CHAPTER 2

Preliminaries

In this chapter, we give a brief survey on graph labeling and we collect all basic definitions and results on graphs which are needed in subsequent chapters. In this thesis we consider only simple, finite, undirected graphs. For graph theoretic terminology, we refer to F. Harary [15] and Douglas B. West [9]. For a detailed survey of graph labeling we refer to Gallian [12].

2.1. Brief Survey of Graph Labeling

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A common theme in graph labeling papers is to build up graphs that have desired labeling from pieces with particular properties. Graph labeling was first introduced in the late 1960’s.

In 1966, Rosa defined an \( \alpha \)-labeling (or \( \alpha \)-valuation) as a graceful labeling with the additional property that there exist an integer \( k \) such that for each edge \( xy \) either \( f(x) \leq k \leq f(y) \) (or) \( f(y) \leq k < f(x) \). Wu [48] has shown that a necessary condition for a bipartite graph with \( n \) edges and degree sequence \( d_1, d_2, \ldots, d_p \) to have an \( \alpha \) – labeling is that \( \gcd(d_1, d_2, \ldots, d_p, n) \) divides \( \frac{n(n-1)}{2} \).

Most graph labeling methods trace their origin to one introduced by Rosa [25] in 1967 or one given by Graham and Sloane [14] in 1980. Rosa introduced a labeling of a graph \( G \) with \( p \) vertices and \( q \) edges known as \( \beta \)-valuation which is an injection
of the set of its vertices into the set of integers \{0, 1, 2, \ldots, q\} such that, when each edge \(xy\) is assigned the label \(|f(x) - f(y)|\), the resulting edge labels are distinct. Golomb subsequently called such labeling graceful.

In 1981, Bange, Barkauskas and Slater [1] introduced the concept of \textit{k-sequential labeling} \(f\) of a graph \(G = (V, E)\) as one for which \(f\) is a bijection from \(V \cup E\) to \(\{k, k+1, \ldots, |V \cup E| + k - 1\}\) such that for each edge \(xy\) in \(E\), \(f(xy) = |f(x) - f(y)|\). In 1980, Graham and Sloane suggested a new labeling known as \textit{harmonious labeling}. A natural generalization of graceful graphs is the notion of \(k\)-graceful graphs introduced independently by Slater in 1982 and by Maheo and Thuillier in 1982. A graph \(G\) with \(q\)-edges is \(k\)-graceful if there is labeling \(f\) from the vertices of \(G\) to \(\{0, 1, 2, \ldots, q + k - 1\}\) such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is \(\{k, k + 1, \ldots, q + k - 1\}\).

In 1985, Lo [18] introduced the notion of edge graceful graphs. A graph \(G = (V, E)\) is said to be \textit{edge graceful} if there exists a bijection \(f\) from \(E\) to \(\{1, 2, \ldots, |E|\}\) so that the induced mapping \(f^+\) from \(V\) to \(\{0, 1, 2, \ldots, |V| - 1\}\) given by \(f^+(x) = (\sum f(xy) \mod |V|)\) taken over all edges \(xy\) is a bijection.

In 1987, Cahit [2] has introduced an analog of graceful labeling called \textit{cordial labeling}. Let \(f\) be a function from the vertices of \(G\) to \(\{0, 1\}\) and for each edge \(xy\) assigns the label \(|f(x) - f(y)|\). \(f\) is called a cordial labeling of \(G\) if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. Cahit has introduced a variation of both graceful and harmonious labeling. Cahit
called a graph $H$-cordial if it is possible to label the edges with the numbers from the set $\{1, -1\}$ in such a way that, for some $k$, at each vertex $v$ the sum of the labels of the edges incident with $v$ is either $k$ or $-k$ and the inequalities $|v(k) - v(-k)| \leq 1$ and $|e(1) - e(-1)| \leq 1$ are also satisfied, where $v(i)$ and $e(j)$ are respectively, the number of vertices labeled with $i$ and the number of edges labeled with $j$.

Hovey has introduced a simultaneous generalization of harmonious and cordial labeling. For any abelian group $A$ (under addition) and graph $G = (V, E)$, he defines $G$ to be a cordial if there is a labeling of $V$ with elements of $A$ such that for all $x$ and $y$ in $A$ when the edge $xy$ is labeled with $f(x) + f(y)$, the number of vertices labeled with $x$ and the number of vertices labeled with $y$ differ by atmost one and number of edges labeled with $x$ and number of edges labeled with $y$ differ by atmost one.

In 1997, Yilmaz and Cahit [47] introduced a weaker version of edge graceful labeling called E-Cordial. Let $G$ be a graph with vertex set $V$ and the edge set $E$ and let $f$ be a function from $E(G)$ to $\{0, 1\}$. Define $f$ on $V$ by $f(v) = \{f(uv) | uv \in E\} (\text{mod} 2)$. The function $f$ is called an E-cordial labeling of $G$ if the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by atmost 1 and number of edges labeled 0 and the number of edges labeled 1 differ by atmost 1. A graph that admits an E-cordial labeling is called E-cordial graph.

In 2004, Sundaram, Ponraj and Somasundaram [40] introduced the notion of product cordial labeling. A product cordial labeling of a graph $G$ with vertex set $V$ is
a function \( f \) from \( V \) to \( \{0, 1\} \) such that if each edge \( xy \) is assigned the label \( f(x)f(y) \), the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a \textit{product-cordial graph}. Sundaram, Somasundaram and Ponraj [41] also have introduced the notion of total product cordial labeling. A total product cordial labeling of a graph \( G \) with vertex set \( V \) is a function \( f \) from \( V \) to \( \{0, 1\} \) such that if each edge \( xy \) is assigned the label \( f(x)f(y) \) the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a \textit{total product cordial graph}.

In 2005, Sundaram, Ponraj and Somasundaram [42] have introduced the notion of prime cordial labeling. A \textit{prime cordial labeling} of a graph \( G \) with vertex set \( V \) is a bijection \( f \) from \( V \) to \( \{1, 2, \ldots, |V|\} \) such that if each edge \( xy \) is assigned the label 1 if \( \gcd(f(x), f(y)) = 1 \) and 0 if \( \gcd(f(x), f(y)) > 1 \), then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most one.

In 2003, Somasundaram and Ponraj [36] have introduced the notion of mean labeling of graphs. A graph \( G \) with \( p \) vertices and \( q \) edges is called a \textit{mean graph} if there is an injective function \( f \) from the vertices of \( G \) to \( \{0, 1, 2, \ldots, q\} \) such that when each edge \( xy \) is labeled with \( \frac{f(x)+f(y)}{2} \) if \( f(x) + f(y) \) is even and with \( \frac{f(x)+f(y)+1}{2} \) if \( f(x) + f(y) \) is odd, then the resulting edge labels are distinct. Ramya, Ponraj and Jeyanthi [20] called a \textit{super mean graph} if vertex labels and the edge labels are \( \{1, 2, \ldots, q\} \).
..., p+q}. Gayathri and Amuthavalli [13] say a (p, q)-graph G has a \((k, d)\)-odd mean labeling if there exists an injection \(f\) from the vertices of \(G\) to \(\{0, 1, 2, \ldots, 2k - 1 + 2(q - 1)d\}\) such that the induced map \(f^*\) defined on the edges of \(G\) by 
\[
f^*(uv) = \left\lfloor \frac{(f(u) + f(v))/2} \right\rfloor
\]
is a bijection from edges of \(G\) to \(\{2k - 1, 2k - 1 + 2d, 2k - 1 + 4d, \ldots, 2k - 1 + 2(q - 1)d\}\). When \(d = 1\), a \((k, d)\)-odd mean labeling is called \(k\)-odd mean. Gayathri and Tamilselvi [13] say a \((p, q)\)-graph G has a \((k, d)\)-super mean labeling if there exists an injection \(f\) from the vertices of \(G\) to \(\{k, k + 1, \ldots, p + q + k - 1\}\) such that the induced map \(f^*\) defined on the edge set of \(G\) by 
\[
f^*(uv) = \left\lfloor \frac{(f(u) + f(v))/2} \right\rfloor
\]
has the property that the vertex labels and the edge labels together are the integers from \(k\) to \(p + q + k - 1\). When \(d = 1\), a \((k, d)\)-super mean labeling is called \(k\)-super mean. Gayathri and Tamilselvi [13] say a \((p, q)\)-graph G has a \(k\)-super edge mean labeling if there exists an injection \(f\) from the edges of \(G\) to \(\{k, k + 1, \ldots, k + 2(p + q)\}\) such that the induced map \(f^*\) from the vertices of \(G\) to \(\{k, k + 1, \ldots, k + 2(p + q)\}\) defined by 
\[
f^*(v) = \left\lfloor \frac{\sum f(uv)}{2} \right\rfloor
\]
taken all edges uv incident to \(v\) is an injection.

In 2012, Sandhya, Somasundaram and Ponraj [31] call a graph \(G\) with \(p\) vertices and \(q\) edges a harmonic mean graph if there is an injective function \(f\) from the vertices of the graph \(G\) to the integers from 1 to \(q + 1\) such that when each edge \(uv\) is labeled with 
\[
\left\lfloor \frac{2f(u)f(v)}{f(u)+ f(v)} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{2f(u)f(v)}{f(u)+ f(v)} \right\rfloor,
\]
then the edge labels are distinct.

2.2. Basic definitions

Here we collect all basic definitions which are needed for future references.
**Definitions 2.2.1.** A graph $G$ consists of a set of vertices $V(G)$ and a set of edges $E(G)$, we denote a graph by $G = (V(G), E(G))$. If $e = \{u, v\}$ is an edge, then we write $e = uv$, we say that $e$ joins the vertices $u$ and $v$, $u$ and $v$ are *adjacent* vertices, $u$ and $v$ are *incident* with $e$. If two vertices are not joined then we say that they are nonadjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

**Definition 2.2.2.** The number $p = |V(G)|$ is called the *order* of $G$ and $q = |E(G)|$ is called the *size* of $G$. A graph of order $p$ and size $q$ is called a $(p, q)$ graph.

**Definition 2.2.3.** A *loop* is an edge whose end points are equal.

**Definition 2.2.4.** *Multiple edges* are edges having the same pair of end points.

**Definition 2.2.5.** A *simple graph* is a graph without loops and multiple edges.

**Definition 2.2.6.** A graph $H$ is called a *subgraph* of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

**Definition 2.2.7.** A *spanning subgraph* $H$ of $G$ is a subgraph of $G$ with $V(H) = V(G)$.

**Definition 2.2.8.** For any set $S$ of vertices of $G$, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $\langle S \rangle$ if and only if they are adjacent in $G$. $\langle S \rangle$ is also denoted by $G[S]$.

**Definition 2.2.9.** Let $v$ be a vertex of a graph $G$. The induced subgraph $\langle V(G) - \{v\} \rangle$ is denoted by $G - v$; it is the subgraph of $G$ obtained by the removal of $v$ and
edges incident with $v$. If $e \in E(G)$, then the spanning subgraph with edge set $E(G) - \{e\}$ is denoted by $G - e$; it is a subgraph of $G$ obtained by the removal of the edge $e$.

**Definition 2.2.10.** The *degree* of a vertex of a graph is the number of edges incident on it, except the fact that a loop contributes twice to the degree of that vertex. The degree of a vertex $v$ in a graph $G$ may be denoted by $\deg_G v$ or $\deg v$. The minimum and maximum degrees of vertices of $G$ are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

**Definition 2.2.11.** A vertex of degree 0 in $G$ is called an *isolated* vertex.

**Definition 2.2.12.** A vertex of degree 1 is called a *pendent* vertex or an end vertex of $G$. Any vertex which is adjacent to a pendent vertex is called a support.

**Definition 2.2.13.** A graph $G$ is said to be *complete*, if every pair of its distinct vertices are adjacent. A complete graph on $p$ vertices is denoted by $K_p$.

**Definition 2.2.14.** The *complement* $\overline{G}$ of a graph $G$ also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

**Definition 2.2.15.** Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomorphic, if there exists a bijection $f : V \to V'$ such that $(u, v) \in E$ if and only if $(f(u), f(v)) \in E'$.

**Definition 2.2.16.** An isomorphism from a graph $G$ to itself is called an *automorphism* of $G$.

**Definition 2.2.17.** A *self-complementary* graph is a graph that is isomorphic with its complement.
**Definition 2.2.18.** A *bipartite graph or bigraph* is a graph whose vertex set \( V(G) \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that every edge of \( G \) has one end in \( V_1 \) and other end in \( V_2 \). \((V_1, V_2)\) is called a bipartition of \( G \).

**Definition 2.2.19.** A *complete bipartite* graph is a bipartite graph with bipartition \((V_1, V_2)\) such that every vertex of \( V_1 \) is joined to all the vertices of \( V_2 \). It is denoted by \( K_{m, n} \), where \(|V_1| = m\) and \(|V_2| = n\).

**Definition 2.2.20.** A *star* graph is a complete bigraph \( K_{1, n} \).

**Definition 2.2.21.** A graph \( G \) is *regular* of degree \( r \) if every vertex of \( G \) has degree \( r \).

Such graphs are called \( r \)-regular graphs.

**Definition 2.2.22.** Any 3-regular graph is called a *cubic* graph.

**Definition 2.2.23.** The graphs \( \bar{K}_p \) are *totally disconnected* and regular of degree 0.

**Definition 2.2.24.** Let \( u \) and \( v \) be (not necessarily distinct) vertices of a graph \( G \).

A *walk* of \( G \) is a finite, alternating sequence \( u = u_0, e_1, u_1, e_2, \ldots, e_n, u_n = v \) of vertices and edges beginning with vertex \( u \) and ending with vertex \( v \) such that \( e_i = u_{i-1}u_i, i = 1, 2, \ldots, n \). The number \( n \) is called the length of the walk.

**Definition 2.2.25.** A \( u \)–\( v \) walk is said to be *open* if \( u \) and \( v \) are distinct vertices; it is *closed* otherwise. A walk \( u_0, e_1, u_1, e_2, \ldots, e_n, u_n \) is denoted by \( (u_0, u_1, \ldots, u_n) \).

**Definition 2.2.26.** A \( u \)–\( v \) *trail* of \( G \) is a \( u \)–\( v \) walk in which no edge is repeated.

**Definition 2.2.27.** A walk in which \( u_0, u_1, \ldots, u_n \) are distinct is called a *path*. A path on \( n \) vertices is denoted by \( P_n \).
**Definition 2.2.28.** A closed path is called a *cycle* of G. A cycle on n vertices is denoted by $C_n$.

**Definition 2.2.29.** A graph G is said to be *connected* if any two distinct vertices of G are joined by a path. G is said to be disconnected if G is not connected.

**Definition 2.2.30.** A maximal connected subgraph of G is called a component of G. The number of *component* of a graph G is denoted by $\omega(G)$.

**Definition 2.2.31.** A graph G is called *acyclic* if it has no cycles.

**Definition 2.2.32.** A connected acyclic graph is called a *tree*.

**Definition 2.2.33.** If the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are such that $V_1 \cap V_2 = \emptyset$, then their sum $G_1 + G_2$ is defined as the graph G in which vertex set is $V_1 + V_2$ and the edge set consists of the edges in $E_1$ and $E_2$, and the edge joining each vertex of $V_1$ with each vertex of $V_2$.

**Definition 2.2.34.** The *complete n-partite* graph $K_{p_1, p_2, \ldots, p_n}$ is defined as the iterated join $K_{p_1} + K_{p_2} + \ldots + K_{p_n}$.

**Definition 2.2.35.** The *corona* of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \bowtie G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where $i^{th}$ vertex of $G_1$ is adjacent to every vertices in the $i^{th}$ copy of $G_2$.

**Definition 2.2.36.** *Unicyclic* graphs are graphs which are connected and have just one cycle.
**Definition 2.2.37.** A tree, which yields a path when its pendent vertices are removed is called a *caterpillar*.

**Definition 2.2.38.** The length of a smallest cycle in a graph $G$ is referred to as its *girth* and is denoted by $gr(G)$.

**Definition 2.2.39.** By a *vertex-cut* in a connected graph $G$, we mean a set $U$ of vertices of $G$ such that $G - U$ is disconnected or trivial.

**Definition 2.2.40.** If $v$ and $w$ are vertices in $G$, then $d(v, w)$ denotes the length of a shortest path between $v$ and $w$.

**Definition 2.2.41.** Given two graphs $G_1$ and $G_2$, their union will be another graph $G$ such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

**Definition 2.2.42.** Two graphs $G_1$ and $G_2$ have disjoint vertex sets $V_1, V_2$ and edge sets $E_1, E_2$ respectively. Their *join* is denoted by $G_1 + G_2$ and it consists of $G_1 \cup G_2$ and all edges joining the vertices of $V_1$ with the vertices of $V_2$.

**Definition 2.2.43.** The *Cartesian product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G(V, E)$ with $V = V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ whenever $(u_1 = v_1$ and $u_2$ is adjacent to $v_2$) or $(u_2 = v_2$ and $u_1$ is adjacent to $v_1$). It is denoted by $G_1 \times G_2$.

**Definition 2.2.44.** The graph $W_n = C_{n-1} + K_1$ is called a *wheel* with $n$ spokes.

**Definition 2.2.45.** The product $P_m \times P_n$ is called a *planar grid* and $C_m \times P_n$ is called a *prism*. The product $P_2 \times P_n$ is called a *ladder*, and it is denoted by $L_n$. 
**Definition 2.2.46.** Any cycle with a pendent edge attached at each vertex is called a *crown*.

**Definition 2.2.47.** A *dragon* is formed by joining the end point of a path to a cycle.

**Definition 2.2.48.** $mG$ denotes the disjoint union of $m$ copies of $G$.

**Definition 2.2.49.** The graph obtained by joining a single pendent edge to each vertex of a path is called a *comb*.

**Definition 2.2.50.** Let $u$ and $v$ be two distinct vertices of a graph $G$. A new graph $G_1$ is constructed by identifying (fusing) two vertices $u$ and $v$ by a single vertex $x$ is such that every edge which was incident with either $u$ or $v$ in $G$ is now incident with $x$ in $G_1$.

**Definition 2.2.51.** A *triangular snake* $T_n$ is obtained from a path $v_1v_2\ldots v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $1 \leq i \leq n - 1$. That is, every edge of path is replaced by a triangle $C_3$.

**Definition 2.2.52.** A *Quadrilateral snake* $Q_n$ is obtained from a path $u_1u_2\ldots u_n$ by joining $u_i$ and $u_{i+1}$ to two new vertices $v_i$ and $w_i$ respectively and joining $v_i$ and $w_i$. That is, every edge of a path is replaced by a cycle $C_4$.

**Definition 2.2.53.** An *Alternate Triangular snake* $A(T_n)$ is obtained from a path $u_1u_2\ldots u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to a new vertex $v_i$, $1 \leq i \leq n - 1$. That is every alternate edge of a path is replaced by $C_3$. 
Definition 2.2.54. An *Alternate Quadrilateral snake* $A(Q_n)$ is obtained from a path $u_1u_2\ldots u_n$ by joining $u_i$ and $u_{i+1}$ (alternatively) to new vertices $v_i$ and $w_i$ respectively and then joining $v_i$ and $w_i$. That is every alternate edge of a path is replaced by a cycle $C_4$.

Definition 2.2.55. The graph $C_n^{(t)}$ denotes the one point union of $t$ copies of cycle $C_n$. The graph $C_3^{(t)}$ is called a *friendship graph* (or) *Dutch t-wind mill*.

Definition 2.2.56. Consider two copies of $C_n$. Connect a vertex of first copy to a vertex of second copy with a new edge, the new graph thus obtained is called *joint sum* of $C_n$.

Definition 2.2.57. The *Bistar* $B_{m,n}$ is the graph obtained from $K_2$ by joining $m$ pendant edges to one end of $K_2$ and $n$ pendant edges to the other end of $K_2$. The edge of $K_2$ is called the central edge of $B_{m,n}$ and the vertices of $K_2$ are called the central vertices of $B_{m,n}$.

Definition 2.2.58. $\lceil x \rceil$ (ie ceil(x)) returns the smallest integer which is greater than or equal to $x$. $\lfloor x \rfloor$ (ie floor(x)) returns the largest integer which is less than or equal to $x$.

Also we state the following theorems which we use in this thesis.

Theorem 2.2.59. [27] Any path is a Harmonic mean graph.

Theorem 2.2.60. [27] Any cycle is a Harmonic mean graph.

Theorem 2.2.61. [27] If $n > 3$, then $K_n$ is not a Harmonic mean graph.

Theorem 2.2.62. [27] $K_{1,n}$ is a Harmonic mean graph iff $n \leq 7$. 
Theorem 2.2.63. [31] Wheels are not a Harmonic mean graph.

Theorem 2.2.64. [31] nK₃ is a Harmonic mean graph.

Theorem 2.2.65. [31] nC₄ is a Harmonic mean graph.