CHAPTER 1

Introduction

The history of mathematics is long and sound whereas the history of mathematics graph theory has been originated only recently. It stands as a delightful ground for the exploration of proof techniques in discrete mathematics and its results have applications in many areas of computing, social and Natural science [3]. The concept of graph theory originated in 1736 with the work of Euler’s solution of the Konigsberg bridge problem and there are several reasons for the acceleration of interest in graph theory.

We note that graph theory finds many colourful applications, some literally, so as in the map colouring problem. Other fields influenced by graph theory branches are physics, chemistry, communication science, computer technology, electrical and civil engineering, architecture, operation research, genetics, psychology, sociology, economics, linguistics and so on. Graph theory is intimately related to many branches of mathematics including Group theory, Matrix theory, Numerical Analysis, probability, Topology and Combinatorics [15]. The fact is that graph serves as a mathematical model for any system involving binary relation. Partly because of their diagrammatic representation, graphs have an intuitive and aesthetic appeal.

How can we lay cable at minimum cost to make every telephone reachable from one another? What is the fastest route from the national capital to each state
capital? How could n jobs be filled by n people with maximum total utility? What is the maximum flow per unit time from source to sink in a network of pipes? How many layers does a computer chip needs so that wires in the same layer do not cross? How could the seasons of sports league be scheduled into the cities to minimize travel time? Could we colour the regions of every map using four colours so that neighbouring regions receive different colours? These are some of the practical problems involved in graph theory.

Although the origin of graph theory could be traced back to almost 300 years, it is only in the recent years the subject has begun to enter the mainstream of mathematics. The number of papers and books published in this field are evidence to the same. Graph labeling was first introduced in the 1960’s where the vertices and edges are assigned real values or subsets of a set subject to certain conditions. An enormous body of literature has grown around graph labeling in the last four decades. Labeled graphs provide mathematical models for a broad range of applications. The qualitative labeling of a graph elements have been used in diverse fields such as Conflict resolutions, Social Psychology and Energy crisis. Quantitative labeling of graph elements have been used in Missile guidance codes, Radar location codes, Coding theory, X-Ray Crystallography, Radio Astronomy, Circuit design, Communication Network, Database management and Secret sharing schemes.

Graph labeling, an assignment of integers to the vertices or edges or both is subject to certain conditions motivated by practical problems. A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv
a label depending on the vertex labels $f(u)$ and $f(v)$. Similarly an edge labeling of a
graph $G$ is an assignment $f$ of labels to the edges of $G$ that induces for each vertex $v$ a
label depending on the labels of the edges incident to it.

Several variations of graph labeling such as Graceful, Harmonious, Sequential, Prime, Magic, Antimagic, Bimagic, Sum, Integral Sum, Elegant, Radio, Mean, Cordial, Product cordial, Prime Cordial, Geometric, Square Sum and Triangular Sum labeling have been introduced by several authors (See Gallian) [11, 12].

In this thesis, an attempt has been made to study the Harmonic mean labeling of graphs.

This thesis consists of seven chapters. The details are as below.

Chapter 1: Introduction

Chapter 2: Preliminaries

Chapter 3: Harmonic Mean Labeling of Graphs

Chapter 4: Harmonic Mean labeling of Degree Spliting Graphs

Chapter 5: One Modulo Three Harmonic Mean Labeling of Graphs

Chapter 6: Super Harmonic Mean Labeling of Graphs

Chapter 7: Conclusion

At last, we have included the References. The definitions, examples and results of this thesis are numbered serially. The numbering is in the format of A.B.C,
where A denotes the Chapter, B denotes the Section and C denotes the respective definition / example / theorem.

Finally the references are arranged as usual in the alphabetical order of the names of the authors and each item includes the name of the journal or the book, volume number, year of publication and page numbers.

**Notations**

The following are the notations followed in this thesis:

\( G = (V, E) \) simple graph

\( V = V(G) \) vertex set of \( G \)

\( E = E(G) \) edge set of \( G \)

\( P \) number of vertices in \( G \)

\( q \) number of edges in \( G \)

\( \bar{G} \) complement of \( G \)

\( K_n \) complete graph with \( n \) vertices

\( K_{m, n} \) complete bipartite graph

\( K_{1, n} \) star

\( C_n \) cycle with \( n \) vertices

\( P_n \) path with \( n \) vertices

\( W_n \) wheel on \( n \) vertices
\( T_n \)  triangular snake

\( A(T_n) \)  alternate triangular snake

\( DT_n \)  double triangular snake

\( A(DT_n) \)  alternate double triangular snake

\( Q_n \)  quadrilateral snake

\( A(Q_n) \)  alternate quadrilateral snake

\( DQ_n \)  double quadrilateral snake

\( A(DQ_n) \)  alternate double quadrilateral snake

\( TL_n \)  triangular ladder

\( G_1 \times G_2 \)  cartesian product or product of two graphs \( G_1 \) and \( G_2 \)

\( G_1 + G_2 \)  sum or join of two graphs \( G_1 \) and \( G_2 \)

\( G_1 \circ G_2 \)  composition of two graphs \( G_1 \) and \( G_2 \)

\( G_1 \odot G_2 \)  corona of two graphs \( G_1 \) and \( G_2 \)

\( DS(G) \)  degree splitting graph of \( G \)

\( C_n \odot K_1 \)  crown graph

\( P_n \odot K_1 \)  comb graph

\( L_n \)  ladder graph

\( C_n(t) \)  friendship graph
\[ P_m \times P_n \quad \text{planar grid} \]
\[ P_m \times C_n \quad \text{prism} \]
\[ D_n \quad P_2 \times C_n \]
\[ (D_n; P_3) \quad \text{a graph obtained by attaching } P_3 \text{ at each vertex of outer cycle of } D_n \]
\[ S(T_n) \quad \text{step ladder graph} \]
\[ P_n(P_1, 2P_2, 3P_3, \ldots, 2P_n) \quad \text{a graph obtained by attaching paths of length } 0, 1, 2, \ldots, n-1 \text{ respectively on both sides of each vertex of } P_n \]

**Content overview**

Chapter 1 of the thesis briefs the contents of the thesis by specifying certain important results in each and every chapter and also elaborates the structure of the dissertation.

Chapter 2 deals with the preliminaries needed for the development of the thesis. A brief survey has been made on graph labeling which are required for the subsequent chapters and a detailed list of basic definitions, results and theorems on graphs and graph labelings are given in this chapter.

In Chapter 3, definition and examples of Harmonic mean labeling is given and also the Harmonic mean labeling of step ladder graph \( S(T_n), (D_n; P_3), P_n(P_1, 2P_2, 2P_3, \ldots, 2P_n), P_n\circ K_2, \) a graph obtained by attaching the central vertex of \( K_{1,2} \) at each pendent vertex of a comb, a graph obtained by attaching a triangle at each pendent vertex of a comb, \( C_n\circ K_2, \) a graph obtained by attaching \( K_{1,3} \) at each vertex of the
cycle $C_n$, a graph obtained by attaching a triangle at each pendent vertex of a crown, $D_n \odot K_1$, flower graph $f_n \times 3$, $L_n \odot \overline{K}_2$, triangular ladder $TL_n$, $TL_n \odot K_1$, double triangular snake $DT_n$, alternate double triangular snake $A(DT_n)$, $T_n \odot K_1$, double quadrilateral snake $DQ_n$, alternate double quadrilateral snake $A(DQ_n)$, $Q_n \odot K_1$, $(C_m \odot K_1) \cup P_n$, $(C_m \odot K_1) \cup C_n$, $C_n \cup (P_n \odot K_1)$, $(C_m \odot \overline{K}_2) \cup P_n$, $(C_m \odot \overline{K}_2) \cup (P_n \odot K_1)$, $C_m \cup (P_n \odot \overline{K}_2)$, $(C_m \odot K_1) \cup (P_n \odot \overline{K}_2)$ and $(C_m \odot \overline{K}_2) \cup (P_n \odot \overline{K}_2)$ are obtained. Most of the results in this chapter have been published or communicated [6, 7, 16, 17, 28, 29].

In chapter 4, we introduced Harmonic mean labeling for Degree splitting graphs. In this chapter we prove that $DS(K_n)$, $n > 2$, and $DS(C_n)$ are not Harmonic mean graphs. We also prove that $DS(\overline{K}_n)$, $n \leq 7$, $nDS(K_{1, 2})$, $nDS(K_2)$, $nDS(P_3)$, $nDS(P_4)$, $nDS(P_2 \odot K_1)$, $nDS(P_2 \odot K_1)$, $nDS(P_2 \odot \overline{K}_2)$ and $nDS(K_{1, 3})$ are Harmonic mean graphs. All the results of this chapter are published in [30].

In chapter 5, a new concept namely, one modulo three harmonic mean labeling is introduced. In this chapter, we prove that any $K$-regular graph, $K > 1$, any cycle $C_n$, $K_n$, $n > 2$, and $K_{1, n}$, $n > 6$ are not one modulo three harmonic mean graphs. We also prove that $P_n$, $nP_m$, $P_n \odot K_1$, $P_n \odot \overline{K}_2$, a graph obtained by attaching $P_3$ at each vertex of $P_n$, $P_n \odot \overline{K}_3$, a graph obtained by attaching the central vertex of $K_{1, 2}$ at each pendent vertex of comb $P_n \odot K_1$, $C_n \odot K_1$, $C_n \odot \overline{K}_2$, a graph obtained by attaching $P_3$ at each vertex of $C_n$, $C_n \odot \overline{K}_3$, a graph obtained by attaching $K_{1, 3}$ at each vertex of a cycle $C_n$ and $L_n \odot K_1$ are one modulo three harmonic mean graphs. Almost all the results of this chapter have been published in [4, 8].
In chapter 6, we introduce a new concept Super harmonic mean labeling of graphs and also we prove that $P_n, P_n \odot K_1$, a graph obtained by joining a pendent vertex with a vertex of degree two in a comb, $nP_m, nK_1, 3$, a graph obtained by attaching $C_3$ to the end vertex of $P_n$, $C_n \odot K_1, C_n \odot K_2$, a graph obtained by attaching an edge at each vertex of an outer cycle of $D_n$, $C_n \cup P_n, C_n \cup (P_n \odot K_1), (C_m \odot K_1) \cup P_n, (C_m \odot K_1) \cup C_n$, a graph obtained by adding an edge at an arbitrary vertex of cycle $C_n$ and $(C_m \odot K_1) \cup (P_n \odot K_1)$ are Super harmonic mean graphs. Some results in this chapter have been published in [5].

In chapter 7, we summarise our results.