CHAPTER 5

One Modulo Three Harmonic Mean Labeling of Graphs

5.1. Introduction

V. Swaminathan and C. Sekar introduced the concept of one modulo three graceful labeling in [43]. S. Somasundaram and R. Ponraj introduced mean labeling of graphs in [36]. P. Jeyanthi and A. Maheswari introduced the concept one modulo three mean labeling of graphs. A graph $G$ is said to be one modulo three mean graph if there is a function $\varphi$ from the vertex set of $G$ to $\{0, 1, 3, 4, \ldots, 3(q-1), 3q-2\}$ with $\varphi$ is one-one and $\varphi$ induces a bijection $\varphi^*$ from the edge set of $G$ to $\{1, 4, 7, \ldots, 3q-2\}$, where $\varphi^*(uv) = \left\lfloor \frac{\varphi(u) + \varphi(v)}{2} \right\rfloor$ and the function $\varphi$ is called as one modulo three mean labeling of $G$. In this chapter, we introduce a new type of labeling called as one modulo three harmonic mean labeling and we investigate one modulo three harmonic mean labeling of some standard graphs.

In section 5.2, we give definition and example of one modulo three harmonic mean graph. In section 5.3, we investigate one modulo three harmonic mean labeling of some trees. In section 5.4, we investigate one modulo three harmonic mean graphs which contains cycles.
5.2. One modulo three harmonic mean graphs

Definition 5.2.1. A graph $G$ is said to be one modulo three harmonic mean graph if there is a function $\phi$ from the vertex set of $G$ to $\{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$, with $\phi$ is one-one and $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, 4, 7, \ldots, 3q - 2\}$, where $\phi^*(e = uv) = \left\lfloor \frac{2\phi(u)\phi(v)}{\phi(u) + \phi(v)} \right\rfloor$ or $\left\lceil \frac{2\phi(u)\phi(v)}{\phi(u) + \phi(v)} \right\rceil$ and the function $\phi$ is called as one modulo three harmonic mean labeling of $G$.

Example 5.2.2. One modulo three harmonic mean labeling of a graph $G$ is shown in figure 5.1.

![Graph G](image)

Figure 5.1.G

Remark 5.2.3. If $G$ is one modulo three harmonic mean graph, then 1 must be a label of one of the vertices of $G$, since an edge must get the label 1.

Theorem 5.2.4. If $G$ is one modulo three harmonic mean graph, then $G$ has atleast one vertex of degree one and an end vertex gets label 1.

Proof. Suppose $G$ is one modulo three harmonic mean graph. Then by remark 5.2.3, 1 must be the label of one of the vertices of $G$. If the vertex $u$ gets the label 1, then any edge incident with $u$ gets the label 1 or 2, since $1 \leq \frac{2m}{m+1} < 2$, where $m$ is the label
of a vertex adjacent to u. But 2 is not in the set \{1, 3, 4, \ldots, 3q - 2, 3q\} and hence any edge incident with u must have label 1. Since \(\varphi^*\) is a bijection from an edge set \(G\) to \(\{1, 3, 4, \ldots, 3q - 2, 3q\}\), only one edge incident with u. Hence the degree of u is 1.

**Theorem 5.2.5.** Any k-regular graph, \(k > 1\), is not one modulo three harmonic mean graph.

**Proof.** Let \(G\) be a k-regular graph. Suppose \(G\) is one modulo three harmonic mean graph. Then by Theorem 5.2.4, there is at least one vertex of degree one which is a contradiction to \(k > 1\). Hence the theorem.

**Corollary 5.2.6.** Any cycle \(C_n\) is not one modulo three harmonic mean graph.

**Proof.** \(C_n\) is a 2-regular graph and hence corollary follows from theorem 5.2.5.

**Corollary 5.2.7.** If \(n > 2\), \(K_n\) is not one modulo three harmonic mean graph.

**Proof.** \(K_n\), \(n > 2\) is a k-regular graph and hence the corollary follows from theorem 5.2.5.

### 5.3. One modulo three harmonic mean labeling of some trees

In this section, we investigate one modulo three harmonic mean labeling of some classes of trees like path, star, comb etc.

**Theorem 5.3.1.** Any path \(P_n\) is one modulo three harmonic mean graph.

**Proof.** Let \(P_n\) be the path \(u_1u_2\ldots u_n\). Define a function \(\varphi : V(P_n) \rightarrow \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}\)

\[\varphi(u_1) = 1, \varphi(u_i) = 3(i - 1), \quad 2 \leq i \leq n.\]
Then $\varphi$ induces a bijection $\varphi^*: \text{E}(\text{P}_n) \rightarrow \{1, 4, \ldots, 3q - 2\}$, where

$$\varphi^*(u_iu_{i+1}) = 3i - 2, \ 1 \leq i \leq n - 1.$$ 

Therefore, $\varphi$ is an one modulo three harmonic mean graph.

**Example 5.3.2.** One modulo three harmonic mean labeling of $\text{P}_8$ is shown in figure 5.2.

![Figure 5.2. P_8](image)

**Theorem 5.3.3.** $n\text{P}_m$ is one modulo three harmonic mean Graph.

**Proof.** Let $v_{i,1} v_{i,2} \ldots v_{i,m}$ be the $i^{th}$ $\text{P}_m$ of $n\text{P}_m$, $1 \leq i \leq n$. Then $V = \{v_{i,j} / 1 \leq i \leq n, 1 \leq j \leq m\}$ is the vertex set and $E = \{v_{i,j}v_{i,j+1} / 1 \leq i \leq n, 1 \leq j \leq m - 1\}$ is the edge set of $n\text{P}_m$. Define a function $\varphi : V(n\text{P}_m) \rightarrow \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$ by

$$\varphi(v_{1,1}) = 1; \varphi(v_{1,j}) = 3(j - 1), 2 \leq j \leq m - 1;$$

$$\varphi(v_{i,j}) = 3(m - 1)(i - 1) + 3(j - 1), 2 \leq i \leq n, 1 \leq j \leq m - 1;$$

$$\varphi(v_{i,m}) = 3im - 3(i + 1) + 1, 1 \leq i \leq n.$$ 

Then $\varphi$ induces a bijective function $\varphi^*: \text{E}(n\text{P}_m) \rightarrow \{1, 4, 7, \ldots, 3q - 2\}$, where

$$\varphi^*(v_{i,j}v_{i,j+1}) = 3(m - 1)(i - 1) + 3(j - 1) + 1, 1 \leq i \leq n, 1 \leq j \leq m - 1.$$ 

Thus $\varphi$ provides one modulo three harmonic mean labeling for $n\text{P}_m$. Hence $n\text{P}_m$ is one modulo three harmonic mean graph.
Example 5.3.4. One modulo three harmonic mean labeling of $4P_7$ is shown in figure
5.3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_5.3.4.png}
\caption{4P_7}
\end{figure}

Theorem 5.3.5. $P_n \circ K_1$ is one modulo three harmonic mean graph.

Proof. Let $u_1u_2 \ldots u_n$ be the path $P_n$. Join vertex $v_i$ with $u_i$, $1 \leq i \leq n$. Then the resultant graph is $P_n \circ K_1$. Define a function $\varphi : V(P_n \circ K_1) \to \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$ by

\begin{align*}
\varphi(u_1) &= 3; \\
\varphi(u_i) &= 6i - 3 \text{ for all even } i \text{ and } i \leq n; \\
\varphi(u_i) &= 6(i - 1) \text{ for all odd } i, i \neq 1 \text{ and } i \leq n; \\
\varphi(v_1) &= 1; \\
\varphi(v_i) &= 6(i - 1) \text{ for all even } i \text{ and } i \leq n; \\
\varphi(v_i) &= 6i - 3 \text{ for all odd } i, i \neq 1 \text{ and } i \leq n.
\end{align*}

Then $\varphi$ induces a bijective function $\varphi^* : E(P_n \circ K_1) \to \{1, 4, 7, \ldots, 3q - 2\}$, where

$$
\varphi^*(u_i u_{i+1}) = 6i - 2, 1 \leq i \leq n - 1;
$$
\[ \varphi^*(u_i v_i) = 6i - 5, \ 1 \leq i \leq n. \]

Thus the edges get the distinct labels 1, 4, ..., 3q – 2. Therefore, \( \varphi \) is one modulo three harmonic mean labeling. Hence \( P_n \Theta K_1 \) is one modulo three harmonic mean graph.

**Example 5.3.6.** One modulo three harmonic mean labeling of \( P_6 \Theta K_1 \) and \( P_7 \Theta K_1 \) are shown in figure 5.4 and 5.5 respectively.

![Figure 5.4. P_6 \Theta K_1](image)

![Figure 5.5. P_7 \Theta K_1](image)

**Theorem 5.3.7.** \( K_{1, n} \) is one modulo three harmonic mean graph if and only if \( n \leq 6 \).

**Proof.** \( K_{1, 1} \) is same as \( P_2 \) and \( K_{1, 2} \) is \( P_3 \). Hence by Theorem 5.3.1, \( K_{1, 1} \) and \( K_{1, 2} \) are one modulo three harmonic mean graphs. One modulo three harmonic mean labeling
for $K_{1,3}$, $K_{1,4}$, $K_{1,5}$ and $K_{1,6}$ are shown in figures 5.6, 5.7, 5.8 and 5.9 respectively.

Figure 5.6. $K_{1,3}$

Figure 5.7. $K_{1,4}$

Figure 5.8. $K_{1,5}$

Figure 5.9. $K_{1,6}$

Suppose $K_{1,n}$ has one modulo three harmonic mean labeling for $n > 6$. Let $u$ be the central vertex of $K_{1,n}$. By theorem 5.2.4, $\varphi(u)$ is not equal to 1. Here we consider two cases.

Case 1. $2 \leq \varphi(u) \leq 13$.

Then clearly there is no edge with label $3q - 2$, $q > 6$, since the highest edge label is $\frac{2 \cdot 13}{13 + 3q}$. Hence in this case $K_{1,n}$ is not one modulo three harmonic mean graph.

Case 2. $\varphi(u) \geq 15$.

Then there is no edge with label 4 as shown in the following figure 5.10.
Figure 5.10

Hence $K_{1, n}$ is not one modulo three harmonic mean graph for $n > 6$.

**Theorem 5.3.8.** $P_n \circ \overline{K_2}$ is one modulo three harmonic mean graph.

**Proof.** Let $u_1 u_2 \ldots u_n$ be the path $P_n$. Let $v_i, w_i$ be the vertices of $i$:th copy of $\overline{K_2}$. Join $v_i$ and $w_i$ with the vertex $u_i$, $1 \leq i \leq n$. The resultant graph is $P_n \circ \overline{K_2}$ whose edge set is $E = \{u_iu_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_i v_i, u_i w_i / 1 \leq i \leq n\}$. Define a function $\phi : V(P_n \circ \overline{K_2}) \to \{1, 3, 4, 6, \ldots, 3q \}$ by

$\phi(u_1) = 4; \phi(u_2) = 13; \phi(u_i) = 9i - 6, 3 \leq i \leq n;$

$\phi(v_1) = 1; \phi(v_2) = 9; \phi(v_i) = 9(i - 1) - 2, 3 \leq i \leq n;$

$\phi(w_1) = 3; \phi(w_2) = 12; \phi(w_i) = 9i - 5, 3 \leq i \leq n.$

Then $\phi$ induces a bijective function $\phi^\ast : E(P_n \circ \overline{K_2}) \to \{1, 3, 4, 6, \ldots, 3q - 2\}$, where

$\phi^\ast(u_i u_{i+1}) = 9i - 2, 1 \leq i \leq n - 1;$

$\phi^\ast(u_i v_i) = 9i - 8, 1 \leq i \leq n;$

$\phi^\ast(u_i w_i) = 9i - 5, 1 \leq i \leq n.$
Therefore, \( \phi \) is one modulo three harmonic mean labeling of \( P_n \odot \overline{K}_2 \). Hence \( P_n \odot \overline{K}_2 \) is one modulo three harmonic mean graph.

**Example 5.3.9.** One modulo three harmonic mean labeling of \( P_5 \odot \overline{K}_2 \) is shown in figure 5.11.

![Figure 5.11. \( P_5 \odot \overline{K}_2 \)](image)

In a similar manner, we prove the following theorem.

**Theorem 5.3.10.** A graph obtained by attaching \( P_3 \) at each vertex of \( P_n \) is one modulo three harmonic mean graph.

**Proof.** Let \( u_1u_2\ldots u_n \) be the path \( P_n \). Let \( x_i \) be the \( i \)th copy of \( P_3 \), \( 1 \leq i \leq n \). Identifying the vertices \( u_i \) and \( x_i \), we get the required graph \( G \) with edge set \( E = \{u_iu_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_iv_i, v_iw_i / 1 \leq i \leq n\} \). Define a function \( \phi : V(G) \to \{1, 3, 4, 6, \ldots, 3q - 2, 3q\} \) by

\[
\begin{align*}
\phi(u_i) &= 9i - 5, 1 \leq i \leq 2; \\
\phi(u_i) &= 9i - 6, 3 \leq i \leq n; \\
\phi(v_i) &= 9i - 6, 1 \leq i \leq 2; \\
\phi(v_i) &= 9i - 5, 3 \leq i \leq n; \\
\phi(w_1) &= 1; \phi(w_2) = 9;
\end{align*}
\]
\[ \varphi(w_i) = 9(i - 1) - 2, \ 3 \leq i \leq n. \]

Then \( \varphi \) induces a bijective function \( \varphi^*: E(G) \to \{1, 4, \ldots, 3q - 2\} \), where

\[ \varphi^*(u_iu_{i+1}) = 9i - 2, \ 1 \leq i \leq n - 1; \]
\[ \varphi^*(u_iv_i) = 9i - 5, \ 1 \leq i \leq n; \]
\[ \varphi^*(v_iw_i) = 9i - 8, \ 1 \leq i \leq n. \]

Therefore, \( \varphi \) is one modulo three harmonic mean labeling of \( G \). Hence \( G \) is one modulo three harmonic mean graph.

**Example 3.3.11.** One modulo three harmonic mean labeling of \( G \) when \( n = 7 \) is shown in figure 5.12.

![Figure 5.12. G](image)

**Theorem 5.3.12.** \( P_n \Theta \overline{K}_3 \) is one modulo three harmonic mean graph.

**Proof.** Let \( P_n \) be the path \( u_1u_2 \ldots u_n \). Let \( x_i, y_i, z_i \) be the vertices of \( i^{th} \) copy of \( \overline{K}_3 \) which are adjacent to the vertex \( u_i \) of \( P_n \), \( 1 \leq i \leq n \). The resultant graph is \( P_n \Theta \overline{K}_3 \) whose edge set is \( E = \{u_iu_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_ix_i, u_iy_i, u_iz_i / 1 \leq i \leq n\} \). Define a function \( \varphi: V(P_n \Theta \overline{K}_3) \to \{1, 3, 4, 6, \ldots, 3q - 2, 3q\} \) by

\[ \varphi(u_1) = 7; \ \varphi(u_2) = 18; \ \varphi(u_3) = 30; \ \varphi(u_i) = 12i - 8, \ 4 \leq i \leq n; \]
\[
\varphi(x_1) = 1; \varphi(x_2) = 10; \varphi(x_i) = 12(i-1) - 3, 3 \leq i \leq 4;
\]
\[
\varphi(x_i) = 12(i-1) - 2, 5 \leq i \leq n;
\]
\[
\varphi(y_1) = 3; \varphi(y_i) = 12(i-1) + 3, 2 \leq i \leq n;
\]
\[
\varphi(z_1) = 6; \varphi(z_i) = 12i - 5, 2 \leq i \leq 3; \varphi(z_i) = 12i - 3, 4 \leq i \leq n.
\]

Then \( \varphi \) induces a bijective function \( \varphi^*: E(P_n \circ K_3) \rightarrow \{1, 4, 7, \ldots, 3q - 2\} \), where
\[
\varphi^*(u_iu_{i+1}) = 12i - 2, 1 \leq i \leq n - 1;
\]
\[
\varphi^*(u_ix_i) = 12(i-1) + 1, 1 \leq i \leq n;
\]
\[
\varphi^*(u_iy_i) = 12(i-1) + 4, 1 \leq i \leq n;
\]
\[
\varphi^*(u_iz_i) = 12(i-1) + 7, 1 \leq i \leq n.
\]

Thus \( \varphi \) provides one modulo three harmonic mean labeling for \( P_n \circ K_3 \). Hence \( P_n \circ K_3 \) is one modulo three harmonic mean graph.

**Example 5.3.13.** One modulo three harmonic mean labeling for \( P_6 \circ K_3 \) is given in figure 5.13.
**Theorem 5.3.14.** A graph obtained by attaching the central vertex of $K_{1,2}$ at each pendant vertex of a comb $P_n\bigcirc K_1$ is one modulo three harmonic mean graph.

**Proof.** Let $P_n$ be the path $u_1u_2\ldots u_n$. Let $v_i$ be a vertex adjacent to $u_i$, $1 \leq i \leq n$. The resultant graph is $P_n\bigcirc K_1$. Let $x_i, w_i, y_i$ be the vertices of $i^{th}$ copy of $K_{1,2}$ with $w_i$ is the central vertex. Identify the vertex $w_i$ with $v_i$, $1 \leq i \leq n$, we get the required graph $G$ whose edge set is $E = \{u_iu_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_iw_i, v_ix_i, v_iy_i / 1 \leq i \leq n\}$. Define a function $\varphi: V(G) \to \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$ by

- $\varphi(u_1) = 7$; $\varphi(u_i) = 12(i - 1) + 6$, $2 \leq i \leq 3$; $\varphi(u_i) = 12(i - 1) + 4$, $4 \leq i \leq n$;
- $\varphi(v_1) = 6$; $\varphi(v_2) = 19$; $\varphi(v_i) = 12i - 3$, $3 \leq i \leq 4$; $\varphi(v_i) = 12i - 3$, $5 \leq i \leq n$;
- $\varphi(x_1) = 1$; $\varphi(x_2) = 9$; $\varphi(x_i) = 21$; $\varphi(x_4) = 31$; $\varphi(x_i) = 12(i - 1) - 6$, $5 \leq i \leq n$;
- $\varphi(y_1) = 3$; $\varphi(y_2) = 13$; $\varphi(y_i) = 12(i - 1)$, $3 \leq i \leq n$.

Then $\varphi$ induces a bijective function $\varphi^*: E(G) \to \{1, 4, 7, \ldots, 3q - 2\}$, where

- $\varphi^*(u_iv_i) = 12i - 5$, $1 \leq i \leq n$;
- $\varphi^*(u_iu_{i+1}) = 12i - 2$, $1 \leq i \leq n - 1$;
- $\varphi^*(v_ix_i) = 12(i - 1) + 1$, $1 \leq i \leq n$;
- $\varphi^*(v_iy_i) = 12(i - 1) + 4$, $1 \leq i \leq n$.

In the view of the above labeling pattern, $\varphi$ provides one modulo three harmonic mean labeling for $G$. Hence $G$ is one modulo three harmonic mean graph.
Example 5.3.15. One modulo three harmonic mean labeling for G when \( n = 7 \) is given in figure 5.14.

![Graph](image)

Figure 5.14.

Theorem 5.3.16. Let G be a graph obtained by joining a pendent vertex with a vertex of degree two of a comb \( P_n \odot K_1 \). Then G is one modulo three harmonic mean graph.

Proof. Let \( P_n \) be the path \( u_1u_2 \ldots u_n \). Let \( v_i \) be the vertex adjacent to \( u_i, 1 \leq i \leq n \). The resultant graph is the comb \( P_n \odot K_1 \). Let \( w \) be the vertex joined with \( u_n \). The resultant graph is G.

Case 1. \( n \) is odd.

Define a function \( \phi : V(G) \rightarrow \{1, 3, 4, 6, \ldots, 3q - 2, 3q\} \) by

\[
\phi(u_1) = 3; \quad \phi(u_i) = 6(i - 1) + 3 \text{ for all odd } i \text{ and } 3 \leq i \leq n;
\]

\[
\phi(u_2) = 7; \quad \phi(u_i) = 6(i - 1) \text{ for all even } i \text{ and } 4 \leq i \leq n;
\]

\[
\phi(w) = 6n;
\]

\[
\phi(v_1) = 1; \quad \phi(v_i) = 6(i - 1) \text{ for all odd } i \text{ and } 3 \leq i \leq n;
\]

\[
\phi(v_2) = 6; \quad \phi(v_i) = 6(i - 1) + 3 \text{ for all even } i \text{ and } 4 \leq i \leq n.
\]
Then $\phi$ induces a bijective function $\phi^*: E(G) \to \{1, 4, 7, \ldots, 3q - 2\}$, where

$$
\phi^*(u_iu_{i+1}) = 6i - 2, \; 1 \leq i \leq n - 1; \; \phi^*(u_iw) = 6n - 2;
$$

$$
\phi^*(u_iw) = 6i - 5, \; 1 \leq i \leq n.
$$

Thus the edges get distinct labels 1, 4, ..., 3q - 2. In this case $\phi$ is one modulo three harmonic mean labeling for $G$.

**Case 2.** $n$ is even.

Define a function $\phi: V(G) \to \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$ by

$$
\phi(u_1) = 3; \; \phi(u_i) = 6(i - 1) \text{ for all odd } i \text{ and } 3 \leq i \leq n;
$$

$$
\phi(u_i) = 6(i - 1) + 3 \text{ for all even } i \text{ and } 2 \leq i \leq n;
$$

$$
\phi(w) = 6n;
$$

$$
\phi(v_1) = 1; \; \phi(v_i) = 6(i - 1) + 3 \text{ for all odd } i \text{ and } 3 \leq i \leq n;
$$

$$
\phi(v_i) = 6(i - 1) \text{ for all even } i \text{ and } 2 \leq i \leq n.
$$

Then $\phi$ induces a bijective function $\phi^*: E(G) \to \{1, 4, 7, \ldots, 3q - 2\}$, where

$$
\phi^*(u_iu_{i+1}) = 6i - 2, \; 1 \leq i \leq n - 1; \; \phi^*(u_iw) = 6n - 2;
$$

$$
\phi^*(u_iw) = 6i - 5, \; 1 \leq i \leq n.
$$

Thus the edges get distinct labels 1, 4, ..., 3q - 2. In this case $\phi$ is one modulo three harmonic mean labeling for $G$. From case 1 and case 2, we conclude that $G$ is one modulo three harmonic mean graph.
Example 5.3.17. One modulo three harmonic mean labeling of \( G \), when \( n = 6 \) and \( n = 7 \) are given in figure 5.15 and figure 5.16 respectively.

![Figure 5.15]

![Figure 5.16]

5.4. One modulo three harmonic mean labeling of cycle related graphs

In this section, we investigate one modulo three harmonic mean graphs which contains cycles.

**Theorem 5.4.1.** \( C_n \bigodot K_1 \) is one modulo three harmonic mean graph for \( n \geq 4 \).

**Proof.** Let \( u_1u_2 \ldots u_nu_1 \) be the cycle \( C_n \). For \( 1 \leq i \leq n \), let \( v_i \) be the vertex of \( i^{th} \) copy of \( K_1 \) which is adjacent to \( u_i \). The resultant graph is \( C_n \bigodot K_1 \) whose edge set is \( E = \{u_iu_1, u_iu_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_i v_i / 1 \leq i \leq n\} \).

Case 1. \( 4 \leq n \leq 7 \).

Define a function \( \varphi : V(C_n \bigodot K_1) \rightarrow \{1, 3, 4, 6, \ldots, 3q - 2, 3q\} \) by

\[
\varphi(u_1) = 6; \quad \varphi(u_i) = 6i - 2, \quad 2 \leq i \leq n;
\]
\( \varphi(v_1) = 1; \varphi(v_2) = 3; \varphi(v_i) = 6i - 3, 3 \leq i \leq n. \)

Then \( \varphi \) induces a bijective function \( \varphi^*: E(C_n \Theta K_1) \to \{1, 4, \ldots, 3q - 2\} \), where

\[
\begin{align*}
\varphi^*(u_1u_2) &= 7; \varphi^*(u_iu_{i+1}) = 6i + 1, 2 \leq i \leq n - 1; \varphi^*(u_nu_1) = 10; \\
\varphi^*(u_1v_1) &= 1; \varphi^*(u_2v_2) = 4; \varphi^*(u_iv_i) = 6i - 2, 3 \leq i \leq n.
\end{align*}
\]

**Case 2.** \( n > 7. \)

Define a function \( \varphi : V(C_n \Theta K_1) \to \{1, 3, 4, 6, \ldots, 3q - 2, 3q\} \) by

\[
\begin{align*}
\varphi(u_1) &= 7; \varphi(u_i) = 6i \text{ for odd } i \text{ and } 3 \leq i \leq n; \\
\varphi(u_2) &= 13; \varphi(u_i) = 6i - 3 \text{ even } i \text{ and } 3 \leq i \leq n; \\
\varphi(v_1) &= 1; \varphi(v_2) = 3; \varphi(v_3) = 4; \\
\varphi(v_i) &= 6i \text{ for even } i \text{ and } 4 \leq i \leq n; \\
\varphi(v_i) &= 6i - 3 \text{ for odd } i \text{ and } 5 \leq i \leq n.
\end{align*}
\]

Then \( \varphi \) induces a bijective function \( \varphi^*: E(C_n \Theta K_1) \to \{1, 4, \ldots, 3q - 2\} \), where

\[
\begin{align*}
\varphi^*(u_1u_{i+1}) &= 6i + 4, 1 \leq i \leq 2; \\
\varphi^*(u_iu_{i+1}) &= 6i + 1, 3 \leq i \leq n - 1; \varphi^*(u_nu_1) = 13; \\
\varphi^*(u_iv_i) &= 3i - 2, 1 \leq i \leq 3; \\
\varphi^*(u_iv_i) &= 6i - 2, 4 \leq i \leq n.
\end{align*}
\]

Therefore, \( \varphi \) is one modulo three harmonic mean labeling of \( C_n \Theta K_1 \). Hence \( C_n \Theta K_1 \) is one modulo three harmonic mean graph.
Example 5.4.2. One modulo three harmonic mean labeling of $C_8 \circ K_1$ and $C_9 \circ K_1$ are given in figure 5.17 and figure 5.18 respectively.

Figure 5.17. $C_8 \circ K_1$

Figure 5.18. $C_9 \circ K_1$
Theorem 5.4.3. $C_n \odot \overline{K}_2$ is one modulo three harmonic mean graph.

Proof. Let $u_1u_2\ldots u_nu_1$ be the cycle $C_n$. For $1 \leq i \leq n$, let $v_i, w_i$ be the vertices of $i^{th}$ copy of $\overline{K}_2$ which are adjacent to $u_i$. The resultant graph is $C_n \odot \overline{K}_2$ whose edge set is $E = \{u_nu_1, u_iu_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_iv_i, u_iw_i / 1 \leq i \leq n\}$.

Case 1. $3 \leq n \leq 5$.

Define a function $\varphi : V(C_n \odot \overline{K}_2) \rightarrow \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$ by

$$\varphi(u_1) = \begin{cases} 12 & \text{for } n = 3 \\ 10 & \text{for } n \neq 3 \end{cases};$$

$$\varphi(u_2) = 16; \varphi(u_i) = 9i - 3, 3 \leq i \leq n;$$

$$\varphi(v_1) = 1; \varphi(v_2) = 4; \varphi(v_i) = 9i - 6, 3 \leq i \leq n;$$

$$\varphi(w_1) = 3; \varphi(w_2) = 7; \varphi(w_i) = 9i - 2, 3 \leq i \leq n.$$ 

Then $\varphi$ induces a bijective function $\varphi^* : E(C_n \odot \overline{K}_2) \rightarrow \{1, 4, \ldots, 3q - 2\}$, where

$$\varphi^*(u_1u_2) = 13; \varphi^*(u_iu_{i+1}) = 9i + 1, 2 \leq i \leq n - 1; \varphi^*(u_nu_1) = 16;$$

$$\varphi^*(u_1v_1) = 1; \varphi^*(u_2v_2) = 7; \varphi^*(u_iv_i) = 9i - 5, 3 \leq i \leq n;$$

$$\varphi^*(u_1w_1) = 4; \varphi^*(u_2w_2) = 10; \varphi^*(u_iw_i) = 9i - 2, 3 \leq i \leq n.$$ 

Case 2. $n > 5$.

Define a function $\varphi : V(C_n \odot \overline{K}_2) \rightarrow \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$ by

$$\varphi(u_1) = 7; \varphi(u_i) = 9i - 3, 2 \leq i \leq n;$$

$$\varphi(v_1) = 1; \varphi(v_2) = 4; \varphi(v_i) = 9i - 6, 3 \leq i \leq n;$$
\[ \varphi(w_1) = 3; \varphi(w_i) = 9i, \ 2 \leq i \leq n. \]

Then \( \varphi \) induces a bijective function \( \varphi^*: E(C_n \circ \overline{K}_2) \rightarrow \{1, 4, \ldots, 3q - 2\} \), where

\[ \varphi^*(u_iu_{i+1}) = 9i + 1, \ 1 \leq i \leq n - 1; \varphi^*(u_nu_1) = 13; \]

\[ \varphi^*(u_1v_1) = 1; \varphi^*(u_2v_2) = 7; \varphi^*(u_iv_i) = 9i - 5, \ 3 \leq i \leq n; \]

\[ \varphi^*(u_1w_1) = 4; \varphi^*(u_iw_i) = 9i - 2, \ 2 \leq i \leq n. \]

Therefore, \( \varphi \) is one modulo three harmonic mean labeling of \( C_n \circ \overline{K}_2 \). Hence \( C_n \circ \overline{K}_2 \) is one modulo three harmonic mean graph.

**Example 5.4.4.** One modulo three harmonic mean labeling of \( C_6 \circ \overline{K}_2 \) is shown in figure 5.19.

![Figure 5.19. \( C_6 \circ \overline{K}_2 \)](image-url)
**Theorem 5.4.5.** Let $G$ be a graph obtained by attaching $P_3$ at each vertex of $C_n$. Then $G$ is one modulo three Harmonic mean graph.

**Proof.** Let $u_1u_2\ldots u_n u_1$ be the cycle $C_n$. For $1 \leq i \leq n$, let $x_i v_i w_i$ be the $i^{th}$ copy of $P_3$. Identify the vertex $x_i$ with $u_i$, $1 \leq i \leq n$. The resultant graph is $G$ whose edge set is $E = \{u_i u_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_i v_i, v_i w_i / 1 \leq i \leq n\}$.

**Case 1.** $3 \leq n \leq 5$.

Define a function $\phi: V(G) \to \{1, 3, 4, \ldots, 3q - 2, 3q\}$ by

\[
\phi(u_1) = \begin{cases} 12 & \text{for } n = 3; \\ 10 & \text{for } n \neq 3; \end{cases}
\]

$\phi(u_2) = 16; \phi(u_i) = 9i - 3, 3 \leq i \leq n$;

$\phi(v_1) = 3; \phi(v_2) = 7; \phi(v_i) = 9i - 2, 3 \leq i \leq n$;

$\phi(w_1) = 1; \phi(w_2) = 6; \phi(w_i) = 9(i - 1) + 1, 3 \leq i \leq n$.

Then $\phi$ induces a bijective function $\phi^*: E(G) \to \{1, 4, \ldots, 3q - 2\}$, where

$\phi^*(u_1 u_2) = 13; \phi^*(u_i u_{i+1}) = 9i + 1, 2 \leq i \leq n - 1; \phi^*(u_i u_1) = 16$;

$\phi^*(u_1 v_1) = 4; \phi^*(u_2 v_2) = 10; \phi^*(u_i v_i) = 9i - 2, 3 \leq i \leq n$;

$\phi^*(v_1 w_1) = 1; \phi^*(v_2 w_2) = 7; \phi^*(v_i w_i) = 9i - 5, 3 \leq i \leq n$.

**Case 2.** $n > 5$.

Define a function $\phi: V(G) \to \{1, 3, 4, \ldots, 3q - 2, 3q\}$ by

$\phi(u_1) = 7; \phi(u_i) = 9i - 3, 2 \leq i \leq n$;
\( \varphi(v_1) = 3; \varphi(v_i) = 9i - 2, 2 \leq i \leq n; \)

\( \varphi(w_1) = 1; \varphi(w_2) = 4; \varphi(w_i) = 9i - 8, 3 \leq i \leq n. \)

Then \( \varphi \) induces a bijective function \( \varphi^*: E(G) \to \{1, 4, \ldots, 3q - 2\}, \) where

\( \varphi^*(u_iu_{i+1}) = 9i + 1, 1 \leq i \leq n - 1; \varphi^*(u_1u_i) = 13; \)

\( \varphi^*(u_1v_1) = 4; \varphi^*(u_iv_i) = 9i - 2, 2 \leq i \leq n; \)

\( \varphi^*(v_1w_1) = 1; \varphi^*(v_2w_2) = 7; \varphi^*(v_iw_i) = 9i - 5, 3 \leq i \leq n. \)

Therefore, \( \varphi \) is one modulo three harmonic mean labeling of \( G. \) Hence \( G \) is one modulo three harmonic mean graph.

**Example 5.4.6.** One modulo three harmonic mean labeling of \( G \) when \( n = 6 \) is shown in figure 5.20.

![Figure 5.20. G](image-url)
**Theorem 5.4.7.** $C_n \odot \overline{K}_3$ is one modulo three harmonic mean graph.

**Proof.** Let $C_n$ be the cycle $u_1 u_2 \ldots u_n u_1$. Let $x_i$, $y_i$, $z_i$ be the vertices of $i^{th}$ copy of $\overline{K}_3$ which are adjacent to the vertex $u_i$ of $C_n$, $1 \leq i \leq n$. The resultant graph is $C_n \odot \overline{K}_3$ whose edge set is $E = \{u_i u_{i+1}, u_n u_1 / 1 \leq i \leq n - 1\} \cup \{u_i x_i, u_i y_i, u_i z_i / 1 \leq i \leq n\}$.

**Case 1.** $n = 3$.

One modulo three harmonic mean labeling of $C_3 \odot \overline{K}_3$ is shown in figure 5.21.

![Figure 5.21. C_3 \odot \overline{K}_3](image)

**Case 2.** $n \geq 4$.

Define a function $\varphi$: $V(C_n \odot \overline{K}_3) \rightarrow \{1, 3, 4, 6, ..., 3q - 2, 3q\}$ by

$\varphi(u_1) = 7; \varphi(u_2) = 21; \varphi(u_i) = 12i - 5, 3 \leq i \leq n;$

$\varphi(x_i) = 12i - 11, 1 \leq i \leq n;$

$\varphi(y_1) = 3; \varphi(y_i) = 12i - 6, 2 \leq i \leq n;$

$\varphi(z_1) = 6; \varphi(z_i) = 12i, 2 \leq i \leq n.$
Then \( \varphi \) induces a bijective function \( \varphi^*: E(C_n \odot \overline{K}_3) \rightarrow \{1, 4, \ldots, 3q - 2\} \), where

\[
\varphi^*(u_1u_2) = 10; \varphi^*(u_iu_{i+1}) = 12i + 1, \ 2 \leq i \leq n - 1; \ \varphi^*(u_1u_n) = 13;
\]

\[
\varphi^*(u_1x_1) = 1; \ \varphi^*(u_ix_i) = 12i - 8, \ 2 \leq i \leq n;
\]

\[
\varphi^*(u_1y_1) = 4; \ \varphi^*(u_iy_i) = 12i - 5, \ 2 \leq i \leq n;
\]

\[
\varphi^*(u_1z_1) = 7; \ \varphi^*(u_iz_i) = 12i - 2, \ 2 \leq i \leq n.
\]

In the view of the above labeling pattern, \( \varphi \) provides one modulo three harmonic mean labeling for \( C_n \odot \overline{K}_3 \). Hence \( C_n \odot \overline{K}_3 \) is one modulo three harmonic mean graph.

**Example 5.4.8.** One modulo three harmonic mean labeling of \( C_5 \odot \overline{K}_3 \) is given in figure 5.22.
**Theorem 5.4.9.** A graph obtained by attaching $K_{1, 3}$ at each vertex of a cycle $C_n$ is one modulo three harmonic mean graph.

**Proof.** Let $C_n$ be the cycle $u_1u_2...u_nu_1$. Let $v_i$, $x_i$, $y_i$, $z_i$ be the vertices of $i^{th}$ copy of $K_{1, 3}$ in which $v_i$ is the central vertex. Identify $z_i$ with $u_i$, $1 \leq i \leq n$. Let the resultant graph be $G$.

**Case 1.** $n = 3$.

One modulo three harmonic mean labeling of $G$ is shown in figure 5.23

![Figure 5.23. G](image)

**Case 2.** $n \geq 4$.

Define a function $\phi: V(G) \to \{1, 3, 4, 6, \ldots, 3q - 2, 3q\}$ by

$\phi(u_1) = 7; \phi(u_2) = 21; \phi(u_i) = 12i - 5$, $3 \leq i \leq n$;

$\phi(v_1) = 6; \phi(v_2) = 24; \phi(v_3) = 37; \phi(v_i) = 12i$, $4 \leq i \leq n$;

$\phi(x_1) = 1; \phi(x_2) = 12; \phi(x_3) = 22; \phi(x_i) = 12(i - 1) - 3$, $4 \leq i \leq n$;
\( \varphi(y_1) = 4; \varphi(y_2) = 16; \varphi(y_i) = 12(i - 1) + 3, \ 3 \leq i \leq n. \)

Then \( \varphi \) induces a bijective function \( \varphi^* : E(G) \rightarrow \{1, 4, \ldots, 3q - 2\} \), where

\[ \varphi^*(u_1u_2) = 10; \varphi^*(u_iu_{i+1}) = 12i + 1, \ 2 \leq i \leq n - 1; \varphi^*(u_nu_1) = 13; \]

\[ \varphi^*(u_1v_1) = 7; \varphi^*(u_2v_2) = 22; \]

\[ \varphi^*(u_iu_j) = 12i - 2, \ 3 \leq i \leq n; \]

\[ \varphi^*(v_1x_1) = 1; \varphi^*(v_ix_i) = 12(i - 1) + 4, \ 2 \leq i \leq n; \]

\[ \varphi^*(v_1y_1) = 4; \varphi^*(v_2y_2) = 19; \varphi^*(v_iy_i) = 12i - 5, \ 3 \leq i \leq n. \]

Figure 5.24. G
Thus $\phi$ provides one modulo three harmonic mean labeling for $G$. Hence $G$ is one modulo three harmonic mean graph.

**Example 5.4.10.** One modulo three harmonic mean labeling of $G$ when $n = 7$ is given in figure 5.24.

**Theorem 5.4.11.** $L_n \circ K_1$ is one modulo three harmonic mean graph.

**Proof.** Let $u_1u_2...u_n, v_1v_2...v_n$ be two paths of equal length. Join $u_i$ and $v_i, 1 \leq i \leq n$. The resultant graph is $L_n$. Add two new vertices $x_i$ and $y_i$ and join them with $u_i$ and $v_i$, respectively, $1 \leq i \leq n$. The resultant graph is $L_n \circ K_1$. Define a function $\phi : V(L_n \circ K_1) \rightarrow \{1, 3, 4, 6, ..., 3q - 2, 3q\}$ by

\[
\phi(u_1) = 7; \quad \phi(u_2) = 21; \quad \phi(u_i) = 15(i - 1) + 3, 3 \leq i \leq n;
\]

\[
\phi(x_1) = 1; \quad \phi(x_i) = 15(i - 1) - 2, 2 \leq i \leq n;
\]

\[
\phi(v_1) = 9; \quad \phi(v_2) = 22; \quad \phi(v_i) = 15(i - 1) + 6, 3 \leq i \leq n;
\]

\[
\phi(y_1) = 3; \quad \phi(y_2) = 16; \quad \phi(y_i) = 15(i - 1) + 9, 3 \leq i \leq n.
\]

Then $\phi$ induces a bijective function $\phi^* : E(L_n \circ K_1) \rightarrow \{1, 4, 7, ..., 3q - 2\}$, where

\[
\phi^*(u_iu_{i+1}) = 15i - 5, 1 \leq i \leq n - 1;
\]

\[
\phi^*(u_ix_i) = 15i - 14, 1 \leq i \leq n;
\]

\[
\phi^*(u_iv_i) = 15i - 8, 1 \leq i \leq 2;
\]

\[
\phi^*(u_{i+1}v_{i+1}) = 15i - 11, 3 \leq i \leq n;
\]

\[
\phi^*(v_{i+1}v_{i+1}) = 15i - 2, 1 \leq i \leq n - 1;
\]
\[ \varphi^*(v_i y_i) = 15i - 11, \quad 1 \leq i \leq 2; \]

\[ \varphi^*(v_i y_i) = 15i - 8, \quad 3 \leq i \leq n. \]

In the view of the above labeling pattern, \( \varphi \) provides one modulo three harmonic mean labeling for \( L_n \Theta K_1 \). Hence \( L_n \Theta K_1 \) is one modulo three harmonic mean graph.

**Example 5.4.12.** One modulo three harmonic mean labeling for \( L_8 \Theta K_1 \) is given in figure 5.25.

![Figure 5.25. \( L_8 \Theta K_1 \)](image)

In this chapter, we used number theoretical approach to label the edges. In this labeling techniques edges get labels from 1, 4, \( \ldots, \) 3q – 2. Therefore it is very interesting to investigate graphs which have one modulo three harmonic mean labeling. In the next chapter, we introduce a new concept Super harmonic mean labeling of graphs.