CHAPTER-4

EDGE SUM NUMBER OF SOME SPECIAL GRAPHS

In this chapter we find the edge sum number of some special graphs such as
\[ (P_2 \times C_{2n+1}), (J_{2,n}), (J_{3,3}), (J_{3,4}) \text{ and } s_n^0. \]

In 1990, Harary [9] introduced the notion of a sum graph. A graph \( G(V, E) \) is said to be a sum graph if there is a bijective labeling from \( V \) to a set of positive integers \( S \) such that \( xy \in E \) if and only if \( f(x) + f(y) \in S \). The edge sum graph, the edge analogue of sum graph was defined by D.S.T. Ramesh and et.al and studied its properties [12, 16, 17, and 18].

4.1 Definition: Let \( G(V, E) \) be a graph. A bijection \( f: E \rightarrow S \) where \( S \) is a set of positive integers is called an edge function of the graph \( G \). Define \( F(v) = \sum \{f(e): e \text{ is incident on } v\} \) on \( V \). Then \( F \) is called the edge sum function of the edge function \( f \).

\( G \) is said to be an edge sum graph if there exists an edge function \( f: E \rightarrow S \) such that \( f \) and its corresponding edge sum function \( F \) on \( V \) satisfy the following conditions:

1. \( F \) is into \( S \). That is, \( F(v) \in S \) for every \( v \in V \).
2. If \( e_1, e_2, \ldots, e_n \in E \) such that \( f(e_1) + f(e_2) + \ldots + f(e_n) \in S \), then \( e_1, e_2, \ldots, e_n \) are incident on a vertex.

**4.1 Example:** Let \( V = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\} \) be the vertex set and \( E = \{u_1u_2, u_2u_3, u_3u_4, u_2u_5, u_3u_6, u_7u_8\} \) be the edge set of the graph \( G \).

The edge function \( f: E \rightarrow S \) is defined by \( f(u_1u_2) = 6, f(u_2u_3) = 3, f(u_3u_4) = 4, f(u_2u_5) = 5, f(u_3u_6) = 7 \) and \( f(u_7u_8) = 14 \). The corresponding edge sum function \( F \) is given by \( F(u_1) = 6, F(u_2) = 14, F(u_3) = 14, F(u_4) = 4, F(u_5) = 5, F(u_6) = 7, F(u_7) = 14 \) and \( F(u_8) = 14 \). Clearly \( G \) is an edge sum graph.

![Figure 4.1](image)

We give some of the results which have already been proved.

**4.1 Theorem:** Let \( G(V, E) \) be an edge sum graph. Then \( K_2 \) is a component of \( G \).

**4.1 Remark:** The graph \( K_2 \) is the only connected edge sum graph.
4.2 Theorem: Let $G(V, E)$ be an edge sum graph with edge function $f: E \rightarrow S$ and edge sum function $F$. Let $e_1, e_2, \ldots, e_m$ where $m > 1$ be a collection of edges incident on a vertex $u \in V$. Let $\ell_1, \ell_2, \ldots, \ell_n$ be another collection of edges none of them incident on $u$ such that $f(e_1) + f(e_2) + \ldots + f(e_m) = f(\ell_1) + f(\ell_2) + \ldots + f(\ell_n)$. Then all of $\ell_1, \ell_2, \ldots, \ell_n$ are incident on a vertex (say) $v$ (with $v \neq u$) such that $(\deg u, \deg v) \in \{m, (m+1)\} \times \{n, (n+1)\}$ and one of the following statements holds:

1. $u$ and $v$ are adjacent and $(\deg u, \deg v) \neq (m, n)$.

2. $u$ and $v$ are not adjacent and $(\deg u, \deg v) = (m, n)$. ■

4.3 Theorem: Let $G(V, E)$ be an edge sum graph with edge function $f: E \rightarrow S$ and edge sum function $F$ of $f$. Let $w$ be a nonpendent vertex and $e = uv \in E$ be such that $F(w) = f(e)$. Then one of the following holds:

1. $\{u, v\}$ forms a $K_2$ component in $G$.

2. $\langle\{u, v, w\}\rangle$ is either $K_3$ or $P_2$ with one of $u, v$ as a pendant vertex in $G$. ■

4.4 Theorem: Let $G(V, E)$ be an edge sum graph with edge function $f: E \rightarrow S$ and edge sum function $F$ of $f$. Let $\ell_1, \ell_2, \ldots, \ell_m$ where $m > 1$ be a collection of edges incident on a vertex
w (say). Let \( w_i = \ell_i \) for \( 1 \leq i \leq m \). If there exists an edge \( e = uv \) such that 
\[
 f(\ell_1) + f(\ell_2) + \ldots + f(\ell_m) = f(e),
\]
then one of the following holds:

1. \( \{u, v\} \) forms a \( K_2 \) component in \( G \).

2. \( <\{u, v, w\}> \) is \( K_3 \) or \( P_2 \) or \( P_1 \) with one of \( u, v \) as a pendant vertex in \( G \).

**4.2 Definition:** The smallest number \( r \) so that \( G \cup rK_2 \) becomes an edge sum graph is called the **edge sum number** of the graph \( G \) and is denoted by \( \sigma_E(G) \).

For any connected graph \( G \) other than \( K_2 \), \( \sigma_E(G) \geq 1 \).

**Example:** Consider the graph \( K_3 \). The edge function given in Figure 4.2 shows that \( \sigma_E(K_3) = 2 \).

![Figure 4.2](image)

**4.3 Definition:** Let \( \sigma_E(G) = r \). An edge function \( f: E \rightarrow S \) and its corresponding edge sum function \( F \) which make \( G \cup rK_2 \) an edge sum graph are respectively called an **optimal edge function** and an **optimal edge sum function** of \( G \).
For a graph $G$ with $\sigma_E(G) = r$, there can be many optimal edge functions. Let $E_1$ be the edge set of $G$ and $E_2$ be that of $rK_2$. We denote $F(V) = \{F(v) : v \in V\}$ and $f(E_i) = \{f(e) : e \in E_i\}$ for $1 \leq i \leq 2$. Then, $\sigma_E(G) = \text{Cardinality of the set } \{F(v) : v \in V; F(v) \notin f(E_1)\}$. $F$ is said to be an **outer edge sum function** if $F(V) \cap f(E_1) = \emptyset$ and an **inner edge sum function** if $F(V) \cap f(E_1) \neq \emptyset$. The range of $F$ has at least $r$ elements. It has exactly $r$ elements if and only if $F$ is an outer edge sum function.

**4.5 Theorem:** Let $f : E \to S$ be an optimal edge function. If $G$ has no pendent vertex and is triangle free, then $F$ is an outer edge sum function.

**Proof:** Let $E_1$ be the edge set of $G$ and $E_2$ that of $rK_2$. Let $u \in V$.

Since $F(u) \in S$, there is an edge $vw$ such that $F(u) = f(vw)$. If $vw \in E_1$, then $\langle \{u, v, w\}\rangle$ is $K_3$ or $P_2$ or $P_1$ with $v$ or $w$ as a pendent vertex which is a contradiction. Hence $vw \in E_2$ so that $F$ is an outer edge sum function.

**4.2 Remark:** It is easily seen that every optimal edge sum function $F$ of a graph $G$ is inner if $G$ has a pendent vertex and is outer if $G$ contains no pendent vertex and triangle free. If $G$ has no pendent
vertex but contains a triangle then F can be either inner (See Figure 4.2(a)) or outer (See Figure 4.2(b)).

Let $E = E_1 \cup E_2$ where $E_1$ is the edge set of $G$ and $E_2$ that of $rK_2$. Then, $\sigma_E(G) = \text{Cardinality of the set } \{F(v) : v \in V; F(v) \notin f(E_1)\}$. F is said to be **outer edge sum function** if $F(V) \cap f(E_1) = \phi$. If $F(V) \cap f(E_1) \neq \phi$, then F is said to be an **inner edge sum function**. The range of F has atleast $r$ elements. It has exactly $r$ elements if and only if F is outer edge sum function.
4.6 Theorem: \( \sigma_E(P_2 \times C_{2n+1}) = 2 \).

**Proof:** Let \( G = P_2 \times C_{2n+1} \) where \( V(G) = \{u_i, v_i : 1 \leq i \leq 2n + 1\} \) and 
\[ E(G) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i : 1 \leq i \leq 2n\} \cup \{v_{2n+1}v_1, u_{2n+1}u_1, u_{2n+1}v_{2n+1}\} \.

Suppose \( \sigma_E(G) = 1 \).

Let \( f \) be an edge function and \( F \) be the corresponding edge sum function that makes \( G \cup K_2 \) is an edge sum graph. Let \( w_1w_2 \) be the \( K_2 \) component of \( G \cup K_2 \) is an edge sum graph. Since \( G \) is triangle free and has no pendent vertex, \( F \) is an outer edge sum function.

That is, \( F(u) = f(w_1w_2) \) for all \( u \in V \)

That is, let \( f(u_iu_{i+1}) = a_i \) for \( 1 \leq i \leq 2n \)
\[ f(v_iv_{i+1}) = b_i \] for \( 1 \leq i \leq 2n \)
\[ f(u_{2n+1}u_1) = a_{2n+1} \]
\[ f(v_{2n+1}v_1) = b_{2n+1} \]
\[ f(u_iv_i) = z_i \] for \( 1 \leq i \leq 2n + 1 \)
\[ f(w_1w_2) = z \]

Now \( F(u_1) = f(u_{2n+1}u_1) + f(u_1u_2) + f(u_1v_1) \)
\[ = a_{2n+1} + a_1 + z_1 = z \]
\( F(v_1) = f(v_{2n+1}v_1) + f(v_1v_2) + f(u_1v_1) \)
\[ = b_{2n+1} + b_1 + z_1 = z \]
Therefore, $a_{2n+1} + a_1 = b_{2n+1} + b_1$

Similarly, $a_1 + a_2 = b_1 + b_2$

\[
a_2 + a_3 = b_2 + b_3
\]

\[
a_3 + a_4 = b_3 + b_4
\]

Proceeding like this we get

\[
a_{2n} + a_{2n+1} = b_{2n} + b_{2n+1}
\]

\[
a_{2n+1} + a_1 = b_{2n+1} + b_1
\]

Suppose $a_1 < b_1 \Rightarrow a_2 > b_2$

\[
\Rightarrow a_3 < b_3
\]

Proceeding like this we get,

$a_i < b_i$ if $i$ is odd

$a_i > b_i$ if $i$ is even

Therefore, $a_{2n} > b_{2n} \Rightarrow a_{2n+1} < b_{2n+1}$

\[
\Rightarrow a_1 > b_1
\]

This is a contradiction. Therefore $\sigma_E(G) > 1$.

We prove $\sigma_E(G) = 2$ by taking the graph $G \cup 2K_2$. Let

$x = 2^{2n+7}$, $y = 2^{2n+4}$, $z = 2^{2n+5}$. Let $S = \{x + 2^{2n+2-i} \mid 1 \leq i \leq 2n+1\}$

\[
\cup \left\{ y + (2n+1)x + (2^{2n+2} - 2) + \sum_{j=1}^{i-1} 2^{2n+1-2j} \text{ for } 0 \leq i \leq n - 1 \right\}
\]

\[
\cup \left\{ y + (2n+1)x + (2^{2n+2} - 2) + \sum_{j=1}^{n} 2^{2n+1-2j} + \sum_{j=1}^{i} 2^{2n+2-2j} \text{ for } 1 \leq i \leq n \right\}
\]
\[
\begin{align*}
\bigcup \left\{ y + (2n + 1)x + (2^{2n+2} - 2) + \sum_{j=1}^{n} 2^{2n+1-2j} \right\} \\
\bigcup \left\{ z + (2n + 1)x + (2^{2n+2} - 2) + \sum_{j=1}^{i} 2^{2n+1-2j} \text{ for } 0 \leq i \leq n - 1 \right\} \\
\bigcup \left\{ z + (2n + 1)x + (2^{2n+2} - 2) + \sum_{j=1}^{n} 2^{2n+1-2j} + \sum_{j=1}^{i} 2^{2n+2-2j} \text{ for } 1 \leq i \leq n \right\} \\
\bigcup \left\{ z + (2n + 1)x + (2^{2n+2} - 2) + \sum_{j=1}^{n} 2^{2n+1-2j} \right\}.
\end{align*}
\]

Consider the graph \( G \cup 2K_2 \) where \( V(G) = \{u_i, v_i : 1 \leq i \leq 2n + 1\} \)

\( \cup \{w_i \text{ for } 1 \leq i \leq 4\} \) and \( E(G) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i : 1 \leq i \leq 2n\} \)

\( \cup \{v_{2n+1}v_1, u_{2n+1}u_1, u_{2n+1}v_{2n+1}\} \cup \{w_1w_2, w_3w_4\} \).

We define \( f(u_i, v_i) = x + 2^{2n+2-i} \text{ for } 1 \leq i \leq 2n + 1 \)

\[
f(u_{2i+1}u_{2i+2}) = 2^{2n+5} + (2n + 1)x + 2^{2n+2} - 2 + \sum_{j=1}^{i} 2^{2n+1-2j} \text{ for } 0 \leq i \leq n - 1
\]

\[
f(u_{2i}u_{2i+1}) = 2^{2n+5} + (2n + 1)x + 2^{2n+2} - 2 + \sum_{j=1}^{n} 2^{2n+1-2j} + \sum_{j=1}^{i} 2^{2n+2-2j} \text{ for } 1 \leq i \leq n
\]

\[
f(u_{2n+1}u_1) = 2^{2n+5} + (2n + 1)x + 2^{2n+2} - 2 + \sum_{j=1}^{n} 2^{2n+1-2j}
\]

\[
f(v_{2i+1}v_{2i+2}) = 2^{2n+4} + (2n + 1)x + 2^{2n+2} - 2
\]
+\sum_{j=1}^{i} 2^{2n+1-2j} \text{ for } 0 \leq i \leq n - 1

f(v_{2i},v_{2i+1}) = 2^{2n+4} + (2n+1)x + 2^{2n+2} - 2 + \sum_{j=1}^{n} 2^{2n+1-2j}

+\sum_{j=1}^{i} 2^{2n+2-2j} \text{ for } 1 \leq i \leq n

f(v_{2n+1},v_1) = 2^{2n+4} + (2n+1)x + 2^{2n+2} - 2 + \sum_{j=1}^{n} 2^{2n+1-2j}

F(u_i) = f(u_{i-1}u_i) + f(u_{i+1}u_i) + f(u_{i}v_{i}) \text{ for } 2 \leq i \leq 2n

F(u_1) = f(u_{2n+1}u_1) + f(u_{1}u_2) + f(u_{1}v_{1})

F(u_{2n+1}) = f(u_{2n}u_{2n+1}) + f(u_{2n}u_1) + f(u_{2n+1}v_{2n+1})

F(v_i) = f(v_{i-1}v_i) + f(v_{i+1}v_i) + f(u_{i}v_{i}) \text{ for } 2 \leq i \leq 2n

F(v_1) = f(v_{2n+1}v_1) + f(v_1v_2) + f(u_2v_{1})

F(v_{2n+1}) = f(v_{2n}v_{2n+1}) + f(v_{2n}v_1) + f(u_{2n+1}v_{2n+1})

Hence we get,

F(u_i) = (4n+3)x + 2^{2n+6} + 2^{2n+4} + 2^{2n+3} + 2^{2n-1} + 2^{2n-3} + 2^{n-5} - 4

\text{ for } 1 \leq i \leq 2n + 1.

F(v_i) = (4n+3)x + 2^{2n+5} + 2^{2n+3} + 2^{2n+1} + 2^{2n-1} + 2^{2n-3} + 2^{n-5} - 4

\text{ for } 1 \leq i \leq 2n + 1.

We have proved that \( F(v) \in \{ f(w_1w_2), f(w_2w_3) \} \) for all \( v \in G \).

Also we have given labels to the edges in such a way that none of
the other sums of the labels except those incidents on a single vertex is a label of an edge.

Hence $\sigma_E(P_2 \times C_{2n+1}) = 2$.

The edge function given in Figure 4.6 shows that $\sigma_E(P_2 \times C_7) = 2$.

\[ \begin{align*}
\text{where } & x = 2^{31}; y = 2^{10}; z = 2^{31}; A = 7x + 2^8 - 2; B = 2^5 + 2^3 + 2^1; \\
& C = 15x + 2^0 + 2^7 + 2^5 + 2^1 + 2^1 - 4.
\end{align*} \]
4.7 Theorem: $\sigma_E(J_{2,n}) = 2$.

**Proof:** Let $G = J_{2,n}$ be a graph where $V(G) = \{v\}$
$\cup\{v_{i,j} : 1 \leq i \leq n; 1 \leq j \leq 2\}$ and $E(G) = \{v_{i,j}, v_{i,j+1} : 1 \leq i \leq n; 1 \leq j \leq 2\}$
$\cup\{v_{i,2}, v_{i+1,1} : 1 \leq i \leq n - 1\}$ $\cup\{v_{n,2}, v_{1,1}\}$ $\cup\{vv_i : 1 \leq i \leq n\}$.

To prove $\sigma_E(J_{2,n}) > 1$

Suppose $\sigma_E(J_{2,n}) = 1$.

Then there exists an optimal edge function $f$ and its corresponding edge sum function $F$ such that $G \cup K_2$ is an edge sum graph. Let $w_1w_2$ be the $K_2$ component of $G \cup K_2$. Since $G$ is triangle free and has no pendent vertex, $F$ is an outer edge sum function.

That is, $F(u) = f(w_1w_2) = a$ (say) for all $u \in V$

$F(v_{1,1}) = f(v_{n,2}v_{1,1}) + f(vv_{1,1}) + f(v_{1,1}v_{1,2}) = a$ ............ (1)

$F(v_{1,2}) = f(v_{1,1}v_{1,2}) + f(v_{1,2}v_{2,1}) = a$ ............ (2)

$F(v_{2,1}) = f(v_{12}v_{21}) + f(vv_{21}) + f(v_{21}v_{22}) = a$ ............ (3)

$F(v_{2,2}) = f(v_{2,1}v_{2,2}) + f(v_{2,2}v_{3,1}) = a$ ............ (4)

$F(v_{3,1}) = f(v_{2,2}v_{3,1}) + f(vv_{3,1}) + f(v_{3,1}v_{3,2}) = a$ ............ (5)

$F(v_{3,2}) = f(v_{3,1}v_{3,2}) + f(vv_{3,2}) = a$ ............ (6)

Proceeding like this we get,
\[ F(v_{n,1}) = f(v_{n-1,2}v_{n,1}) + f(vv_{n,1}) + f(v_{n,1}v_{n,2}) = a \] \hspace{1cm} (7)

\[ F(v_{n,2}) = f(v_{n,1}v_{n,2}) + f(v_{n,2}v_{1,1}) = a \] \hspace{1cm} (8)

From equations (1) & (2)

\[ f(v_{n,2}v_{1,1}) + f(vv_{1,1}) + f(v_{1,1}v_{1,2}) = f(v_{1,1}v_{1,2}) + f(v_{1,2}v_{2,1}) \]

\[ \Rightarrow f(v_{n,2}v_{1,1}) + f(vv_{1,1}) = f(v_{1,2}v_{2,1}) \]

\[ \Rightarrow f(v_{n,2}v_{1,1}) < f(v_{1,2}v_{2,1}) \] \hspace{1cm} (9)

From equations (3) & (4)

\[ f(v_{1,2}v_{2,1}) + f(vv_{2,1}) + f(v_{2,1}v_{2,2}) = f(v_{2,1}v_{2,2}) + f(v_{2,2}v_{3,1}) \]

\[ \Rightarrow f(v_{1,2}v_{2,1}) + f(vv_{2,1}) = f(v_{2,2}v_{3,1}) \]

\[ \Rightarrow f(v_{1,2}v_{2,1}) < f(v_{2,2}v_{3,1}) \] \hspace{1cm} (10)

From equations (9) & (10)

\[ f(v_{n,2}v_{1,1}) < f(v_{2,2}v_{3,1}) \] \hspace{1cm} (11)

From equations (5) & (6)

\[ f(v_{2,2}v_{3,1}) + f(vv_{3,1}) + f(v_{3,1}v_{3,2}) = f(v_{3,1}v_{3,2}) + f(v_{3,2}v_{4,1}) \]

\[ \Rightarrow f(v_{2,2}v_{3,1}) + f(vv_{3,1}) = f(v_{3,2}v_{4,1}) \]

\[ \Rightarrow f(v_{2,2}v_{3,1}) < f(v_{3,2}v_{4,1}) \] \hspace{1cm} (12)

From equations (11) & (12)

\[ f(v_{n,2}v_{1,1}) < f(v_{3,2}v_{4,1}) \] \hspace{1cm} (13)
Proceeding like this we get,
\[ f(v_{n,2}v_{1,1}) < f(v_{4,2}v_{5,1}) \]
\[ f(v_{n,2}v_{1,1}) < f(v_{5,2}v_{6,1}) \]

Continuing like this we get,
\[ f(v_{n,2}v_{1,1}) < f(v_{n-1,2}v_{n,1}) \] \[\ldots \ldots \ (I)\]

From equations (7) & (8)
\[ f(v_{n-1,2}v_{n,1}) + f(vv_{n,1}) + f(v_{n,1}v_{n,2}) = f(v_{n,1}v_{n,2}) + f(v_{n,2}v_{1,1}) \]
\[ f(v_{n-1,2}v_{n,1}) + f(vv_{n,1}) = f(v_{n,2}v_{1,1}) \]
\[ \Rightarrow f(v_{n,2}v_{1,1}) > f(v_{n-1,2}v_{n,1}) \] \[\ldots \ldots \ (II)\]

This is a contradiction to (I). Hence \( \sigma_E(J_{2,n}) > 1 \).

Now we prove that \( \sigma_E(J_{2,n}) = 2 \) by taking the graph \( G \cup 2K_2 \).

Consider the graph \( G \cup 2K_2 \) where \( V(G) = \{v\} \)
\[ \cup \{v_{i,j} : 1 \leq i \leq n; 1 \leq j \leq 2\} \cup \{w_i \text{ for } 1 \leq i \leq 4\} \] and \( E(G) = \)
\[ \{v_{i,j}, v_{i,j+1} : 1 \leq i \leq n; 1 \leq j \leq 2\} \cup \{v_{i,2}v_{i+1,1}, v_{i+1,1} : 1 \leq i \leq n-1\} \cup \{v_{n,2}v_{1,1}\} \]
\[ \cup \{vv_{i,1} : 1 \leq i \leq n\} \cup \{w_1w_2, w_3w_4\}. \]

Let \( x = 2^{4n} \). Let \( S = \left\{ \frac{n}{2}x - 2^{3n+i-1} : 1 \leq i \leq n \right\} \)
\[ \cup \left\{ \frac{n}{2}x + 2^{3n+i-1} : 1 \leq i \leq n-1 \right\} \cup \left\{ \frac{n}{2}x + 2^{3n+n-1} \right\} \]
\[ \cup \{ x + 2^{3n} - 2^{3n+n-1} \} \cup \{ x + 2^{3n+i-1} - 2^{3n+i-2} : 2 \leq i \leq n \}. \]

We define
\[ f(v_{i,1}v_{i,2}) = \frac{n}{2} x - 2^{3n+i-1} \text{ for } 1 \leq i \leq n \]
\[ f(v_{i,2}v_{i+1,1}) = \frac{n}{2} x + 2^{3n+i-1} \text{ for } 1 \leq i \leq n - 1 \]
\[ f(v_{n,2}v_{1,1}) = \frac{n}{2} x + 2^{3n+n-1} \]
\[ f(vv_{1,1}) = x + 2^{3n} - 2^{3n+n-1} \]
\[ f(vv_{i,1}) = x + 2^{3n+i-1} - 2^{3n+i-2} \text{ for } 2 \leq i \leq n \]

The corresponding edge sum functions are
\[ F(v_{i,1}) = f(v_{i-1,2}v_{i,1}) + f(v_{i,1}v_{i,2}) + f(vv_{i,1}) \text{ for } 2 \leq i \leq n \]
\[ F(v_{1,1}) = f(v_{n,2}v_{1,1}) + f(v_{1,1}v_{1,2}) + f(vv_{1,1}) \]
\[ F(v_{i,2}) = f(v_{i,1}v_{i,2}) + f(v_{i,2}v_{i+1,1}) \text{ for } 1 \leq i \leq n - 1 \]
\[ F(v_{n,2}) = f(v_{n,1}v_{n,2}) + f(v_{n,2}v_{1,1}) \]
\[ F(v) = \sum_{i=1}^{n} f(vv_{i,1}) \]

Now, \( F(v_{1,1}) = F(v_{2,1}) = \ldots = F(v_{n,1}) = F(v) = (n+1)x \) and
\[ F(v_{1,2}) = F(v_{2,2}) = \ldots = F(v_{n,2}) = nx. \]

We have proved that \( F(v) \in \{ f(w_1w_2), f(w_3w_4) \} \) for all \( v \in G \). Also, we have given labels to the edges in such a way that
none of the other sums of the labels except those incidents on a single vertex is a label of an edge. Hence $\sigma_E(J_{2,n}) = 2$.

The edge function given in Figure 4.7 shows that $\sigma_E(J_{2,7}) = 2$.

4.8 Theorem: $\sigma_E(J_{3,3}) = 3$.

Proof: Let $G = J_{3,3}$ where $V(G) = \{v\} \cup \{v_{i,j}: 1 \leq i \leq 3; 1 \leq j \leq 3\}$ and $E(G) = \{v_{i,j}v_{i,j+1}: 1 \leq i \leq 3; 1 \leq j \leq 2\} \cup \{v_{i,2}v_{i+1,1}: 1 \leq i \leq 2\}$

$\cup \{v_{3,3}v_{i,1}\} \cup \{vv_{i,1}: 1 \leq i \leq 3\}$. 

where $x = 2^{28}$. 

Figure 4.7
First let us prove that $\sigma_E(J_{3,3}) > 1$.

Suppose $\sigma_E(J_{3,3}) = 1$.

Then there exists an optimal edge function $f$ and its corresponding edge sum function $F$ such that $G \cup K_2$ is an edge sum graph. Let $w_1w_2$ be the $K_2$ component of $G \cup K_2$. Since $G$ is triangle free and has no pendent vertex, $F$ is an outer edge sum function.

That is, $F(u) = f(w_1w_2) = a$ (say) for all $u \in V$

$F(v_{1,2}) = f(v_{1,1}v_{1,2}) + f(v_{1,2}v_{1,3}) = a$

$F(v_{1,3}) = f(v_{1,2}v_{1,3}) + f(v_{1,3}v_{2,1}) = a$

That is, $f(v_{1,1}v_{1,3}) = f(v_{1,3}v_{2,1})$

This is not possible as $f$ is a bijection. Hence $\sigma_E(J_{3,3}) > 1$.

Suppose $\sigma_E(J_{3,3}) = 2$.

Then there exists an optimal edge function $f$ and an optimal edge sum function $F$ such that $G \cup 2K_2$ is an edge sum graph. Let $w_1w_2$ and $w_3w_4$ be the edges of the $K_2$ component of $G \cup 2K_2$.

Let $f(w_1w_2) = a$ and $f(w_3w_4) = b$.

Suppose $F(v_{1,1}) = a$.

That is, $f(vv_{1,1}) + f(v_{3,3}v_{1,1}) + f(v_{1,1}v_{1,2}) = a$
This cannot be equal to \( F(v_{1,2}) = f(v_{1,1}v_{1,2}) + f(v_{1,2}v_{1,3}) \) as \( f(vv_{1,1}) + f(v_{3,3}v_{1,1}) + f(v_{1,1}v_{1,2}) = f(v_{1,1}v_{1,2}) + f(v_{1,2}v_{1,3}) \)

\[ \Rightarrow f(vv_{1,1}) + f(v_{3,3}v_{1,1}) = f(v_{1,2}v_{1,3}) \ldots \ldots \text{(1)} \]

\( F(v_{1,3}) = f(v_{1,2}v_{1,3}) + f(v_{1,3}v_{2,1}) \)

\[ = f(vv_{1,1}) + f(v_{3,3}v_{1,1}) + f(v_{1,3}v_{2,1}) \in S. \]  This is not possible as \( vv_{1,1}, v_{3,3}v_{1,1}, v_{1,3}v_{2,1} \) are not incident on a vertex.

Therefore, \( F(v_{1,1}) \neq F(v_{1,2}). \)

Similarly, \( F(v_{1,1}) \neq F(v_{3,3}). \)

Let \( F(v_{1,2}) = b. \)

Since we have assumed \( \sigma_E(J_{3,3}) = 2 \), we have \( F(v) \in \{a, b\}. \)

Therefore, as \( F(v_{1,2}) \neq F(v_{1,3}), F(v_{1,3}) = a \)

Proceeding like this we get

\( F(v_{2,1}) \neq F(v_{1,3}), F(v_{2,1}) = b \)

\( F(v_{2,2}) \neq F(v_{2,1}), F(v_{2,2}) = a \)

\( F(v_{2,3}) \neq F(v_{2,2}), F(v_{2,3}) = b \)

\( F(v_{3,1}) \neq F(v_{2,3}), F(v_{3,1}) = a \)

\( F(v_{3,2}) \neq F(v_{3,1}), F(v_{3,2}) = b \)
Since $F(v_{1,1}) \neq F(v_{3,3})$, this is a contradiction.

Therefore, $\sigma_E(J_{3,3}) > 2$.

The edge function given in Figure 4.8 shows that $\sigma_E(J_{3,3}) = 3$.

**4.9 Theorem:** $\sigma_E(J_{3,4}) = 3$.

**Proof:** Let $G = J_{3,4}$ where $V(G) = \{v\} \cup \{v_{i,j} : 1 \leq i \leq 4; 1 \leq j \leq 3\}$ and $E(G) = \{v_{i,j}v_{i,j+1} : 1 \leq i \leq 4; 1 \leq j \leq 2\} \cup \{v_{i,3}v_{i+1,1} : 1 \leq i \leq 3\} \cup \{v_{4,3}v_{1,1}\} \cup \{vv_{i,1} : 1 \leq i \leq 4\}$. 

101
First let us prove that $\sigma_E(G) > 1$.

Suppose $\sigma_E(G) = 1$.

Then there exists an optimal edge function $f$ and its corresponding edge sum function $F$ such that $G \cup K_2$ is an edge sum graph. Let $w_1w_2$ be the $K_2$ component of $G \cup K_2$. Since $G$ is triangle free and has no pendent vertex, $F$ is an outer edge sum function.

That is, $F(u) = f(w_1w_2) = a$ (say) for all $u \in V$

\[
F(v_{1,2}) = f(v_{1,1}v_{1,2}) + f(v_{1,2}v_{1,3}) = a
\]

\[
F(v_{1,3}) = f(v_{1,2}v_{1,3}) + f(v_{1,3}v_{2,1}) = a
\]

That is, $f(v_{1,1}v_{1,3}) = f(v_{1,3}v_{2,1})$

This is not possible as $f$ is a bijection. Hence $\sigma_E(G) > 1$.

Suppose $\sigma_E(G) = 2$.

Then there exists an optimal edge function $f$ and an optimal edge sum function $F$ such that $G \cup 2K_2$ is an edge sum graph. Let $w_1w_2$ and $w_3w_4$ be the edges of the $K_2$ component of $G \cup 2K_2$.

Let $f(w_1w_2) = z$ and $f(w_3w_4) = y$ where $z = 2x$.

Let $f(v_{1,1}v_{1,2}) = x - b_1 \Rightarrow f(v_{1,2}v_{1,3}) = x + b_1$

$f(v_{1,2}v_{1,3}) = x + b_1 \Rightarrow f(v_{1,3}v_{2,1}) = y - x - b_1$
\[
f(v_{2,2}v_{2,3}) = x - b_2 \Rightarrow f(v_{2,3}v_{3,1}) = x + b_2
\]
\[
f(v_{2,2}v_{2,3}) = x - b_2 \Rightarrow f(v_{2,1}v_{2,2}) = y - x + b_2
\]
\[
f(v_{3,1}v_{3,2}) = x - b_3 \Rightarrow f(v_{3,2}v_{3,3}) = x + b_3
\]
\[
f(v_{3,2}v_{3,3}) = x + b_3 \Rightarrow f(v_{3,3}v_{4,1}) = y - x - b_3
\]
\[
f(v_{4,2}v_{4,3}) = x - b_4 \Rightarrow f(v_{4,3}v_{1,1}) = x + b_4
\]
\[
f(v_{4,2}v_{4,3}) = x - b_4 \Rightarrow f(v_{4,1}v_{4,2}) = y - x + b_4
\]

Let \( f(v_{1,1}) = x_1 \)

\( f(v_{2,1}) = x_2 \)

\( f(v_{3,1}) = x_3 \)

\( f(v_{4,1}) = x_4 \)

Suppose

\[
F(v_{1,2}) = F(v_{2,1}) = F(v_{2,3}) = F(v_{3,2}) = F(v_{4,1}) = F(v_{4,3}) = 2x \text{ and}
\]
\[
F(v_{1,1}) = F(v_{1,3}) = F(v_{2,2}) = F(v_{3,1}) = F(v_{3,3}) = F(v_{4,2}) = y
\]
\[
F(v_{1,1}) = f(v_{1,1}) + f(v_{4,3}v_{1,1}) + f(v_{1,1}v_{1,2}) = y
\]
\[
\Rightarrow x_1 + 2x - b_1 + b_4 = y
\]
\[
\Rightarrow x_1 = y - 2x + b_1 - b_4
\]
\[
F(v_{2,1}) = f(v_{2,1}) + f(v_{1,3}v_{2,1}) + f(v_{1,1}v_{2,2}) = 2x
\]
\[
\Rightarrow x_2 + 2y - 2x - b_1 + b_2 = 2x
\]
\[ x_2 = 4x - 2y + b_1 - b_2 \]

\[ F(v_{3,1}) = f(vv_{3,1}) + f(v_{2,3}v_{3,1}) + f(v_{3,1}v_{3,2}) = y \]

\[ x_3 + 2x + b_2 - b_3 = y \]

\[ x_3 = y - 2x + b_3 - b_2 \]

\[ F(v_{4,1}) = f(vv_{4,1}) + f(v_{3,3}v_{4,1}) + f(v_{4,1}v_{4,2}) = 2x \]

\[ x_4 + 2y - 2x - b_3 + b_4 = 2x \]

\[ x_4 = 4x - 2y + b_3 - b_4 \]

\[ F(v) = f(vv_{1,1}) + f(vv_{2,1}) + f(vv_{3,1}) + f(vv_{4,1}) \]

\[ = x_1 + x_2 + x_3 + x_4 \]

**Case (i)**

If \( F(v) = y \) where \( y < 2x \)

Let \( 2x - y = a \)

Therefore, \( a = 2x - y > 2x \)

Now \( x_1 = b_1 - b_4 - a \Rightarrow b_1 > a + b_4 \)

\[ b_1 - b_4 > a \]

\( x_3 = b_3 - b_2 - a \Rightarrow b_3 > a + b_2 \)

\[ b_3 - b_2 > a \]

\( x_2 = 2a + b_1 - b_2 \)

\( x_4 = 2a + b_3 - b_4 \)

\( x_2 + x_4 = 4a + b_1 - b_2 + b_3 - b_4 \)

104
\[ = 4a + b_1 - b_4 + b_3 - b_2 \]
\[ > 4a + a + a = 6a \]
\[ x_1 + x_2 + x_3 + x_4 > x_1 + x_3 + 6a \]
\[ > x_1 + x_3 + 12x \text{ (since } a > 2x) \]
\[ > 12x \]

This is a contradiction.

Case (ii)

If \( F(v) = 2x \) where \( 2x < y \)

Let \( y - 2x = b \)

Therefore \( b > y > 2x \)

\[ x_1 = b + b_1 - b_4 \]
\[ x_3 = b + b_3 - b_2 \]
\[ x_2 = b_1 - b_2 - 2b \]
\[ \implies b_1 - b_2 > 2b \]
\[ x_4 = b_3 - b_4 - 2b \]
\[ \implies b_3 - b_4 > 2b \]
\[ x_1 + x_3 = 2b + b_1 - b_2 + b_3 - b_4 \]
\[ > 2b + 2b + 2b \]
\[ = 6b > y > 2x \]

This is a contradiction.

\[ y \geq x_1 + x_2 + x_3 + x_4 > x_1 + x_3 > 6b > y \]
Hence $\sigma_E(G) > 2$. 

The edge function given in Figure 4.9 shows that $\sigma_E(J_{3,4}) = 3$.

4.4 Definition: The Crown graph $s^0_n$ for any integer $n \geq 3$ is the graph with vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, w_1, w_2\}$ and the edge set $E = \{u_i v_j : 1 \leq i, j \leq n \text{ for } i \neq j\}$. $s^0_n$ is therefore equivalent to the complete bipartite graph $K_{n,n}$ with horizontal edges viz. $\{u_i v_i : 1 \leq i \leq n\}$ removed.
4.10 Theorem: $\sigma_E(s_n^0) = 1$ for $n \geq 3$.

Proof: Let $G = s_n^0 \cup K_2$ where $V(G) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{w_1, w_2\}$ and $E(G) = \{u_i v_j : 1 \leq i, j \leq n; i \neq j\} \cup \{w_1 w_2\}$. Let $x = 2^{(n-2)^2+2}$.

The edge function matrix $A = (a_{i,j})$ of order $n$ is defined as follows:

$$a_{i,j} = \begin{cases} 0 & 1 \leq i = j \leq n \\ x + (n-2)2^{(i-1)+(j-1)(n-2)} & 1 \leq i < j \leq n-1 \\ x + (n-2)2^{(i-2)+(j-1)(n-2)} & 1 \leq j < i \leq n-1 \\ y - (n-2) \left( 2^{(j-1)(n-2)} \sum_{i=1}^{j-1} 2^{(i-1)} + 2^{(j-1)(n-2)} \sum_{i=j+1}^{n-1} 2^{(i-2)} \right) & i = n \text{ and } 1 \leq j \leq n-1 \\ y - (n-2) \left( 2^{(i-2)} \sum_{j=1}^{i-1} 2^{(j-1)(n-2)} + 2^{(i-1)} \sum_{j=i+1}^{n-1} 2^{(j-1)(n-2)} \right) & 1 \leq i \leq n-1 \text{ and } j = n \\ \end{cases}$$

$$a_{i,n} = y + (n-2)x - \sum_{j=1}^{n-1} a_{i,j} = y + (n-2)x - \sum_{j=1}^{i-1} a_{i,j} - \sum_{j=i+1}^{n-1} a_{i,j}$$

$$= y + (n-2)x - \sum_{j=1}^{i-1} \left( x + (n-2)2^{(i-2)+(j-1)(n-2)} \right)$$

$$- \sum_{j=i+1}^{n-1} \left( x + (n-2)2^{(i-1)+(j-1)(n-2)} \right)$$

$$= y - (n-2) \left( 2^{(i-2)} \sum_{j=1}^{i-1} 2^{(j-1)(n-2)} + 2^{(i-1)} \sum_{j=i+1}^{n-1} 2^{(j-1)(n-2)} \right)$$
\[ a_{n,j} = y + (n-2)x - \sum_{i=1}^{n-1} a_{i,j} = y + (n-2)x - \sum_{i=1}^{j-1} a_{i,j} - \sum_{i=j+1}^{n-1} a_{i,j} \]

\[ = y + (n-2)x - \sum_{i=1}^{j-1} \left( x + (n-2)2^{(i-1)+(j-1)(n-2)} \right) \]

\[ - \sum_{j=i+1}^{n-1} \left( x + (n-2)2^{(i-2)+(j-1)(n-2)} \right) \]

\[ = y - (n-2) \left( 2^{(j-1)(n-2)} \sum_{i=1}^{j-1} 2^{(i-1)} + 2^{(j-1)(n-2)} \sum_{i=j+1}^{n-1} 2^{(i-2)} \right) \]

where \( y = x + 2^{(n-1)(n-2)} - 1 \)

Let \( S = \{ a_{i,j} : 1 \leq i, j \leq n \} \cup \{ y + (n-2)x \} \)

where \( x = 2^{(n-2)^2+2} \), \( y = x + 2^{(n-1)(n-2)} - 1 \).

The edge function \( f : E \to S \) is defined as

\[ f(u_i v_i) = a_{i,j} \text{ for } 1 \leq i, j \leq n \text{ and } f(w_1 w_2) = (n-2)x + y. \]

The corresponding edge sum function is defined as follows:

For \( 1 \leq i \leq n-1; \)

\[ F(u_i) = \sum_{j=1}^{n} f(u_i v_j) \]

\[ = \sum_{j=1}^{i-1} a_{i,j} + \sum_{j=i+1}^{n-1} a_{i,j} + a_{i,n} \]

\[ = \sum_{j=1}^{i-1} \left( x + (n-2)2^{(i-2)+(j-1)(n-2)} \right) \]

108
\[+ \sum_{j=i+1}^{n-1} \left( x + (n - 2)2^{(i-1)+(j-1)(n-2)} \right)\]

\[+ y - (n - 2) \left( 2^{(i-2)} \sum_{j=1}^{i-1} \left( 2^{(j-1)(n-2)} \right) + 2^{(i-1)} \sum_{j=i+1}^{n-1} \left( 2^{(j-1)(n-2)} \right) \right)\]

\[= y + (n - 2)x + (n - 2) \left( 2^{(i-2)} \sum_{j=1}^{i-1} \left( 2^{(j-1)(n-2)} \right) + 2^{(i-1)} \sum_{j=i+1}^{n-1} \left( 2^{(j-1)(n-2)} \right) \right)\]

\[= y + (n - 2)x\]

For \(1 \leq j \leq n - 1;\)

\[F(v_j) = \sum_{i=1}^{n-1} a_{i,j} + a_{n,j} = \sum_{i=1}^{j-1} a_{i,j} + \sum_{i=j+1}^{n-1} a_{i,j} + a_{n,j}\]

\[= \sum_{i=1}^{j-1} \left( x + (n - 2)2^{(i-1)+(j-1)(n-2)} \right)\]

\[+ \sum_{i=j+1}^{n-1} \left( x + (n - 2)2^{(i-2)+(j-1)(n-2)} \right)\]

\[+ y - (n - 2) \left( 2^{(j-1)(n-2)} \sum_{i=1}^{j-1} \left( 2^{(i-1)} \right) + 2^{(j-1)(n-2)} \sum_{i=j+1}^{n-1} \left( 2^{(i-2)} \right) \right)\]

\[= y + (n - 2)x + (n - 2) \left( 2^{(j-1)(n-2)} \sum_{i=1}^{j-1} \left( 2^{(i-1)} \right) + 2^{(j-1)(n-2)} \sum_{i=j+1}^{n-1} \left( 2^{(i-2)} \right) \right)\]

\[= y + (n - 2)x\]
\[ F(u_n) = \sum_{j=1}^{n} f(u_n v_j) \] for \( 1 \leq i \leq n - 1 \)

\[ = \sum_{j=1}^{n} a_{nj} \]

\[ = \sum_{j=1}^{n-1} a_{nj} + a_{nn} \]

\[ = \sum_{j=1}^{n-1} a_{nj} + 0 \]

\[ = \sum_{j=1}^{n-1} \left( y - (n - 2) \left( 2^{(j-1)} \sum_{i=1}^{j-1} \left( 2^{(j-1)(n-2)} \right) + 2^{(j-2)} \sum_{i=j+1}^{n-1} \left( 2^{(j-1)(n-2)} \right) \right) \right) \]

\[ = (n - 1)y - (n - 2) \left( 2^{(n-1)(n-2)} - 1 \right) \]

\[ = (n - 2)x + y. \]

Similarly, \( F(v_n) = (n - 2)x + y. \)

\[ F(w_1) = F(w_2) = f(w_1 w_2) = (n-2)x + y. \]

We have got that \( F(v) = f(w_1 w_2) = (n-2)x + y \) for all the vertices \( v \in V \). Hence \( F \) is into \( S \). The row sums and column sums are equal and the column value is \( ((n - 2)x + y) \). The elements of \( S \) are such that except for the row sums and column sums of \( A_{n \times n} \), no other sum of any number of elements is in \( S \). Hence \( S_n^0 \cup K_2 \) with this edge function becomes an edge sum graph. That is,
\( \sigma_E(s_n^0) \leq 1 \) for \( n \geq 3 \). Since \( K_2 \) is the only connected edge sum graph, we get that \( \sigma_E(s_n^0) = 1 \) for \( n \geq 3 \).

The following \( 6 \times 6 \) matrix gives edge function of \( s_6^0 \).

\[
\begin{bmatrix}
0 & x + 4(2^4) & \cdots & x + 4(2^{16}) & y - 4(2^4 + 2^8 + 2^{12} + 2^{16}) \\
x + 4(2^0) & 0 & \cdots & x + 4(2^{17}) & y - 4(2^6 + 2^9 + 2^{13} + 2^{17}) \\
x + 4(2^1) & x + 4(2^5) & \cdots & x + 4(2^{18}) & y - 4(2^1 + 2^5 + 2^{14} + 2^{18}) \\
x + 4(2^2) & x + 4(2^6) & \cdots & x + 4(2^{19}) & y - 4(2^2 + 2^6 + 2^{10} + 2^{19}) \\
x + 4(2^3) & x + 4(2^7) & \cdots & 0 & y - 4(2^3 + 2^7 + 2^{11} + 2^{15}) \\
y - 4(2^0 + \cdots + 2^3) & y - 4(2^4 + \cdots + 2^7) & \cdots & y - 4(2^{16} + \cdots + 2^{19}) & 0 \\
\end{bmatrix}
\]

where \( x = 2^{18} \), \( y = x + 2^{20} - 1 \).