7.1 **Definitions and Notations.** We adopt the definition of summability \(|J, p_n|_k, k \geq 1\), and other notations given in Section 6.1 of Chapter VI.

Throughout we write

\( g(t) = \int_{\mathbb{R}} \frac{2^u}{t} \sin u/2 \, du \); \hspace{1cm} \text{(7.1.1)}

where

\( \phi(u) = \frac{1}{2} f(x_0 + u) + f(x_0 - u) - 2e^i \).

7.2 **Introduction.** Huhman and Patnaik 1) have proved the following theorem for an even function \( f \).

**Theorem A.** If the function

\( g(t) = \frac{1}{t \log(2\pi/t)} \in L(0, x), \hspace{1cm} \text{(7.2.1)} \)

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* Huhman (1).

1) Hobbsy and Patnaik (1).
than the Fourier series of \( f \) is \( |L| \)-summable at the point \( x_0 \).

Recently I. I. I. I. and I. I. II. I. 1) gave an alternative proof of this theorem. Generalizing this, they \(^2\) have also proved the following theorem:

**Theorem 2.** Suppose that (i) \( \{ n p_n \} \) and \( \{ n^p p_n \} \) are monotone and concave or convex and that

\[
(1-x)^2 p''(x)/ p(x) \in L((0,1)).
\]

If

\[
(7.7.2) \int_0^1 d(t) t^{-3} dt \int_0^1 \left( (l-x)^n p''(x)/ p(x) \right) dx < \infty,
\]

where \( d(t) = \int_0^x |e(u)| du \), then the Fourier series of \( f \) is

\[
|J, p_n| -\text{summable at the point } x_0.
\]

Our object here is to establish a theorem for summability \( |J, p_n| k, k \geq 1 \) of Fourier series, which contains theorem 2 as a special case when \( k = 1 \).

7. To establish the following theorem.

1) Isumi and Isumi (2).
2) Isumi and Isumi (1).
Theorem. Suppose that (i) \( \{ n p_n \} \) and \( \{ n^p p_n \} \) are monotone and concave or convex and that (ii) 
\[(1-x)^{3-1/k} p^n(x) / p(x) \in \mathbb{R}^k(0, 1). \]
If

\[
(7.5.1) \quad \int_0^1 \frac{G^k(t)}{t^{k+1}} dt \leq \int_0^1 (1-x)^{3k-1} \left( \frac{p^n(x)}{p(x)} \right)^k dx < \infty
\]

where \( G(t) = \left( \int_0^t |g(u)|^k du \right)^{1/k} \), then the Fourier series of \( f \) is \( \{ n p_n \} \)-summable at the point \( x_0 \).

Remark. The condition (7.5.1) is satisfied when

\[
(7.5.2) \quad \left( \int_0^t u^{3k-1} \left( \frac{p^n(1-u)}{p(1-u)} \right)^k du \right)^{1/k} \leq t^3 \frac{p^n(1-t)}{p(1-t)}
\]

for all \( t > 0 \), and

\[
(7.5.3) \quad t^{1-1/k} \frac{p^n(1-t)}{p(1-t)} G(t) \in \mathbb{R}^k(0, 1).
\]

7.4 For the proof of our theorem, we need the following lemma.

Lemma. For \( 0 < c < 1 \),

\[
\left| \sum_{n=1}^{\infty} n p_n \cos(n^{1/2}/2) \frac{x^n}{p(x)} \right| \leq \frac{k(1-x)^{2p}}{(1-x)^2 + t^p} \cdot \frac{p^n(x)}{p(x)}
\]
on the interval \((c, 1)\), where \(k\) is a constant.

The proof of this lemma is contained in the proof of Theorem 2 of I. 1 and 2.1. 1)

7.5 Proof of the Theorem. We can suppose that

\[
\int_0^\infty \beta(u) \, du = 0 \quad \text{and} \quad p_1 = p_2 = 0.
\]

The sequence \(\{n \gamma_n; \, n \geq 3\}\) is also monotone and concave or convex. Let \(e_n(x_0)\) be the \(n\)-th partial sum of the Fourier series of \(f\) at the point \(x_0\). Then

\[
e_n(x_0) = \frac{1}{x} \int_0^\infty \frac{\sin(n+\frac{1}{2})t}{\sin t/2} \, dt.
\]

Therefore,

\[
j(x) = \frac{1}{p(x)} \sum_{n=1}^\infty \gamma_n \, e_n(x_0) \, x^n
\]

\[
= \frac{1}{np(x)} \int_0^\infty \frac{\hat{f}(t)}{\sin t/2} \left( \sum_{n=1}^\infty \frac{n \gamma_n \sin(n+\frac{1}{2})t}{p(x)} \, x^n \right) \, dt.
\]

Differentiating with respect to \(x\), we get

\[
j'(x) = \frac{1}{x} \int_0^\infty \frac{\hat{f}(t)}{\sin t/2} \left( \sum_{n=1}^\infty \frac{n \gamma_n \sin(n+\frac{1}{2})t}{p(x)} \, (x^{n-1}/p(x))' \right) \, dt
\]

1) Izumi and Izumi (1), pages 650-651.
(7.5.1) \[ \frac{1}{\pi} \int_0^\pi g(t) \left( \sum_{n=1}^\infty \frac{1}{n} \cos(n+1/2) t \frac{\partial^x}{\partial x} \right) dt, \]

where \(^'\) denotes the differentiation with respect to \( x \).

Let us write

\[ J'(x) = \frac{1}{\pi} \int_0^\pi g(t) \cdot \cdot'(x, t) \ dt, \]

where

\[ \cdot'(x, t) = \sum_{n=1}^\infty \frac{1}{n} \cos(n+1/2) t \left( \frac{\partial^x}{\partial x} \right) . \]

Now, the Fourier series of \( f \) is summable \[ J, p_n \] if

\[ \int_0^1 (1-x)^{k-1} \left| J'(x) \right|^k dx < \infty. \]

Since,

\[ J'(x) = \frac{1}{\pi} \int_0^\pi g(t) \cdot \cdot'(x, t) \ dt \]

\[ = \frac{1}{\pi} \int_0^\pi g(t) \cdot \cdot_1'(x, t) \ dt + \frac{1}{\pi} \int_0^\pi g(t) \cdot \cdot_2'(x, t) \ dt \]

\[ = J_1'(x) + J_2'(x), \]

say, where
\[ i' (x, t) = \sum_{n=1}^{\infty} p_n \cos(n \pi x / p) t (x^p / p(x))' \]

and

\[ j' (x, t) = \frac{1}{2} \sum_{n=1}^{\infty} p_n \cos (n \pi x / p) t (x^p / p(x))'. \]

In order to prove (7.5.8), by Hille-C. - l's inequality, it is enough to show that

\[ (7.5.8) \quad 1_x = \int_0^1 (1-x)^{k-1} | J'_1(x) |^k dx < \infty, \quad (r = 1, \infty). \]

**Proof of (7.5.3).** We have

\[ l_1 = \int_0^1 (1-x)^{k-1} | J'_1(x) |^k dx \]

\[ \leq \int_0^1 (1-x)^{k-1} \left( \int_0^1 g(t) \left| J'_1(x, t) \right| dt \right)^k dx \]

\[ \leq \left( \frac{1}{k} \right)^k \int_0^1 dx \int_0^1 (1-x)^{k-1} g(t) \left| J'_1(x, t) \right|^k dt \int_0^1 (1-x)^{k-1} \]

\[ = \frac{k-1}{k} \int_0^1 dx \int_0^1 (1-x)^{k-1} g(t) \left| J'_1(x, t) \right|^k dt \]

\[ \leq \frac{1}{k} \int_0^1 g(t) | J'_1(x, t)| dx \int_0^1 \left(1-x)^{k-1} | J'_1(x, t)|^k dt \]

\[ \leq \frac{1}{k} \int_0^1 g(t) | J'_1(x, t)| dx \int_0^1 \left(1-x)^{k-1} | J'_1(x, t)|^k dt \]
\[
\begin{align*}
I_11 &= \frac{1}{n} \int_0^1 \left| s(t) \right|^k dt \int_0^1 (1-x)^{k-1} \left| \frac{\partial}{\partial x} (x,y,t) \right|^k dx \\
&= \frac{1}{n} \int_0^1 \left| s(t) \right|^k dt \int_0^1 (1-x)^{k-1} \left| \frac{\partial}{\partial x} h_1(x,t) \right|^k dx \\
&= \frac{1}{n} \int_0^1 \left| s(t) \right|^k dt \int_0^1 (1-x)^{k-1} \left| y'_1(x,t) \right|^k dx \\
&= I_{111} + I_{112} + I_{113}, \text{ say.}
\end{align*}
\]

Now, we see that

\[I_{11} = \frac{1}{n} \int_0^1 \left| s(t) \right|^k dt \int_0^1 (1-x)^{k-1} \left| \frac{\partial}{\partial x} h_1(x,t) \right|^k dx\]

\[\leq \frac{K}{n} \int_0^1 \left| s(t) \right|^k dt \int_0^1 (1-x)^{k-1} \left( \frac{p'(x)}{p(x)} \right)^k dx\] (by the lemma)

\[= k \int_0^1 (1-x)^{k-1} \left( \frac{p'(x)}{p(x)} \right)^k dx \left( \int_0^1 \left| s(t) \right|^k dt \right)^{1/k} \cdot k^{1/k} \]

\[= k \int_0^1 (1-x)^{k-1} \left( \frac{p'(x)}{p(x)} \right)^k dx \left( \int_0^1 \left| s(t) \right|^k dt \right)^{1/k} \cdot k^{1/k} \]

\[= k \int_0^1 (1-x)^{k-1} \left( \frac{p'(x)}{p(x)} \right)^k \left( \int_0^1 \left| s(t) \right|^k dt \right)^{1/k} dx \]

\[= -k \int_0^1 (1-x)^{k-1} \left( \frac{p''(1-t)}{p(1-t)} \right)^k (o(t))^k dt\]
\[
\begin{align*}
\frac{1}{t} \int_0^1 t^{k-1} \left( \frac{p''(1-t)}{p(1-t)} \right)^k (\tau(t))^k dt \\
\leq K \int_0^1 \left( \frac{p''(1-t)}{p(1-t)} \right)^k (\tau(t))^k dt \\
\leq C.
\end{align*}
\]
by (7.3.3).

Next,

\[
\begin{align*}
(7.5.5) \quad L_1 f &= \frac{1}{2} \int_0^1 |g(t)|^k dt \int_{1-t}^1 (1-x)^{k-1} |\phi_1(x,t)| dx \\
&\leq K \int_0^1 \frac{|g(t)|}{t^{2k}} dt \int_{1-t}^1 (1-x)^{3k-1} \left( \frac{p''(x)}{p(x)} \right)^k dx \\
&\leq K \int_0^1 (1-x)^{3k-1} \left( \frac{p''(x)}{p(x)} \right)^k dx \int_{1-x}^1 \frac{|g(t)|}{t^{2k}} dt \\
&= K \int_0^1 (1-x)^{3k-1} \left( \frac{p''(x)}{p(x)} \right)^k dx \int_{1-x}^1 \frac{|g(t)|}{t^{2k+1}} dt \\
&= K \int_0^1 \frac{|g(t)|}{t^{2k+1}} dt \int_{1-x}^1 (1-x)^{3k-1} \left( \frac{p''(x)}{p(x)} \right)^k dx \\
&\leq K,
\end{align*}
\]
by the hypothesis (7.3.1).

Finally,

\begin{align*}
I_1 &= \frac{1}{k} \int_{0}^{\infty} \int_{0}^{1} \left| \varphi(t) \right|^k dx \int_{0}^{1-x} \left| p'(x, t) \right|^k dx \\
&\leq \frac{\hat{k}}{k} \int_{0}^{\infty} \left| \varphi(t) \right|^k dx \int_{0}^{1-x} \left( \frac{\varphi(x)}{p(x)} \right)^k dx \\
&\leq \hat{k} \left\{ \mathcal{H}(\cdot) \right\}^k \leq \hat{k},
\end{align*}

again by the hypothesis of the theorem.

Combining the inequalities (7.5.4), (7.5.5) and (7.5.6), we get

\[ I_1 \leq \hat{k}. \]

Similarly, \( I_2 \) is also finite and this completes the proof of our theorem.