CHAPTER IV

THEOREM 4.1: ON THE J, \( p_n \) - UNIQUENESS

4.1 definitions and notations. We suppose throughout that

\[ p_n > 0, \quad \sum_{n=0}^{\infty} p_n = \infty, \]

and that the radius of convergence of the power series

\[ p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad p(\cdot) = p_0, \]

is 1. Given any series \( \{ a_n \} \), with the sequence of partial sums \( \{ s_n \} \), we shall use the notations:

\[ (4.1.1) \quad p_{s}(x) = \sum_{n=0}^{\infty} a_n x^n, \]

and

\[ (4.1.2) \quad J(x) = J_s(x) = p_s(x) / p(x). \]

If the series on the right of (4.1.1) is convergent
in the right open interval \((0, 1)\) and if

\[ J(x) \in (c, 1), \quad (0 < c < 1), \]

we say that the series \(\sum_{n=0}^{\infty} a_n\), or the sequence \(\{a_n\}\), is absolutely summable \((J, \{p_n\})\), or simply summable \((J, p_n)\) \(^1\).

It is known that the \((J, \{p_n\})\)-transform \((4.1.9)\) is both regular \(^2\) and absolutely regular. \(^3\)

In the special case in which \(p(x) = (1-x)\), \(-1 < x < 1\),

\[(J, p_n)\text{-method reduces to the absolute Abel method, } |\cdot| .\]

Now, we write

\[ n = p_0 + p_1 + \ldots + p_n \quad n = 0, 1, 2, \ldots , \]

\[ p_{-1} = p_{-2} = 0 , \]

and

\[ (4.1.5) \quad \sigma_n = \frac{1}{p_n} \sum_{v=0}^{n} p_v a_v \quad (n \geq 0) . \]

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1) Ahmad (1), Chapter VIII; see also Ahmad (4) and 
2) Hardy (1), page 80; see also Horwein (2). 
3) Ahmad (1), (4).
with $p_n > 0$. If $\{ s_n \} \subseteq H$, we say that the series $\sum a_n$, or the sequence $\{ s_n \}$ is absolutely summable $\sum (s, p_n)$, or simply summable $\sum (s, n, \tau_n)$. The $(n, p_n)$-transform (4.1.7) is also both regular $1$ and absolutely regular $p$.

Again, let us write

$$
(4.1.4) \quad t_n = \sum_{v=1}^{n-1} \tau_v a_v, \quad t_0 = 0.
$$

Then, from (4.1.5) and (4.1.4), we get

$$
(4.1.5) \quad \Delta \sigma_n = \sigma_n - \sigma_{n-1} = \frac{p_n}{n-1} t_n(n-1),
$$

and

$$
(4.1.6) \quad s_n = \sigma_n + t_n(n \geq 1).
$$

Thus, the summability $\sum (s, p_n)$ of $\{ s_n \}$ is the same as

$$
(4.1.7) \quad \frac{p_n}{n} | t_n | \leq \infty,
$$

and the summability $\sum (s, p_n)$ of the sequence $\{ \frac{a_n}{p_n} \}$ is the same as

$$
\{ t_n \} \subseteq H.
$$

1) Hardy (1), page 57.

p) Sunouchi (1).
Suppose that \( \{ \lambda_n \} \) be a sequence such that
\[ 0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty \quad \text{as} \quad n \to \infty, \]
and let us write
\[
t_n(t) = \lambda_n(t) = \sum_{n \leq t} a_n.
\]
and, for \( k > 0 \), we write
\[
k(t) = \sum_{n \leq t} \left( t - \lambda_n \right)^k a_n,
\]
with \( \lambda_n(t) = 0 \), for \( t < \lambda_1 \).

If
\[
(4.1.8) \quad k(t) = \lambda_n(t) / t^k \in V(p, \infty),
\]
for some finite positive number \( h \), then we say that the series \( \sum a_n \) is absolutely summable by \( \text{Riesz means} \) \(^1\) of 'type' \( \lambda_n \) and 'order' \( k (k \geq 0) \), or simply summable
\[
[\lambda_n, \lambda_n, k]. \quad \text{(2)}
\]

Now, in \( R_n(t) \), if \( t \) be restricted to the sequence \( \{ \lambda_n \} \) only, then we obtain the 'discrete Riesz means', or \( (\lambda_n, \lambda, k)-\text{means} \):
\[
(4.1.9) \quad \sum_{n=0}^{\infty} \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right)^k a_n, \quad k \geq 0.
\]

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1) Definition of Riesz mean \( R_n(t) \) is due to Riesz (1).

2) Obrechkoff (1), (p).
we say that the series $\sum a_n$ is absolutely summable
by Riesz's discrete means of 'order' $k$ and 'type' $\omega_k$,
or simply summable $\left| \omega^k, \omega_n, k \right|$, $k > 0$, if
\[
\sum_{v=0}^{n} \left( 1 - \frac{v}{n+1} \right)^k a_v = O \left( \frac{1}{n} \right).
\]

It may be noted that $\left| \omega, \omega_n, 0 \right|$ and $\left| \omega^k, \omega_n, 0 \right|$ are
the same as absolute convergence.

It is known that the summability method $\left| i, \omega_n, 1 \right|$ and $\left| \omega^k, \omega_n, 1 \right|$ are equivalent. 1)

We observe that, by definition, $\left( \omega^k, \omega_n, 1 \right)$-mean is
the same as the $(\omega, p_n)$-mean and in effect, $\left| \omega^k, p_n \right|$-method
is equivalent to $\left| \omega, \omega_n, 1 \right|$-method.

Throughout the notation:
\[
(4.1.1) \quad \lambda(a) = \sum_{u=0}^{v-1} \sum_{u=0}^{v} a_n \left( \frac{v}{u} - 1 \right) u \frac{v-u}{v+u}.
\]

Throughout we use $\beta$ as a strictly positive constant; possibly different at each occurrence.

1) see Kohnanty (1). 2) Kohnanty (1), where he states that it was mentioned
to him by Prof. L.S. Bosanquet. An explicit proof due to
Prof. ..Pati is quoted in Iyer (?). Also, see Lorentz and
Macphail (1).
4.9 Introduction. Recently, Ahmad [1] proved the following Abelian theorem for $|J, p_n|$-summability:

**Theorem 5.** $|R, p_n| \subseteq |J, p_n|$.

Concerning summability $|R, p_n|$, we know the following Tauberian theorem:

**Theorem 3.** If (i) $\sum a_n$ is summable $|J, p_n|$ and (ii) $\{t_n\} \subseteq J$ and (iii) $\left\{\frac{\sum_{n=1}^{\infty} t_n}{p_n}\right\} \subseteq BV$, then $\sum a_n$ is absolutely convergent.

It is observed that the condition (iii) in redundant in view of the fact that $|J, p_n|$ and $|R, p_{n-1}, l|$-summability methods are equivalent, since by (4.1.6), we get:

**Theorem 4.** If (i) $\sum a_n$ is summable $|R, p_n|$ and (ii) $\{t_n\} \subseteq BV$, then $\sum a_n$ is absolutely convergent.

The object of the present chapter is to obtain Tauberian theorems for $|J, p_n|$-summability. We prove these theorems in Sections 4.5 and 4.6.

It is interesting to note that, as a special case of our Theorem 3, we get the following Tauberian theorem of

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1) Ahmad (4), Theorem 2(a); see also Ahmad (1), Chapter VIII, Theorem 9.

2) Bhatt (1).
for absolute Abel summability.

Theorem 1. If \( s a_n \) is summable \( J \) and \( \{ n a_n \} \in \mathbb{N} \),
then \( s a_n \) is absolutely convergent.

4.3. We establish the following theorems.

Theorem 1. If \( s a_n \) is summable \( J, p_n \) and \( \{ t_n \} \in \mathbb{N} \),
and if \( \{ p_n \} \) is such that

\[ \frac{n p_n}{n-1} < \infty \quad \text{for} \quad n = 1, 2, \ldots \]

then \( s a_n \) is summable \( J, p_n \).

Theorem 2. If \( s a_n \) is summable \( J, p_n \) and \( \{ t_n \} \in \mathbb{N} \),
and if \( \{ p_n \} \) satisfies the same conditions as in Theorem 1,
then \( s a_n \) is absolutely convergent.

Theorem 3. If \( s a_n \) is summable \( J, p_n \), and
\( \{ a_n n^{-1} / p_n \} \in \mathbb{N} \), and if \( \{ p_n \} \) satisfies the same conditions
as in Theorem 1, then \( s a_n \) is absolutely convergent.

1) Nyelop (1), Theorem 3.
4.4: We require the following lemmas for the proof of our theorem.

Lemma 1. Let \( u_m > 0 \), \( u_m = \sum_{n=1}^{m} u_n \), and

\[ d_m = \sum_{n=1}^{m} u_n c_n. \]

Then, if \( \{c_n\} \subset BV \), \( \{d_n\} \subset BV \).

Lemma 2. If \( \{t_n\} \subset BV \) and

\[ j(\varepsilon) = \lim_{n \to \infty} \sum_{n=0}^{\infty} p_n e^{-n \varepsilon} \quad (\varepsilon > 0), \]

then

\[ j'(\varepsilon) = -\lim_{n \to \infty} \sum_{n=0}^{\infty} \frac{\sum_{v=1}^{n} p_v \sum_{u=n}^{\infty} p_u e^{-u \varepsilon} v (u-v) - v u}{(\sum_{o} p_o e^{-o \varepsilon})^p}. \]

Proof. Since, by (4.1.4), (4.1.5) and (4.1.6),

\[ t_n = \left( \frac{\gamma_n}{\gamma} \right) \sum_{v=1}^{n} \gamma_{v-1} a_v, \quad t_0 = 0, \]

and

\[ a_n = \frac{p_n}{p_{n-1}} t_n + (t_n - t_{n-1}). \]

1) Multiply (1).
for $n \geq 1$,

$$
\epsilon_n = a_0 + t_n + \sum_{v=1}^{n} \frac{p_v}{r_{v-1}} t_v.
$$

Hence

$$
(4.4.1) \quad \sum_{n=0}^{\infty} p_n e^{-ns} = a_0 \sum_{n=0}^{\infty} p_n e^{-ns} + \sum_{n=1}^{\infty} t_n e^{-ns} + \sum_{n=1}^{\infty} \sum_{v=1}^{n} \frac{p_v}{r_{v-1}} t_v.
$$

$$
= a_0 \sum_{n=0}^{\infty} p_n e^{-ns} + \sum_{n=1}^{\infty} t_n e^{-ns} + \sum_{n=1}^{\infty} \sum_{v=1}^{n} \frac{p_v}{r_{v-1}} t_v.
$$

the inversion in the last sum being justified by absolute convergence whenever $\{t_v\} \in R$ (and thus, a fortiori, when $\{t_v\} \in \mathbb{B}$).

Also, for $n=1, 2, 3, \ldots$,

$$
(4.4.2) \quad \left( \sum_{v=n}^{\infty} p_v e^{-vs} \right) \left( \sum_{v=0}^{\infty} e^{-vs} \right) = \left( \sum_{v=0}^{\infty} p_{n+v} e^{-(n+v)s} \right) \left( \sum_{v=0}^{\infty} e^{-vs} \right)
$$

$$
= e^{-ns} \left( \sum_{v=0}^{\infty} p_{n+v} e^{-vs} \right) \left( \sum_{v=0}^{\infty} e^{-vs} \right)
$$
\[ e^{-ns} \sum_{v=0}^{\infty} e^{-vs} (\sum_{u=0}^{n+v} p_{n+v}) \]

\[ = e^{-ns} \sum_{v=0}^{\infty} e^{-vs} (\sum_{n=0}^{v} e^{-ns}) \]

\[ = e^{-ns} \sum_{v=0}^{\infty} e^{-vs} (n+1) \sum_{n=0}^{\infty} e^{-ns} (1 - e^{-s})^{-1} \]

\[ = e^{-ns} \sum_{v=0}^{\infty} e^{-vs} - \sum_{n=0}^{\infty} e^{-ns} (1 - e^{-s})^{-1} \]

Therefore, from (4.4.1) and (4.4.7), for \(0 < e \leq \infty\), we have

\[ J(s) = \sum_{n=0}^{\infty} \frac{\sum_{n=1}^{\infty} p_{n} \cdot t_{n}}{1 - e^{-ns}} \]

\[ = \sum_{n=0}^{\infty} \frac{\sum_{n=1}^{\infty} p_{n} \cdot t_{n}}{1 - e^{-ns}} + \sum_{n=1}^{\infty} \frac{p_{n} \cdot t_{n}}{1 - e^{-ns}} \]

\[ = \sum_{n=0}^{\infty} \frac{\sum_{n=1}^{\infty} p_{n} \cdot t_{n}}{1 - e^{-ns}} + \sum_{n=1}^{\infty} \frac{p_{n} \cdot t_{n}}{1 - e^{-ns}} \]

\[ = a_{0} + \sum_{n=1}^{\infty} \frac{\sum_{n=1}^{\infty} p_{n} \cdot t_{n}}{1 - e^{-ns}} + \sum_{n=1}^{\infty} \frac{p_{n} \cdot t_{n}}{1 - e^{-ns}} \]
\[ \frac{p_n}{p_{n-1}} e^{-ns} \]
\[
= a_0 + \frac{\sum p_n e^{-ne}}{n=0} + \frac{\sum p_{n-1} e^{-ns}}{n=0} - \frac{\sum p_{n-1} e^{-ns}}{n=0} + \frac{\sum p_{n-1} e^{-ns}}{n=0}
\]

Again, since \( \{t_n\} \in BV \), the series

\[ \sum_{n=1}^{\infty} \frac{p_n}{p_{n-1}} t_n e^{-np} x^v \]

is absolutely convergent, for then this series is majorised by

\[ \sum_{n=1}^{\infty} \sum_{v=0}^{\infty} \frac{p_n}{p_{v-1}} t_v x^v = \sum_{v=0}^{\infty} x^v \sum_{v=0}^{\infty} \frac{p_v}{p_{v-1}} x^v = x \sum_{v=0}^{\infty} \frac{p_v}{p_{v-1}} x^v \]

which converges for \( 0 \leq x < 1 \), by hypothesis.
Therefore, differentiating (4.4.7) with respect to \( \alpha \), we get

\[
J'(\alpha) = \sum_{v=0}^{\alpha} \left( \sum_{u=0}^{\alpha - v - 1} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} \right) (\alpha - v)^{\alpha - v - u - 1} e^{-\alpha v} \]

**Lemma 2.** (i) For \( v < n \), \( n > 0 \) (\( 0 \leq v \leq u, v < v \leq u \) and \( \alpha > \beta \)),

\[
\sum_{u=0}^{\alpha - v - 1} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} \leq n.
\]

(ii) For \( n > 1 \) and \( v \geq n \),

\[
\sum_{u=0}^{\alpha - v - 1} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} \geq 0.
\]

**Proof.** (i) In due to McFadden [1], we give its proof here for completeness. We see that

\[
\sum_{u=0}^{\alpha - v - 1} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} = \sum_{u=1}^{\alpha} (\alpha - v - u - 1) \frac{(\alpha - u)!}{(\alpha - v - u)!}.
\]

For \( \alpha > 1 \),

\[
\sum_{v=0}^{\alpha-1} (\alpha - v)^{\alpha - v - u - 1} e^{-\alpha v} = (\alpha - 1)^{\alpha - u - 1} e^{-\alpha} (\alpha - 1)^{\alpha - v - u - 1} e^{-\alpha v}.
\]

\[
= -(\alpha + 1) \frac{(\alpha - v - 1)!}{(\alpha - u)!} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} \cdots \frac{(\alpha - v - u - n)!}{(\alpha - u)!} e^{-(\alpha - 1)v}.
\]

\[
= -(\alpha + 1) \frac{(\alpha - v - 1)!}{(\alpha - u)!} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} \cdots \frac{(\alpha - v - u - n)!}{(\alpha - u)!} e^{-(\alpha - 1)v}.
\]

\[
= -(\alpha + 1) \frac{(\alpha - v - 1)!}{(\alpha - u)!} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} \cdots \frac{(\alpha - v - u - n)!}{(\alpha - u)!} e^{-(\alpha - 1)v}.
\]

\[
= -(\alpha + 1) \frac{(\alpha - v - 1)!}{(\alpha - u)!} \frac{(\alpha - v - u - 1)!}{(\alpha - u)!} \cdots \frac{(\alpha - v - u - n)!}{(\alpha - u)!} e^{-(\alpha - 1)v}.
\]

1) McFadden (1), page 181.
it being understood that \( r = 0 \) when \( r \) is negative. If we consider separately the cases \( 0 \leq v \leq n \), \( n < v \leq \infty \) and \( v > \infty \), it is easily seen that in all cases the given expression is either negative or zero. 1)

(ii) From (i), we infer that, when \( n \geq 1 \), and for \( v \geq 1 \),

\[
\sum_{u=0}^{n-1} (\rho - v)^{u} v^{u} \leq c.
\]

so,

\[
\sum_{u=0}^{n-1} (\rho - v)^{u} v^{u} \leq c.
\]

by (1), since \( v (\rho - v)^{u} v^{u} = c \).

**Lemma 4**. Let

\[
L(n; \lambda) = \sum_{u=0}^{n-1} \frac{p_{n}}{u!} e^{v/u} (\rho - v)^{u} v^{u}.
\]

(4.4.4) \( L(w, \lambda) = \sum_{u=0}^{n-1} \frac{p_{n}}{u!} e^{v/u} (\rho - v)^{u} v^{u}.
\]

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1) See McWadden (1), page 181.
If \( a(w) > c \) for \( w > 1 \), then for \( w > 1 \), \( a(w) > c \).

**Proof.** Since

\[
(j_1 < \left\{ \frac{w}{1-e^{-1/w}} \right\}^{-1} < j_p \quad (w > 1),
\]

where \( j_1 \) and \( j_p \) are two suitable positive constants, we observe that

\[
\sum_{n=0}^{\infty} \frac{p_n}{n!} \frac{e^{-v/w}}{w(1-e^{-1/w})} \frac{v^n}{(v-u)_n} \frac{u^n}{(v-u)_n}.
\]

\[
> \frac{1}{w(1-e^{-1/w})} \sum_{n=0}^{\infty} \frac{p_n}{n!} \frac{e^{-v/w}}{v^n} \frac{u^n}{(v-u+1)_n} (v-u)_n.
\]

\[
= \frac{1}{w(1-e^{-1/w})} \quad (w)
\]

\[\geq c,\]

by hypothesis and the notation \( (4.110) \). This completes the proof of the lemma.

**Lemma 5.** If \( p_n > 0 \) and \( P_n = p_0 + p_1 + \ldots + p_n \), for \( n = 0, 1, 2, \ldots \), then \( \left\{ \frac{p_n}{v=0} \frac{e^{-v/n}}{v} \right\} \) is bounded.
Proof. Since, by hypothesis

\[ \sum_{v=0}^{\infty} p_v e^{-v/n} \geq \frac{1}{\epsilon} \sum_{v=0}^{\infty} p_v e^{-v} \geq \frac{1}{\epsilon} \sum_{v=0}^{\infty} p_v = \frac{n}{\epsilon}, \]

the lemma follows.

4.5 Proof of Theorem 1. We use the following alternative definition for \(|J, p_n|\)-summability.

Let

\[ J(s) = \sum_{v=0}^{\infty} \frac{p_v a_v e^{-vs}}{\sum_{v=0}^{\infty} p_v e^{-vs}}. \]

Then the sequence \( \{a_n\} \), or the series \( \sum a_n \), will be said to be summable \(|J, p_n|\) if

\[ \int_0^\infty |J'(s)| \, ds < \infty. \]

We shall suppose throughout that \( m = [w] \), \( w = \epsilon^{-1} \), and that \( \lambda \) is a positive integer. By using (4.4.4), it is sufficient to prove that

\[ \beta_n = \int_1^h \left| t_m \right| w^{-1} \psi(w) \, dw = O(1), \]

for, by Lemma 4 and (4.1.5),
\[
J = \sum_{n=1}^{\infty} \int_{|t_n|}^{n+1} \frac{1}{n} w^{-1} \psi(w) \, dw
\]

\[
> \frac{k}{C} \sum_{n=1}^{\infty} \left| \frac{p_n}{p_{n-1}} t_n \right| \quad \text{(by hypothesis (i))}
\]

\[
= \frac{k}{C} \sum_{n=1}^{\infty} \left| \tilde{\delta}_n \right|
\]

the result follows.

Let write

\[
J \approx J_1 + \ldots
\]

where

\[
J_1 = \int_{1}^{\infty} w^{-\alpha} \left| \tilde{\delta}(\frac{1}{w}) \right| \, dw
\]

\[
\psi = \int_{1}^{\infty} \left| w^{-\alpha} \tilde{\delta}(\frac{1}{w}) + t_n w^{-1} \psi(w) \right| \, dw.
\]

Now,

\[
J_1 = \int_{1/\beta}^{1} |\tilde{\delta}(s)| \, ds = O(1),
\]

since \( \sum a_n \) is summable \( J, p_n \). Also, since, by Lemma 1,
\[ J'(1/w) = \sum_{n=1}^{\infty} \frac{p_n}{v_n} \sum_{v=0}^{\infty} \frac{e^{-v/w} \Gamma(\tau - v) u \Gamma(v - u)}{v^p \left( \sum_{v=0}^{\infty} e^{-v/w} \right)^p}, \]

we have

\[ \pi = \int \left\{ \frac{\sum_{n=1}^{\infty} p_n}{\sum_{v=0}^{\infty} \Gamma(v - u) u \Gamma(v - u)} \right\} \left( \sum_{v=0}^{\infty} e^{-v/w} \right)^2 \, dw \]

where

\[ \pi, 1 = \int \left\{ \frac{\sum_{n=1}^{\infty} p_n}{\sum_{n=1}^{\infty} \gamma_n (\tau - \gamma_n) \Gamma(\tau - \gamma_n) \Gamma(\tau - \gamma_n) u \Gamma(v - u)} \right\} \left( \sum_{v=0}^{\infty} e^{-v/w} \right)^2 \, dw \]

and

\[ \pi, e^{\tau} = \int \left\{ \frac{\sum_{n=1}^{\infty} p_n}{\sum_{n=1}^{\infty} \gamma_n (\tau - \gamma_n) \Gamma(\tau - \gamma_n) \Gamma(\tau - \gamma_n) u \Gamma(v - u)} \right\} \left( \sum_{v=0}^{\infty} e^{-v/w} \right)^2 \, dw. \]

Now

\[ \pi, 1 = \int \left\{ \frac{\sum_{n=1}^{\infty} p_n}{\sum_{n=1}^{\infty} \gamma_n (\tau - \gamma_n) \Gamma(\tau - \gamma_n) \Gamma(\tau - \gamma_n) u \Gamma(v - u)} \right\} \left( \sum_{v=0}^{\infty} e^{-v/w} \right)^2 \, dw. \]
\[
\sum_{v=0}^{\infty} \frac{\sum_{v=0}^{\infty} P_v e^{-v/w}}{\sum_{v=0}^{\infty} P_v e^{-v/w}}
\]

\[
\sum_{v=0}^{\infty} \frac{\sum_{v=0}^{\infty} P_v e^{-v/w}}{\sum_{v=0}^{\infty} P_v e^{-v/w}}
\]

\[
\sum_{v=0}^{\infty} \frac{\sum_{v=0}^{\infty} P_v e^{-v/w}}{\sum_{v=0}^{\infty} P_v e^{-v/w}}
\]

\[
\sum_{v=0}^{\infty} \frac{\sum_{v=0}^{\infty} P_v e^{-v/w}}{\sum_{v=0}^{\infty} P_v e^{-v/w}}
\]

\[
\sum_{v=0}^{\infty} \frac{\sum_{v=0}^{\infty} P_v e^{-v/w}}{\sum_{v=0}^{\infty} P_v e^{-v/w}}
\]

\[
\sum_{v=0}^{\infty} \frac{\sum_{v=0}^{\infty} P_v e^{-v/w}}{\sum_{v=0}^{\infty} P_v e^{-v/w}}
\]

\[
\sum_{v=0}^{\infty} \frac{\sum_{v=0}^{\infty} P_v e^{-v/w}}{\sum_{v=0}^{\infty} P_v e^{-v/w}}
\]
\[ t \leq \sum_{r=1}^{\infty} \left| \Delta t_r \right| \leq \frac{\pi}{2} \int_0^\infty \frac{e^{-v/r}}{p_r} \, dv \]

(by Lemma 5)

\[ k < \infty \]

by hypothesis.

Again,

\[ \sum_{r=1}^{\infty} \sum_{n=m+1}^{\infty} \frac{p_n}{n-1} \sum_{r=m+1}^{n-1} \left| \Delta t_r \right| \leq \sum_{r=1}^{\infty} \sum_{n=r}^{\infty} \frac{p_n}{n-1} \int_0^\infty \frac{e^{-v/w}}{v} \left( \sum_{v=0}^{\infty} \frac{p_n}{n} e^{-v/w} \right) dv \]

\[ = \sum_{r=1}^{\infty} \left| \Delta t_r \right| \sum_{n=r}^{\infty} \frac{p_n}{n-1} \int_0^\infty \frac{e^{-v/w}}{v} \left( \sum_{v=0}^{\infty} \frac{p_n}{n} e^{-v/w} \right) dv \]

\[ < \sum_{r=1}^{\infty} \left| \Delta t_r \right| \sum_{n=r}^{\infty} \frac{p_n}{n-1} \frac{\sum_{v=n}^{\infty} e^{-v/w}}{\sum_{v=0}^{\infty} \frac{p_n}{n} e^{-v/w}} \]
\[
\sum_{r=1}^{N} |\Delta t_r| \cdot \frac{\sum_{n=1}^{\infty} p_n e^{-n/r}}{(1-e^{-1/r}) \sum_{v=0}^{\infty} p_v e^{-v/r}} + \\
\sum_{r=1}^{N} |\Delta t_r| \cdot \frac{\sum_{n=1}^{\infty} p_n e^{-n/r}}{(\sum_{n=0}^{\infty} e^{-n/r}) (1-e^{-1/r})} \\
\sum_{r=1}^{N} |\Delta t_r| \cdot \frac{\sum_{n=0}^{\infty} p_n e^{-n/r}}{\sum_{n=0}^{\infty} p_n e^{-n/r}} + \\
\sum_{r=1}^{N} |\Delta t_r| \cdot \frac{\sum_{v=0}^{\infty} p_v e^{-v/r}}{r(1-e^{-1/r}) (\sum_{v=0}^{\infty} r_v e^{-v/r})} \\
- \sum_{r=1}^{N} |\Delta t_r| \cdot \frac{\sum_{v=0}^{\infty} p_v e^{-v/r}}{\sum_{v=0}^{\infty} p_v e^{-v/r}} \\
\sum_{r=1}^{N} |\Delta t_r| + \sum_{r=1}^{N} |\Delta t_r| \cdot \frac{\sum_{v=0}^{\infty} p_v e^{-v/r}}{r(1-e^{-1/r}) \sum_{v=0}^{\infty} r_v e^{-v/r}} \\
\sum_{r=1}^{N} |\Delta t_r| \\
\sum_{r=1}^{N} |\Delta t_r| < \infty
\]
by hypothesis and by the fact that

\[ J_1 < \frac{1}{r(1 - e^{-1/r})} < J_p \quad \text{for } r \geq 1, \]

where \( J_1 \) and \( J_p \) are suitable positive constants.

This completes the proof of Theorem 1.

4.6 Proof of Theorems 2 and 3.

Theorem 2 is obtained by combining the results of Theorem 1 and Theorem C.

Theorem 3 is obtained from Theorem 2, by an appeal to Lemma 1.