates an element $d$ of $S$ if for every semigroup $T$ and for all homomorphisms $\beta, \gamma : S \rightarrow T$, $u\beta = u\gamma$ for each $u \in U$ implies $d\beta = d\gamma$. The set of all elements of $S$ dominated by $U$ is called the dominion of $U$ in $S$ and we denote it by $\text{Dom}(U, S)$.

Remark 1.4.5. It can be easily seen that $\text{Dom}(U, S)$ is a subsemigroup of $S$ containing $U$.

Definition 1.4.6. Following Howie and Isbell [27], a semigroup $U$ is said to be saturated if it cannot be properly epimorphically embedded in any properly containing semigroup $S$, that is, $\text{Dom}(U, S) \neq S$ for every properly containing semigroup $S$.

Definition 1.4.7. A class $\mathcal{C}$ of semigroups is said to be epimorphically closed if $S \in \mathcal{C}$ and $\alpha : S \rightarrow T$ is epi implies $T \in \mathcal{C}$. Further, a class $\mathcal{C}$ of semigroups is called saturated if all of its members are saturated.

Clearly, the condition of being epimorphically closed or closed under epis is weaker than being saturated.

Remark 1.4.8. One can easily show that a morphism $\alpha : S \rightarrow T$ is epi if and only if the inclusion $i : S\alpha \rightarrow T$ is epi and the inclusion $i : U \rightarrow S$ is epi if and only if $\text{Dom}(U, S) = S$.

Definition 1.4.9. A semigroup identity $u = v$ is said to be epimorphically preserved or preserved under epis if whenever $S$ satisfies $u = v$ and $\alpha : S \rightarrow T$ is epi, then $T$ also satisfies $u = v$. Or equivalently, if whenever $U$ satisfies $u = v$ and $\text{Dom}(U, S) = S$ implies that $S$ also satisfies $u = v$.

One may consider three conditions that a semigroup identity $\phi$ may satisfy:

(A) $\phi$ is preserved under epis;

(B) each variety admitting $\phi$ is epimorphically closed;

(C) each variety admitting $\phi$ is saturated.
Condition (C) clearly implies (B), which in turn implies (A), but the reverse implications are not true in general. In [19], Higgins gave a necessary condition for a semigroup identity to satisfy condition (A): a semigroup identity is preserved under epis only if one of its sides contains no repeated variable. Isbell [28] showed that commutativity is stable under dominions, which yields that commutativity satisfies condition (A). Khan [29, 33] generalized this result in two directions by showing that commutativity satisfies condition (B), and that all permutation identities satisfy condition (A). In [33], Khan further showed, jointly with Higgins [18], that left and right semicommutative identities satisfy condition (B). Therefore, it is natural to try to determine all those semigroup identities that satisfy condition (A) in conjunction with seminormal identities. Khan [31, 32] showed that all semigroup identities in which both sides do not contain repeated variables satisfy condition (A) in conjunction with any nontrivial permutation identity, while Higgins [18] has shown that seminormal permutation identities do not satisfy condition (B) by showing that the identity $xyx = yxy$ does not satisfy (A) in conjunction with the normality identity.

It is well known that all subvarieties of a saturated variety are saturated, but the same is not true for epimorphically closed varieties in general. Thus, determining all semigroup identities whose both sides contain repeated variables and satisfy condition (A) in conjunction with any seminormal identity becomes much more difficult.

**Remark 1.4.10.** It is clear that every saturated class of semigroups is epimorphically closed, but the converse is not true in general. For example, the variety of all commutative semigroups is epimorphically closed [28 corollary 2.5], but not saturated as the inclusion map of an infinite monogenic semigroup into an infinite cyclic group is epi. [25, Chapter vii, Exercise 2(i)]

For further details and other related results on Epimorphisms and Dominions, one may refer to Burgess [3], Gardner [8, 9], Hall [12], Higgins [13, 14, 16, 17, 20-22, 24] and Khan and Shah [39].

**§ 1.5. SOME IMPORTANT RESULTS**

In this section, we give some important results which we shall be using throughout the thesis. The following celebrated result due to Isbell, known as Isbell’s Zigzag
Theorem, is of basic importance to our investigations and is the main tool used for studying semigroup dominions. This theorem is so important for our purposes that we include a full proof of it, although this proof may be found in the introductory text of Howie [25].

**Result 1.5.1** ([28, Theorem 2.3] or [25, Theorem VII. 2.13]). Let \( U \) be a subsemigroup of a semigroup \( S \) and let \( d \in S \). Then \( d \in Dom(U, S) \) if and only if \( d \in U \) or there exists a series of factorization of \( d \) as follows:

\[
d = a_0 t_1 = y_1 a_1 t_1 = y_1 a_2 t_2 = y_2 a_3 t_2 = \cdots = y_m a_{2m-1} t_m = y_m a_{2m},
\]

where \( m \geq 1 \), \( a_i \in U \) \((i = 0, 1, \ldots, 2m)\), \( y_i, t_i \in S \) \((i = 1, 2, \ldots, m)\), and

\[
\begin{align*}
a_0 &= y_1 a_1, & a_{2m-1} t_m &= a_{2m}, \\
a_{2i-1} t_i &= a_{2i} t_{i+1}, & y_i a_{2i} &= y_{i+1} a_{2i+1} & (1 \leq i \leq m-1).
\end{align*}
\]

Such a series of factorization is called a zigzag in \( S \) over \( U \) with value \( d \), length \( m \) and spine \( a_0, a_1, \ldots, a_{2m} \).

**Proof.** The proof in the reverse direction is just a straightforward zigzag manipulation. Suppose \( Z \) is a zigzag with value \( d \) in \( S \) over \( U \) and that \( \alpha, \beta : S \rightarrow T \) are two semigroup morphisms such that \( \alpha|U = \beta|U \). Then

\[
d \alpha = (a_0 t_1) \alpha = a_0 \alpha t_1 \alpha = a_0 \beta t_1 \alpha = (y_1 a_1) \beta t_1 \alpha = y_1 \beta a_1 t_1 \alpha = y_1 \beta (a_1 t_1) \alpha = y_1 \beta (a_2 t_2) \alpha = \cdots = y_m \beta (a_{2m-1} t_m) \alpha = y_m \beta a_{2m} \alpha = y_m \beta a_{2m} \beta = (y_m a_{2m}) \beta = d \beta,
\]

as required. \( \square \)

Now, we prepare for a proof of the converse part of the Zigzag Theorem. If \( M \) is a set and \( S \) is a semigroup with identity \( 1 \), we say that \( M \) is a right \( S \)-system if there is a mapping \((x, s) \rightarrow xs \) from \( M \times S \) into \( S \) such that \((xs)t = x(st) \) \((x \in M; s, t \in S)\) and \( x1 = x \) \((x \in M)\). A left \( S \)-system is defined dually. If \( S \) and \( T \) are semigroups with identity, we say that \( M \) is an \((S, T)\)-biset if it is a left \( S \)-system, a right \( T \)-system and for all \( s \in S, t \in T \) and \( x \in M \), \((sx)t = s(xt)\).

Let \( M \) be a right \( S \)-system and \( N \) be a left \( S \)-system. Let \( \tau \) be the equivalence relation on \( M \times N \) generated by \( \{(x, y), (x, sy)\} : x \in M, s \in S, y \in N \} \). We denote \((M \times N)/\tau \) by \( M \otimes_S N \), the tensor product over \( S \) of two \( S \)-systems. The
equivalence class \((x, y)\tau\) will be denoted by \(x \otimes y\). Note that \(xs \otimes y = x \otimes sy\) \((x \in M, s \in S, y \in N)\).

Observe that if \(M\) is a \((T, S)\)-bisystem and \(N\) is an \((S, U)\)-bisystem, then \(M \otimes_S N\) becomes a \((T, U)\)-bisystem if we define \(t(x \otimes y) = tx \otimes y, (x \otimes y)u = x \otimes yu\), for \(t \in T, u \in U, x \otimes y \in M \otimes_S N\).

If \(P\) and \(Q\) are right \(S\)-systems, we say that a map \(\alpha : P \rightarrow Q\) is a right \(S\)-system morphism if for every \(x \in P\) and \(s \in S\), \((xs)\alpha = (x\alpha)s\). Similar definitions apply to left \(S\)-system and \((S, T)\)-bisystem morphisms.

Next, suppose that \(U\) is a subsemigroup of a semigroup \(S\) and let \(S^{(1)}\) be the semigroup obtained from \(S\) by adjoining an identity element 1 whether it has 1 or not, and let \(U^{(1)} = U \cup \{1\}\); then \(U^{(1)}\) is a subsemigroup of \(S^{(1)}\). We may, then, clearly form \(A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}\).

**Result 1.5.3** ([25, Chapter VII Theorem 2.5]). If \(U\) is a subsemigroup of a semigroup \(S\) and if \(d \in S\), then \(d \in Dom(U, S)\) if and only if \(d \otimes 1 = 1 \otimes d\) in \(A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}\).

**Proof.** Suppose that \(d \in S\) and \(d \otimes 1 = 1 \otimes d\) in \(A\). The tensor product \(A\) is \((S^{(1)} \otimes S^{(1)})/\tau\), where \(\tau\) is the equivalence relation on \(S^{(1)} \times S^{(1)}\) generated by

\[T = \{(xu, y), (x, uy) : x, y \in S^{(1)}, u \in U^{(1)}\}.

Let \(R\) be a semigroup and let \(\beta, \gamma : S \rightarrow R\) be morphisms coinciding on \(U\). We can regard \(\beta\) and \(\gamma\) as morphisms from \(S^{(1)}\) into \(R^{(1)}\) coinciding on \(U^{(1)}\) by defining \(1\beta = 1\gamma = 1\). Define \(\psi : S^{(1)} \times S^{(1)} \rightarrow R^{(1)}\) by:

\[(x, y)\psi = (x\beta)(y\gamma), ((x, y) \in S^{(1)} \times S^{(1)}).

It can be easily checked that \(T \subset \psi \circ \psi^{-1}\), since \(\psi \circ \psi^{-1}\) is an equivalence relation. Hence the map \(\chi : A \rightarrow R^{(1)}\) denoted by \((x \otimes y)\chi = (x\beta)(y\gamma), (x \otimes y \in A)\) is indeed well defined. But now \((d \otimes 1)\chi = (1 \otimes d)\chi\); that is, \(d\beta = d\gamma\) and so \(d \in Dom(U, S)\).

To prove the converse, we regard the tensor product \(A\) as an \((S^{(1)}, S^{(1)})\)-bisystem by defining

\[s(x \otimes y) = sx \otimes y, (x \otimes y)s = x \otimes ys\] \((s, x, y \in S^{(1)})\).
Let \((Z(A), +)\) be the free abelian group on \(A\). The abelian group \(Z(A)\) inherits an \((S^{(1)}, S^{(1)})\)-bisystem structure from \(A\) if we define
\[
s(\Sigma z_i a_i) = \Sigma z_i (sa_i), \quad (\Sigma z_i a_i)s = \Sigma z_i (a_is),
\]
for all \(s \in S^{(1)}\) and \(\Sigma z_i a_i \in Z(A)\). Observe that, for \(x, y \in Z(A)\) and \(s \in S^{(1)}\), we have
\[
s(x + y) = sx + sy, \quad (x + y)s = xs + ys.
\]
Next, we define a binary operation on \(S^{(1)} \times Z(A)\) by
\[
(p, x)(q, y) = (pq, py + xq).
\]
Using the statements labelled (3) and (4), one verifies that this operation makes \(S^{(1)} \times Z(A)\) a semigroup with identity \((1,0)\).

We now consider two homomorphisms \(\beta\) and \(\gamma\) from \(S^{(1)}\) into \(S^{(1)} \times Z(A)\) and show that \(\beta|U = \gamma|U\). We define \(\beta\) by \(s\beta = (s, 0)\) \((s \in S^{(1)})\), then clearly \(\beta\) is a morphism. We define \(\gamma\) by \(s\gamma = (s, s(1 \otimes 1) - (1 \otimes 1)s)\) \((s \in S^{(1)})\). To show that \(\gamma\) is a morphism, we denote \(1 \otimes 1\) by \(a\), and using the statements (3) and (4), we verify that \((s, sa - as)(t, ta - at) = (st, s(ta - at) + (sa - as)t) = (st, (st)a - a(st))\).

If \(u \in U^{(1)}\), then
\[
u(1 \otimes 1) = u \otimes 1 = 1u \otimes 1 = 1 \otimes u1 = (1 \otimes 1)u,
\]
and so \(u\beta = u\gamma\). Removing identities gives two morphisms \(\beta\) and \(\gamma\) from \(S\) into \(S \times Z(A)\) such that \(u\beta = u\gamma\) for all \(u \in U\). If \(d \in \text{Dom}(U, S)\), we must therefore have that \(d\beta = d\gamma\); that is, \(d \otimes 1 = 1 \otimes d\), as required. \(\Box\)

We may now complete the proof of the Zigzag Theorem. Take \(d \in \text{Dom}(U, S)\). By Result 1.5.3, we have that \(d \otimes 1 = 1 \otimes d\) in the tensor product \(A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}\). Hence, the pair \((1, d)\) and \((d, 1)\) are connected by a finite sequence of steps of the form
\[
(xu, y) \rightarrow (x, uy),
\]
or of the form
\[
(x, uy) \rightarrow (xu, y).
\]
If we have two successive steps
\[
(xu, y) \rightarrow (x, uy) = (zu, uy) \rightarrow (z, vuy),
\]
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of the first type, we may achieve the same effect with a single step of this type:

\[(xu, y) = (zvu, y) \rightarrow (z, \nu y)\].

A similar remark applies to the other case. Consequently, we may assume that steps of the two types occur alternately in the sequence connecting \((1, d)\) to \((d, 1)\).

The first and last steps must have the form

\[(1, d) = (1, uy) \rightarrow (u, y) \quad \text{and} \quad (x, u) \rightarrow (xu, 1) = (d, 1)\] respectively.

Hence the statement that \(1 \otimes d = d \otimes 1\) is equivalent to the statement that \((1, d)\) is connected to \((d, 1)\) by a sequence of the steps as follows:

\[
(1, d) = (1, a_0 t_1) \rightarrow (a_0, t_1)
\]

\[
= (y_1 a_1, t_1) \rightarrow (y_1, a_1 t_1)
\]

\[
= (y_1, a_2 t_2) \rightarrow (y_1 a_2, t_2)
\]

\[
\vdots
\]

\[
= (y_i a_{2i-1}, t_i) \rightarrow (y_i, a_{2i-1} t_i)
\]

\[
= (y_i, a_{2i} t_{i+1}) \rightarrow (y_i a_{2i}, t_{i+1})
\]

\[
\vdots
\]

\[
= (y_m, a_{2m}) \rightarrow (y_m a_{2m}, 1) = (d, 1),
\]

where \(a_0, \ldots, a_{2m} \in U^{(1)}, y_1, \ldots, y_m, t_1, \ldots, t_m \in S^{(1)},\) and where \(d = a_0 t_1, a_0 = y_1 a_1, a_{2m-1} t_m = a_{2m}, y_m a_{2m} = d\) and \(a_{2i-1} t_i = a_{2i} t_{i+1}, y_i a_{2i} = y_{i+1} a_{2i+1} (i = 1, 2, \ldots, m - 1).\)

Without loss we may assume that each \(a_i \in U,\) since a transition of type (5) or of type (6) with \(a = 1\) may be deleted. If any \(y_i = 1,\) let \(y_k\) be the last \(y_i\) that is equal to 1, then we have a subsequence of the sequence above as follows:

\[(1, d) \rightarrow \cdots \rightarrow (1, a_{2k} t_{k+1}) \quad \text{(but ending in \((1, a_{2m})\) when \(k = m).}\]

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Note that, if \((p, q)\) and \((r, s)\) are connected by steps of the form (5) and (6) then \(pq = rs\). In the present instance this gives \(d = a_{2k}t_{2k+1}\) (or \(d = a_{2m}\)); hence this sequence merely connects \((1, d)\) to \((d, 1)\) and so may be deleted. What remains is a sequence in which no \(y_i\) is 1.

A dual argument now ensures that we may construct the sequence from \((1, d)\) to \((d, 1)\) so that no \(t_i\) is 1. This completes the proof of the Zigzag Theorem. \(\Box\)

In whatever follows, we shall refer to equations (2) as the zigzag equations.

**Remark 1.5.4.** The above Zigzag Theorem is also valid in the category of all commutative semigroups (Howie and Isbell [27]).

**Result 1.5.5** ([33, Theorem 3.1]). All permutation identities are preserved under epis.

**Result 1.5.6** ([32, Proposition 3.1]). Let \(S\) be any permutative semigroup satisfying the identity (1) with \(n \geq 3\). Then

(i) For each \(j \in \{2, 3, \ldots, n\}\) such that \(x_{j-1}x_j\) is not a subword of \(x_{i_1}x_{i_2} \cdots x_{i_m}\), \(S\) also satisfies the permutation identity

\[
x_1x_2 \cdots x_{j-1}xyx_j \cdots x_n = x_1x_2 \cdots x_{j-1}yxx_j \cdots x_n.
\]

(ii) If \(x_1 \neq x_{i_1}\), then \(S\) also satisfies the permutation identity

\[
xyx_1x_2 \cdots x_n = yxx_1x_2 \cdots x_n.
\]

In the following result and elsewhere in the text, \(S^{(\ell)}\), for any semigroup \(S\) and for positive integer \(\ell\), denotes the product of \(\ell\)-copies of \(S\).

**Result 1.5.7** ([32, Proposition 6.3]). Let \(S\) be any permutative semigroup satisfying (1) with \(n \geq 3\). Then for each \(j \in \{2, 3, \ldots, n\}\) such that \(x_{j-1}x_j\) is not a subword of \(x_{i_1}x_{i_2} \cdots x_{i_m}\), for all \(m \geq j - 1\), \(p \geq n - j + 1\), and for all \(u \in S^{(m)}\), \(v \in S^{(p)}\), we have \(ux_1x_2v = ux_2x_1v\) for all \(x_1, x_2 \in S\). In particular \(S^{(k)}\) is medial (a semigroup is medial if it satisfies the normality identity) for all \(k \geq \max (j - 1, n - j + 1)\).
**Result 1.5.8** ([28, Corollary 2.5]). If \( U \) is a commutative subsemigroup of any semigroup \( S \), then \( \text{Dom}(U, S) \) is also commutative.

**Result 1.5.9** ([32, Result 3]). Let \( U \) and \( S \) be any semigroups with \( U \) a subsemigroup of \( S \). Take any \( d \in S \setminus U \) such that \( d \in \text{Dom}(U, S) \). Let \( 2 \) be a zigzag of minimum length \( m \) over \( U \) with value \( d \). Then \( y_j, t_j \in S \setminus U \) for \( j = 1, 2, \ldots, m \).

In the following results, let \( U \) and \( S \) be any semigroups with \( U \) dense in \( S \).

**Result 1.5.10** ([32, Result 4]). For any \( d \in S \setminus U \) if \( 2 \) be a zigzag of minimum length \( m \) over \( U \) with value \( d \) and \( k \) any positive integer, then there exist \( b_1, b_2, \ldots, b_k \in U \) and \( d_k \in S \setminus U \) such that \( d = b_1b_2 \cdots b_kd_k \). In particular \( d \in S^k \) for each positive integer \( k \).

**Result 1.5.11** ([32, Corollary 4.2]). Let \( U \) be permutative, then

\[ sx_1x_2 \cdots x_k t = sx_{j_1}x_{j_2} \cdots x_{j_k} t, \]

for all \( x_1, x_2, \ldots, x_k \in S, s, t \in S \setminus U \) and for any permutation \( j \) of the set \( \{1, 2, \ldots, k\} \).

The following corollary easily follows by Result 1.5.11.

**Corollary 1.5.12** ([37, Corollary 1.8]). For any \( d \in S \) and positive integer \( k \), if \( d = b_1b_2 \cdots b_kd_k \) for some \( b_1, b_2, \ldots, b_k \in U \) and \( d_k \in S \setminus U \) such that \( b_1 = y_{1'} c_1 \) for some \( y_{1'} \in S \setminus U, c_1 \in U \), then \( d^p = b_1^p b_2^p \cdots b_k^p d_k^p \) for any positive integer \( p \).

**Result 1.5.13** ([33, Proposition 4.6]). Assume that \( U \) is permutative. If \( d \in S \setminus U \) and \( 2 \) is a zigzag of length \( m \) over \( U \) with value \( d \) such that \( y_1 \in S \setminus U \), then \( d^k = a_0^k t_1^k \) for each positive integer \( k \); in particular, the conclusion holds if \( 2 \) is of minimum length. Symmetrically, if \( d \in S \setminus U \) and \( 2 \) is a zigzag of length \( m \) over \( U \) with value \( d \) such that \( t_m \in S \setminus U \), then \( d^k = y_m^k a_2^k \) for each positive integer \( k \); in particular, the conclusion holds if \( 2 \) is of minimum length.

The following result easily follows from Result 1.5.7.
**Result 1.5.14** ([37, Proposition 2.1]). Let $S$ be any permutative semigroup satisfying (1) with $n \geq 3$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_1x_2 \cdots x_n$, for all $m \geq g - 1$, $p \geq n - g + 1$ and for all $u \in S^{(m)}$, $v \in S^{(p)}$, we have

$$ux_1x_2 \cdots x_{\ell}v = ux_{\lambda_1}x_{\lambda_2} \cdots x_{\lambda_{\ell}}v$$

for all $x_1, x_2, \ldots, x_{\ell} \in S$ ($\ell \geq 2$), where $\lambda$ is any permutation of the set $\{1, 2, \ldots, \ell\}$.

Again using results 1.5.5, 1.5.10 and 1.5.14, we get easily the following:

**Result 1.5.15** ([37, Proposition 2.2]). Let $U$ be any semigroup satisfying (1) with $n \geq 3$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_1x_2 \cdots x_n$, for all $m \geq g - 1$ and for all $u \in S^{(m)}$, $v \in S \setminus U$, we have

$$ux_1x_2 \cdots x_{\ell}v = ux_{\lambda_1}x_{\lambda_2} \cdots x_{\lambda_{\ell}}v$$

for all $x_1, x_2, \ldots, x_{\ell} \in S$ ($\ell \geq 2$), where $\lambda$ is any permutation of the set $\{1, 2, \ldots, \ell\}$.

Symmetrically, for all $p \geq h - 1$ such that $x_{n-h}x_{n-(h-1)}$ is not a subword of $x_1x_2 \cdots x_n$ and for all $v \in S^{(p)}$, $u \in S \setminus U$, we have

$$ux_1x_2 \cdots x_{\ell}v = ux_{\lambda_1}x_{\lambda_2} \cdots x_{\lambda_{\ell}}v$$

for all $x_1, x_2, \ldots, x_{\ell} \in S$ ($\ell \geq 2$), where $\lambda$ is any permutation of the set $\{1, 2, \ldots, \ell\}$. 

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