Chapter 6

Super-Resolution using Optimum Wavelet Filter Coefficients and Sparsity Regularization

In the previous chapter we considered non overlapping degradation matrix entries while regularizing the solution. That means, the considered degradation matrix (PSF) of sensor is depending only on the position of the detectors. In practice this is not true since the aperture of the optical device is limited and it introduces diffraction. In this chapter a new learning based approach for SR using the wavelet transform is proposed. The SR algorithm is applied on the reduced dimension obtained by using the principal component analysis (PCA). The novelty of our approach in this chapter lies in designing application specific wavelet basis. We use low and high spatial resolution image pairs, consisting of materials of interest, to estimate the wavelet filter coefficients (basis). An initial estimate of SR is obtained by using these filter coefficients and by learning the high frequency details in the wavelet domain. The final solution is obtained using a sparsity based regularization framework in which image degradation and the sparseness of SR are estimated using the estimated low pass filter coefficients and the initial SR estimate, respectively. The advantage of the proposed algorithm lies in (1) the use of estimated filter coefficients to represent an optimal PSF to model image degradation process, (2) use of sparsity prior to preserve neighborhood dependencies in SR image, and (3) avoiding the use of registered images while learning the initial estimate. Experiments are conducted
on HSI of natural scene as well as on real HSI collected by Airborne Visible/Infrared Imaging Spectrometer (AVIRIS). Visual inspection and quantitative comparison confirm that our method enhances spatial resolution without introducing considerable spectral distortion.

### 6.1 Related Work

The problem of SR has been attempted by many researchers since early 1980. Tsai and Huang [20] were the first to suggest resolution improvement of an image using several downsampled noise free images of the same scene. There are many ways to improve spatial resolution of hyperspectral images (HSIs) such as SR reconstruction using a few low-resolution images [33, 34, 5], pan-sharpening fusion using coincident HR image (e.g. PAN) followed by super-resolution mapping (SRM) [159, 55], SRM after spectral unmixing [24, 26, 60], wavelet based methods [22, 87, 27], etc. Depending on the number of LR images involved, the SR method is called multi-frame [33, 34, 5] or single-frame SR [159, 55, 24, 26, 60]. Multi-frame based SR methods use subpixel shifted LR observations of the same scene to obtain SR results, while single-frame approaches learn the detail information from image database that has large number of HR (high-resolution) or LR (Low resolution) and HR training images. An accurate registration of the low-resolution images is critical in multi-frame SR since the method is based on exploiting the non-redundancy available in the subpixel shifted LR observations. When working using remotely sensed images, many times it is difficult to obtain subpixel shifted LR observations of the same scene, specifically for highly dynamic scenes. Therefore, in remote sensing single-frame SR image mapping has become a popular area of research.

Several SR techniques have been proposed based on single-frame super-resolution mapping (SRM) techniques [159, 55, 24, 26, 60]. SRM based techniques exploit the spatial information by making use of coincident HR image (e.g. PAN image) [159, 55] or unmixing model that describes the spatial distribution of the contents of mixed pixels [24, 26, 60]. An algorithm proposed by Foody in [159] uses a simple regression based approach to enhance the spatial resolution of LR HSI using coincident HR image. Improved result is obtained by using an SR mapping technique in which location of landcover classes are predicted by fitting class membership contours that results in reducing the blockiness
6.1 Related Work

in final SR output. The main limitation of this algorithm is the need of secondary HR image coincident with LR test image. Nguyen et al. [55] used fused image as an additional source of information for SRM using a Hopfield neural network (HNN). Need of secondary HR coincident image is the limitation of this algorithm as well. Besides this, algorithms based on HNN require higher computational time. In a different approach Gu et al. [24] proposed an SR algorithm that uses an indirect approach based on spectral mixture analysis (SMA) and the learning based SRM is performed by using backpropagation neural network (BPNN). A set of HR training images unassociated with test image are used for training the BPNN. An advantage of this method is that no supplementary source of information associated with LR test image is required. In a similar work, Mianji et al. [26] used LR and its downsampled version to train the BPNN. Then learning based SRM is performed after SMA by considering spatial correlation of different materials present in the HSI. Villa et al. [60] used spatial regularization by simulated annealing to perform SRM which is performed after coarse classification using support vector machine and SMA steps.

There are considerable number of techniques in which wavelet decomposition is used to increase the spatial resolution of remote sensing images [22, 87, 27]. These methods are based on the decomposition of the image into multiple levels based on their local frequency contents. The wavelet transform decomposes images into a number of new images each having different spatial resolution. Need of the coincident HR auxiliary information is the main limitation of these methods. Besides, all these methods use fixed wavelet basis like Db4 in their implementation and they require accurate co-registration to achieve acceptable results. Mertens et al. [76] proposed use of predicted wavelet coefficients to obtain SR image. They learn relation between approximate and detail coefficients using training data in neural network, without making any assumption about data distribution. In a recent approach, Li et al. [88] characterized the wavelet coefficients by a mixed Gaussian distribution and the dependencies between the coarser and the finer scale wavelet coefficients were modeled as prior by using the universal hidden Markov model and the problem was solved as an maximum a posteriori (MAP) framework. Recently, learning based SR approaches for single wideband and multiband images have been explored by the researchers to solve the super-resolution problem [89, 90, 91, 92, 93, 94, 95, 96, 26, 79, 97] in which high frequency details are obtained using the training data. These methods
use a database of HR images or LR-HR image pairs in order to learn the high frequency
details for SR. Use of sparsity as a prior for regularization of ill-posed problems has been
validated by many researchers [80, 81, 82, 84, 79, 61].

Most of the earlier research on SR of HSI assumes implicitly or explicitly that each
LR pixel of individual spectral band is obtained as a equally weighted sum of pixels of
corresponding HR spectral band, and they are perfectly aligned with HR pixels. This
means, the point spread function (PSF) of sensor is same over the entire spatial and
spectral region, depending only on the position of the detectors. But, in practice, PSF
depends on various factors of hyperspectral imager such as fill factor of CCD array, camera
gain, zoom factor, imaging wavelength etc. [17]. The effect of diffraction is significant
at higher wavelength in a hyperspectral imager. This results in spatially and spectrally
varying PSF of the degradation function.

In this chapter, we address the problem of single-frame image super-resolution using
learning based approach in wavelet domain, where we obtain high frequency contents from
HR training images unassociated with test image. This eliminates the need of registration
while obtaining these frequencies. Novelty of our approach lies in estimating the wavelet
filter coefficients that takes care of spectrally varying PSF. Here we are not considering
spatially varying PSF, which is quite involved as this requires the estimation of PSF at
every pixel. The estimated filter coefficients are then used to learn high frequency details
in a given band in wavelet domain, obtaining an initial estimate of SR image. The final
SR image is obtained using the sparsity based regularization that has the observation
model constructed using the estimated filter coefficients.

For the estimation of optimum wavelet filter coefficients, LR-HR pairs of HSIs referred
to as the training images can be created in two different ways:

1. by changing the configurations of the hyperspectral imager. For example, optically
varying the width of the observed target strip projected onto the sensor’s slit fa-
cilitates manipulation of the spatial resolution of the system independent of the
spectral resolution of the system [160].

2. by changing the height of the platform since the spatial resolution of HSI depends
on the platform height. For example, a typical mission, mounting AVIRIS on a
NASA aircraft (ER-2), produces a spatial resolution of about 20 meters, but it can
be improved to 5 meters by flying at lower altitudes.

It has to be noted here that acquiring the HR data and LR data is a one time and offline operation. Once a database is created, the LR images captured by hyperspectral imager can be super-resolved using our approach. This is greatly beneficial, as one can capture low spatial resolution HSIs (even though it is capable of capturing HR HSIs) and reduce the memory, transmission bandwidth and power requirements. One may transmit the LR HSIs from the satellites and aircrafts and obtain super-resolved HSIs at the receiver end by using the available database as training images. In order to reduce the computational complexity due to the use of large data base, we use PCA and work with first few principal components only. In our work, filter coefficients are estimated for each of the reduced set of PCA bands. Use of estimated filter coefficients in learning the initial SR as well as in the degradation model incorporates wavelength dependent i.e. spectrally varying PSF while estimating the SR image. Efficacy of the proposed method is tested by conducting experiments on three different data sets, namely single band natural images, a 31-band hyperspectral image of a natural scene captured under controlled illumination and a 224 band AVIRIS remotely sensed data. The results of proposed approach are compared with bicubic interpolation technique [148], learning based SR method of Jiji et al. [161] that uses fixed basis wavelet coefficients Db4, and two recently published SR methods that are based on sparse representation [79, 61].

6.2 Block Diagram Description of the Proposed Approach

The proposed technique of learning based super-resolution of HSI is illustrated in Figure 6.1. Given LR test image and a database of LR-HR HSIs, the proposed technique is implemented using the following steps

1. Form a training database of registered low-resolution and high-resolution HSIs.

2. Reduce dimensionality of the HSIs using PCA. We now have PCA transformed LR-HR images.

3. Estimate the optimum wavelet filter coefficients using a database created in step 2.
4. Use the estimated filter coefficients and obtain the initial estimate of SR using the wavelet transform based learning. Additional HR HSIs are used while learning the initial SR estimate.

5. Use regularization to obtain the final super-resolved image in PCA domain and subsequently the super-resolved HSI is obtained after inverse PCA.

To estimate wavelet filter coefficients to be used in learning high frequency details in step 4, we first need to create database of registered LR-HR pairs of HSIs (which contains materials of interest in sufficient amount) using any one of the approach described in section I. In practice if it is not feasible to capture LR-HR pairs from the imager, one may use only the HR training images and LR images are obtained by simulation. Here we assume that we have access to HR training images to learn the detail information. In case
of nonavailability of LR-HR pairs one can choose an indirect approach to estimate the optimum wavelet filter coefficients from the available training images at one resolution. For example, one can obtain point spread function of the imager, using one of the methods described in [162]. The estimated PSF may then be used to generate the LR images and the resulting LR-HR pairs can be used to estimate wavelet filter coefficients.

Hyperspectral images are composed of large number of spectral bands (e.g., AVIRIS acquires 224 bands). Applying super-resolution technique to each band separately is prohibitive because of high time complexity. Since the spectral content of HSIs are inherently low dimensional, it can be exploited by using PCA, a standard tool for analysis of multivariate data [8]. Hence, we first apply dimensionality reduction on LR and HR training images having B bands in each LR-HR training set using PCA. This transformation incorporates most of the spectral variability of HSI data in first few principal components. We retain only the first $\kappa$ significant eigen vectors of spectral covariance matrix, corresponding to significant eigen values. Since the number of eigen vectors retained is much less compared to the total number of HSI bands ($B$), one cannot reconstruct original hyperspectral image exactly by inverse PCA, thus causing information loss. However, it is reasonable to assume that the spectral signature of the materials/objects of interest is present in sufficient amount in reasonable number of spectral bands. Note that the number of PCA components retained ($\kappa$) is application dependent and it can be increased at the cost of computational speed, and information loss and the reconstruction error may be made arbitrarily small in order to take care of classification accuracy.

The database of LR-HR images in PCA domain can now be used in wavelet based learning to obtain an initial estimate of SR HSI. However, while using discrete wavelet transform (DWT) it is not a good idea to use the conventional basis such as Haar, Daubechies or Coiflet as they are not optimized over the class of images. This motivates us to estimate the wavelet filter coefficients before learning the initial SR estimate. Using the registered LR-HR PCA data sets of training images, optimum filter coefficients are estimated for each PCA band individually. Thus we obtain a total of $\kappa$ sets of wavelet filter coefficients for PCA bands 1 to $\kappa$. These coefficients are then used in wavelet transform based learning to obtain the initial SR estimate for the $\kappa$ test (LR) HSIs which are also in the PCA domain. These filter coefficients are also used to define the PSF/degradation in the observation model that is used in regularization to obtain the final SR image.
To this end our method for SR uses adaptive wavelet basis which is optimized for a group of HSIs. One may note that there is no need of registration while learning initial SR estimate as we use only the HR database while obtaining the initial SR estimate. This gives us freedom to include additional HR training HSIs in the database (i.e., \( N + 1 \) to \( Q \)) after the dimensionality reduction as shown in Figure 6.1. This inclusion would enhance the accuracy of initial SR estimate. Applying inverse DWT gives us SR initial estimates of LR HSIs in PCA domain. To avoid confusion, we use the word “SR” for algorithmic output only (e.g. initial SR, final SR in Figure 6.1), elsewhere we use the word “HR”.

Our method of obtaining initial SR estimate do not consider the contextual dependencies among pixels as it is patch based. This results in artifacts in initial SR image around the patch boundaries. Hence prior knowledge about HSI is utilized in order to obtain better solution. Regularization based on sparsity as prior is performed in order to obtain final solution for each PCA components. As shown in Figure 6.1, observation model and the sparse coefficients are used in regularization. Observation/degradation model is constructed for each LR image using the already estimated \( \kappa \) sets of estimated filter coefficients. A patch based approach is obtaining the sparse coefficients using the initially estimated SR PCA components and they represent the dependence of an SR patch on its nearby patches. Note that though the individual PCA bands are uncorrelated, spatial dependency exists within the pixels of PCA image [163]. Our final cost function being differentiable, a simple optimization technique like gradient descent is used to minimize the same. This results in final SR HSI in the PCA domain. Applying inverse PCA transformation results in super-resolved HSI.

### 6.3 Estimation of Wavelet Filter Coefficients

For the last two decades discrete wavelet transform has become one of the most important tools in the field of image processing. Many researchers have attempted to increase spatial resolution of natural [161, 164, 165, 166] as well as remotely sensed images [87, 22, 124, 27] using DWT. The limitation of these algorithms is the use of a specific type of wavelet transform where the basis is fixed. That is, they use known filter coefficients and hence do not guarantee the optimum performance. In contrast, in the proposed approach, we derive optimal filter coefficients which are then used in learning the initial SR estimate.
6.3 Estimation of Wavelet Filter Coefficients

A number of general conditions and unbounded degrees of freedom can be used to adapt wavelets to a desired signal processing application. Here DWT basis coefficients are not explicitly specified, instead they are computed from the signal (images in our work) itself by computing the impulse response coefficients of a particular wavelet filter. As already mentioned this is done on the reduced set, i.e. after performing PCA.

To start with, a database of N pairs of LR and HR HSIs is created, generating κ LR-HR pairs of principal components. We choose LR-HR database of HSIs to represent materials and objects of interest. The estimation of wavelet filter coefficients is carried out for each PCA component separately. We first take the wavelet transform of HR PCA images of training pairs and perform one level and two level wavelet decomposition for the magnification factor of 2 and 4, respectively. To estimate wavelet filter coefficients, we use the fact that the coarser part of wavelet transformed HR PCA image should be close to the LR PCA image in the mean squared sense. This has to be true for all LR-HR pairs in the database as shown in Figure 6.1. If these coefficients are used for initial estimate learning, it represents a better approximation to the SR.

Although numerous wavelet basis are available, there exist difficulty in finding optimal length wavelet filter basis for image super-resolution application. The discrete Haar wavelet transform having filter length of 2 for low pass (LP) and high pass (HP) has the advantage of having smaller filter length and simpler to compute, but it is not continuous, resulting in introduction of blockiness in the learned SR image. Hence we need to use higher order filter with overlapping response to make it continuous, which helps to preserve continuity in SR image. But, increase in the filter length adds to computational burden with no significant improvement in the performance. As a compromise between computational burden and the performance, we have chosen a design length of 4 for the wavelet filter to suite our requirement in SR algorithm and use the necessary conditions to obtain the optimal coefficients. In [167] Daubechies described a family of filters for wavelet transform (WT) computation. For details of WT one may refer to [167, 168]. Here we describe the procedure to estimate the wavelet filter coefficients for one of the PCA bands of HSI. The same procedure is repeated for all κ bands. For example, if there are 3 significant PCA bands corresponding to κ = 3, then we will have 3 sets of LP as well as HP filter coefficients.

We now explain the mathematical theory for finding our filter coefficients in terms
of a single coefficient and describe how this coefficient can be estimated from the data. Consider an LR PCA image having size of \( M \times M \) and corresponding HR PCA image be of size \( 2M \times 2M \) giving a resolution difference of \( q = 2 \). We write a system of equations that must be solved to find low pass wavelet filter coefficients \( l = (l_0, l_1, l_2, l_3) \). The high pass filter coefficients \( h_0, ..., h_3 \) can then be determined from \( l_0, ..., l_3 \). Considering 1-D case, the wavelet transformation of vector of length \( 2M \) is given by

\[
\text{Transformed vector} = W_{2M} \times \text{Input vector}, \tag{6.1}
\]

where \( W_{2M} \) is the wavelet transformation matrix given by

\[
W_{2M} = \begin{bmatrix}
    l_3 & l_2 & l_1 & l_0 & 0 & 0 & \cdots & 0 & 0 \\
    0 & 0 & l_3 & l_2 & l_1 & l_0 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    l_1 & l_0 & 0 & 0 & 0 & 0 & \cdots & l_3 & l_2 \\
    h_3 & h_2 & h_1 & h_0 & 0 & 0 & \cdots & 0 & 0 \\
    0 & 0 & h_3 & h_2 & h_1 & h_0 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    h_1 & h_0 & 0 & 0 & 0 & 0 & \cdots & h_3 & h_2 \\
\end{bmatrix}_{2M \times 2M}.
\]

The upper half of rows in \( W_{2M} \) matrix perform the low pass filtering operation and generate coarser part of the signal and the lower half performs high pass filtering operation and generates finer details of the signal. Now consider

\[
W_{2M} = \begin{bmatrix} L \end{bmatrix}, \tag{6.2}
\]

where \( L \) and \( H \) are low pass and high pass filter matrices, each of size \( M \times 2M \). Considering the fact that the two-dimensional DWT is separable, 2-D transform \( W_t \) can be performed in two steps, each of which involves a one-dimensional transform operation.
We can find wavelet transform of 2-D image $I$ of size $2M \times 2M$ as

$$W_t = W_{2M} I W_{2M}^T$$

$$= \begin{vmatrix} LIL^T & LIH^T \\ HIL^T & HIH^T \end{vmatrix}. \quad (6.3)$$

Considering left to right multiplication in equation (6.3), it first computes one dimensional transform along columns of image $I$ (i.e. $W_{2M} I$) and then computes one dimensional transform along rows of image $W_{2M} I$ to obtain $W_t$. This results in the coarser version of image $I$ as block $LIL^T$, of size $M \times M$ and the vertical, horizontal and diagonal details of the image $I$ are contained in blocks $LIH^T$, $HIL^T$, and $HIH^T$ respectively, each of size $M \times M$. To derive the filter coefficients, we consider $LIL^T$ and design a transformation matrix $W_{2M}$, which has the desired filter coefficients. $W_{2M}$ is designed as follows. Considering the orthonormality constraints we have

$$\sum_{n=0}^{3} l_n^2 = 1, \quad (6.4)$$

where $l_0 l_2 + l_1 l_3 = 0. \quad (6.5)$

Taking the discrete time Fourier transform (DTFT) of the sequence $l_n, n = 0, 1, 2, 3$ gives us

$$H(\omega) = \sum_{n=0}^{3} l_n e^{jn\omega}. \quad (6.6)$$

Now for low pass response, $H(\omega) = 1$ at $\omega = 0$ and it is 0 at $\omega = \pi$. Using this in equation (6.6) we obtain,

$$H(0) = \sum_{n=0}^{3} l_n = 1 \quad \text{and} \quad (6.7)$$

$$H(\pi) = \sum_{n=0}^{3} (-1)^n l_n = 0. \quad (6.8)$$

But equation (6.7) violets distance preserving property of orthogonal matrices [169]. To
satisfy orthonormality conditions and the low pass condition we must have,

\[ \sum_{n=0}^{3} l_n = \pm \sqrt{2} \quad \text{and} \quad (6.9) \]

to satisfy equation (6.5) we must have

\[ [l_2, l_3]^T = c[-l_1, l_0]^T \text{ for } c \neq 0. \quad (6.10) \]

Equation (6.10) in conjunction with equation (6.8) leads to

\[ l_1 = \frac{1-c}{1+c} l_0 \text{ for } c \neq -1. \quad (6.11) \]

Using equations (6.4), (6.10) and (6.11) we obtain the following set of equations.

\[
\begin{align*}
    l_0 &= \frac{1+c}{\sqrt{2}(1+c^2)} \\
    l_1 &= \frac{1-c}{\sqrt{2}(1+c^2)} \\
    l_2 &= -\frac{c(1-c)}{\sqrt{2}(1+c^2)} \\
    l_3 &= \frac{c(1+c)}{\sqrt{2}(1+c^2)}.
\end{align*}
\quad (6.12)
\]

Since \( c \) can be any real value except -1, we end up with infinite number of solutions. A unique solution for the coefficient \( c \) is obtained by solving the following optimization problem. Let \( Y_{m1,n1} \) be the LR PCA image in the training set and \( B_{m1,n1} \) be the coarser resolution version of the HR PCA images (i.e. \( LIL^T \) in equation (6.3)), where \( m1, n1 = 1, 2, ..., M \) indicate spatial locations. There are \( N \) such LR - HR pairs. We formulate the minimization problem as

\[ \epsilon = \arg \min_{c} \sum_{P=1}^{N} \sum_{m1=1}^{M} \sum_{n1=1}^{M} (Y_{m1,n1}^{(P)} - B_{m1,n1}^{(P)})^2, \quad (6.13) \]

where \( Y^{(P)} \) is the \( P^{th} \) low-resolution PCA training image and \( B^{(P)} \) is the coarser part of the wavelet transformed HR PCA training image number \( P \). Here \( Y_{m1,n1}^{(P)} \) is known and \( B_{m1,n1}^{(P)} \) can be expressed in terms of \( c \). Equation (6.13) is convex, hence simple optimization technique like gradient descent can be used to find optimum value of \( c \), which in turn can be used to determine optimum \( l_0, ..., l_3 \) (see equation (6.12)). Note that in equation (6.13) \( B_{m1,n1} \) is obtained as 2D convolution of each \( 4 \times 4 \) block of HR
image with $L = [l_0, l_1, l_2, l_3]$ i.e.,

$$
B_{m_1,n_1} = l_2^2 I_{m,n} + l_2 l_3 I_{m,n+1} + l_1 l_3 I_{m,n+2} + l_0 l_3 I_{m,n+3} + l_3 l_2 I_{m+1,n} + l_3^2 I_{m+1,n+1} \\
+ l_1 l_2 I_{m+1,n+2} + l_0 l_2 I_{m+1,n+3} + l_1 l_3 I_{m+2,n} + l_1 l_2 I_{m+2,n+1} + l_1^2 I_{m+2,n+2} \\
+ l_0 l_1 I_{m+2,n+3} + l_0 l_3 I_{m+3,n} + l_0 l_2 I_{m+3,n+1} + l_0 l_1 I_{m+3,n+2} + l_0^2 I_{m+3,n+3}, \quad (6.14)
$$

where $m = 2 \times (m_1 - 1) + 1$ and $n = 2 \times (n_1 - 1) + 1$.

Here $I_{m,n}$ represents the intensity of the HR PCA image pixel at $(m, n)$. For a resolution factor of 4, the above equation can equivalently represented as a convolution of low pass filter coefficients with HR image block of size $16 \times 16$, resulting in a total of 256 terms in the right hand side (RHS) of equation (6.14). The high pass filter coefficients can now be obtained, to satisfy orthonormality condition of matrix $W_{2M}$, as

$$
h_0 = l_3; \quad h_1 = -l_2; \quad h_2 = l_1; \quad h_3 = -l_0. \quad (6.15)
$$

It may be noted that the derived LP filter coefficients from equation (6.13) are optimal in the mean squared sense i.e. they minimize the square of the error between the LR and coarser part of the wavelet transformed HR PCA images. Therefore, the corresponding high pass filter should yield better edge details in the SR image. These estimated filter coefficients are used in learning the initial SR PCA estimate as well as in constructing the degradation model.

### 6.4 Learning Initial SR Estimate

The richness of the texture in the real-world images is difficult to derive analytically. Hence learning based approaches work well while obtaining the missing high frequency details. Use of learning based approaches for super-resolving natural as well as remotely sensed images is considered by many of the researchers [89, 90, 91, 92, 93, 94, 95, 96, 26, 137, 97]. In our approach, once the filter coefficients are estimated, our next task is to learn the high frequency details in terms of detail coefficients. The estimated filter coefficients are used in obtaining the detail wavelet coefficients, by taking the wavelet transform of the test image and the training images using these coefficients. This would
6.4 Learning Initial SR Estimate

Figure 6.2: Illustration of learning of detail wavelet coefficients for $q = 2$ using a database of HR PCA images. (a) Two level wavelet decomposition of test PCA image (LR observation). Dotted lines show wavelet coefficients to be learned. (b) Three level wavelet decomposition of HR PCA training images.

minimize the error while learning that may result when using the known basis. Our learning uses only HR HSI in training database as opposed to LR-HR image pairs used by few researchers [161, 165]. For learning purposes, the detail wavelet coefficients are learned for a decimation factor of $q = 2$ and $q = 4$, respectively. Considering a decimation factor 2, we use two level and three level wavelet decomposition for the test and training images, respectively. Figure 6.2 shows the block schematic for learning of wavelet coefficients for one of the test image for $q = 2$. Note that the test and training images are in the PCA domain. Figure 6.2(a) shows the subbands 0 to $VI$ of the LR test image, while the dotted lines show subbands $VII – IX$ that have to be learned. Subband 0 represents the coarser part of the DWT transformed image and subbands $I – III$ and $IV – VI$ represent the vertical, horizontal, and diagonal details. Figure 6.2(b) shows three level wavelet
decomposition of HR training images having subbands \(0^{(r)} - IX^{(r)}, r = 1, 2, ..., Q\). Here subbands \(I - III, IV - VI,\) and \(VII - IX\) represent the vertical, horizontal, and diagonal details in level 3, 2 and 1, respectively. To learn the wavelet coefficients for subbands \(VII\) of LR test image, we compare the coefficients in subbands \(I\) and \(IV\) of the LR test image with that in subbands \(I^{(r)}\) and \(IV^{(r)}, r = 1, 2, ..., Q\) of the HR training images and obtain best match coefficient from subbands \(VII^{(r)}\) of a training image. Similarly, we learn wavelet coefficients for subbands \(VIII\) and \(IX\) of LR test image. Learning procedure is described in detail as below.

Consider an LR test image of size \(M \times M\) pixels. The corresponding HR image is of size \(2M \times 2M\) pixels giving a resolution difference of 2. We have a total of \(Q\) HR training images. In order to learn the wavelet coefficients we exploit the idea of zero tree concept, i.e. in a multiresolution system, every coefficient at a given scale can be related to a set of coefficients at the next coarser scale of similar orientation [170]. Using this idea we follow the minimum mean squared error between the known DWT coefficients of test and training images to learn the unknown detail wavelet coefficients. Suppose \(\phi(i, j)\) is the wavelet coefficients at location \((i, j)\) in subband 0, where \(0 \leq i, j < M/4\) of the LR test image. Corresponding detail coefficients in subbands \(I, II, III\) are at locations \(\phi(i, j + M/4), \phi(i + M/4, j),\) and \(\phi(i + M/4, j + M/4),\) respectively and the coefficients in subband \(IV, V, VI\) correspond to blocks of size \(2 \times 2\) \(\{\phi(p, q + M/2)_{p=2i+1, q=2j},\) \(\{\phi(p + M/2, q)_{p=2i, q=2j+1},\) and \(\{\phi(p + M/2, q + M/2)_{p=2i+1, q=2j+1}\}\), respectively. These coefficients of \(I - VI\) are used in learning the missing \(4 \times 4\) blocks in subbands \(VII - IX\).

For a pixel at \((i, j)\) in the test image at subband 0 following minimization is carried out to pick the missing block of size \(4 \times 4\) in subband \(VII\) that gives us the horizontal edge details.

\[
\epsilon = \arg \min_{l,m,r} \left[ \phi_I(i, j + M/4) - \phi_I^{(r)}(l, m + M/4) \right]^2 \\
+ \left[ \sum_{p=2i}^{2i+1} \sum_{q=2j}^{2j+1} \phi_{IV}(p, q + M/2) - \sum_{l=2i}^{2i+1} \sum_{m=2j}^{2j+1} \phi_{IV}^{(r)}(l, m + M/2) \right]^2, \tag{6.16}
\]

where \(r = 1, ..., Q,\) and \(0 \leq l, m < M/4,\) and \(\phi_I^{(r)}\) and \(\phi_{IV}^{(r)}\) denote the wavelet coefficients for the \(r^{th}\) training image at \(I^{st}\) and \(IV^{rth}\) subbands. The corresponding wavelet detail coefficients from the subband \(VII\) of HR training image are copied into the subband \(VII\).
of the test image. This is repeated for every location in the subbands. In this way we obtain

$$\phi_{VII}(s,t)_{s=11, t=1} := \phi_{(r)}(s_n, t_n)_{s_n=4l+3, t_n=4m+3+M}$$

(6.17)

where \( t_1 = t + M \), \( i_1 = 4i + 3 \), and \( j_1 = 4j + 3 \). This way we end up learning the unknown (missing) wavelet coefficients in subbands \( VII \) of LR test image. To find the vertical and diagonal details, subscripts of \( \phi \) are changed to \( (II \) and \( V \) and \( (III \) and \( VI \) ), respectively, in place of \( (I \) and \( IV \) ), in addition to appropriate displacement of pixel indices by \( M/4 \) or \( M/2 \) in equation (6.16). Similarly, subscripts of \( \phi \) are changed to \( VIII \) and \( IX \) with appropriate displacement of pixel indices by \( M \) in equation (6.17). Essentially we are searching for the best matching horizontal, vertical and diagonal detail coefficient blocks separately. By applying inverse wavelet transformation to this learned image gives the initial SR estimate. Similar procedure is used on all PCA images to obtain the initial SR estimate for every test image in PCA domain. If the error \( (\epsilon) \) is quite large, it signifies that the \( 4 \times 4 \) patch does not have its corresponding HR representation in database. To avoid such spurious learning, we consider the DWT coefficients only when the error \( (\epsilon) \) is less than a chosen threshold. The goodness of the learning depends on how extensive is the training data set. Our database consists of sufficiently large data set in order to avoid large errors.

### 6.5 Final Solution using Regularization

In the DWT based learning process, the detail coefficients are learned from the training set using block based approach. Thus, spatial correlation is not considered while learning these coefficients. Since we choose the high frequency components of each \( 4 \times 4 \) region independently as per the best fit, corresponding SR image lacks any spatial context dependency which may cause an unwanted abrupt variation across the \( 8 \times 8 \) blocks when we consider a resolution factor \( q = 2 \). This necessitates further refinement of the initial estimate to obtain a better solution by using the prior information about the solution. A better solution can be obtained by formulating the problem in regularization framework. For regularization purpose one needs to have a data fitting term and a regularization
6.5 Final Solution using Regularization

term. Hence we first model the image formation in order to obtain the data fitting term.

6.5.1 Observation Model

Let $Y_\beta, (\beta = 1, 2, ..., \kappa)$, be the PCA transformed LR HSI test image of size $M \times M$ and
and $Z_\beta$ be the corresponding SR PCA image of size $qM \times qM$. Assuming a linear model,
for image formation, the LR observation $y_\beta$ can be expressed as

$$y_\beta = A_\beta z_\beta + n_\beta,$$

(6.18)

where $y_\beta$ and $z_\beta$ represent the lexicographically ordered vectors of size $M^2 \times 1$ and
$q^2M^2 \times 1$ respectively, with $z_\beta$ representing the SR vector. $A_\beta$ is the degradation matrix
of size $M^2 \times q^2M^2$ that takes care of degradation that includes aliasing caused as a result
of downsampling. Generally, the degradation matrix used to obtain the aliased pixel
intensities from the HR pixels has the form as mentioned in [126] that has $q^2$ non-zero
entries in every row having values of $1/q^2$. Before we move on, we would like to point
out that in the earlier research works on SR, matrix with fixed entries [126] was used as
a degradation model for all bands of the multispectral and HSIs [23, 13, 25, 61]. This
means, an LR pixel is the average of light intensity that falls on the HR pixels assuming
that the entire area of a pixel is acting as the light sensing area and fill factor for the
CCD array is unity for all spectral bands. In practice this is not true, and incorporation
of improved degradation model leads to better solution.

In our approach, we use the estimated LP wavelet filter coefficients $l_0, ..., l_3$ to con-
struct the degradation matrix $A_\beta$. Instead of considering LR pixel as the averaging of HR
pixels, we represent it as a linear combination of HR pixels i.e., $z_\beta$ weighted appropriately
by using the estimated low pass filter coefficients. In this case for an integer factor of
$q$, matrix $A_\beta$ consists of $q^4$ non-zero elements along each row at appropriate locations.
For a resolution factor of $q = 2$ the values and locations of each element are determined
from equation (6.14), where $I_{m,n}$ and $B_{m1,n1}$ in equation (6.14) correspond to $Z_\beta$ and $Y_\beta$,
respectively in equation (6.18). Thus, the LR intensity represents the weighted average
of the HR intensities over a neighborhood of $q^4$ pixels corrupted with additive noise.
Here the noise $n_\beta$ is the independent and identically distributed (i.i.d.) vector with zero
mean and variance $\sigma^2_n$ and has same size as that of $y_\beta$. It is important to note that
we estimate LP wavelet filter coefficients for each PCA image separately and hence we are using degradation operator optimized for each band. Doing so, makes the estimated entries of matrix $A_\beta$ closer to the true values for the chosen model. Since the obtained filter coefficients represent the close approximation to SR. Besides, the model has overlap of HR pixels, horizontally and vertically as seen from equation (6.14). This relaxes the assumption that LR HSI is strictly defined by the specific detector area only.

### 6.5.2 Sparsity as a Prior

Super-resolution is an ill-posed inverse problem. There are infinite number of solutions to equation (6.18). Hence selection of appropriate model as the prior information and use of regularization helps to obtain better solution. In the field of image processing and computer vision Markov random field (MRF) is the most general model for including the prior information. But the use of MRF often tend to make the solution smooth. This is because MRF is defined on the basis of local dependencies. In recent years, sparse representations of signals have attracted a great deal of attention in signal and image processing researchers. Olshausen and Field [77] proved that a natural image can be represented with a relatively small number of basis functions chosen from over-complete descriptor sets. Compared to methods based on orthonormal transforms or direct time domain processing, sparse representation usually offers better alternate for efficient signal modeling [78]. This motivates us to use sparsity based regularization. The use of sparsity as a prior for SR is explored by many researchers [79, 80, 81, 82]. They use trained dictionaries of HR and LR patches and assume same sparsity for LR and HR patches in order to obtain SR image. Here, sparse coefficients for each LR patch are found using trained LR patch dictionary, and these coefficients are used in generating the HR output. One may use sparsity constraint by directly imposing the condition that the solution should be sparse [80]. In such cases we need to know the sparseness of SR or it has to be learned using dictionary training. In our work we do not require any kind of dictionary training since we are using the dictionary constructed from the initial estimate itself. The sparsity in our work is imposed in a different way as follows.

In our work the sparseness is represented by the weights of the SR patches when a particular SR patch is represented as a linear combination of other patches. Since our
6.5 Final Solution using Regularization

The objective is to preserve spatial correlation, we consider that a patch in SR can be represented as a sparse linear combination of the other patches, mostly nearby. By imposing the condition that final solution should have the same sparseness as the groundtruth, we obtain an SR solution that preserves the spatial dependencies. But to know the true sparse coefficients we need the groundtruth which is not available since it has to be estimated. However, we do have the close approximation to SR in the form of initial estimate and we have made use of the same in obtaining the necessary sparse coefficients.

Suppose, \( \hat{D}_{H_p} \in R^{n \times K} \) represents an over-complete dictionary of \( K \) atoms \((K \gg n)\) formed by considering the patches/blocks in the initially estimated SR PCA image represented as vectors. Let \( x_p \in R^K \) be the sparse approximation over this dictionary. The atoms represent the lexicographically ordered patches in the initial estimate. Then a measurement vector \( \hat{z}_p \in R^n \), a patch of initial estimate, can be represented as a linear combination of a few atoms from the dictionary \( \hat{D}_{H_p} \) i.e., sparse linear combination of other patches in the image. Thus, \( \hat{z}_p \) can be written as \( \hat{z}_p = \hat{D}_{H_p}x_p \), where \( x_p \) has very few nonzero entries i.e. \( x_p \) is sparse. Note that \( \hat{D}_{H_p} \) has column vectors excluding the patch under consideration i.e. \( \hat{z}_p \). Given \( z_p \) and \( \hat{D}_{H_p} \), \( x_p \) can be obtained by solving the \( l_1 \) minimization using standard optimization tool such as linear programming, by posing the problem as

\[
\min_{x_p \in R^K} ||x_p||_{l_1}\text{ subject to } \hat{z}_p = \hat{D}_{H_p}x_p, \tag{6.19}
\]

where \( ||x_p||_{l_1} = \sum_{i=1}^{K} |x_{p_i}| \).

Using the above formulation we find the sparse coefficients for every patch of initial estimate in terms of other patches in the image, for the considered PCA band. To do this, the initial estimate is divided into patches of size 10 \( \times \) 10 and sparse coefficients are found for each patch, in terms of dictionary atoms \((100 \times 1)\) formed from the same image excluding the patch under consideration. For example if our initial estimate has a size of 128 \( \times \) 128 then each atom is of size 100 \( \times \) 1 and there are 169 \( - \) 1 = 168 atoms in dictionary. For simplicity, the example is given for non-overlapped patches, though the overlap is considered while estimating the sparseness and hence final SR. We exploit the fact that the pixel intensities do not vary much within a local neighborhood i.e., neighboring patches
are correlated, exploiting the contextual dependency. In order to remove unwanted abrupt variations across the patches of SR image, we considered patches overlapped horizontally and vertically by 2 pixel rows and columns, respectively while forming the dictionary. Thus all but boundary patches of the image are overlapped with 8 neighboring patches while the boundary patches are overlapped by 5 or 3 neighboring patches depending on whether they belong to border or corner. This results in maintaining spatial dependencies among patches. As an example, for an image sizing of $128 \times 128$, we obtain a total of $K + 1 = 256$ patches each of size $100 \times 1$. Border pixels of last two rows and columns are reflected in order to meet the size requirement of patches.

### 6.5.3 Regularization with Sparsity Coefficients

Considering one of the PCA bands, regularization is carried out as follows. Sparse coefficients obtained for every patch from the initial SR image serve as weights for the final SR image which is now estimated as the unknown atoms of a dictionary. The regularization is obtained using a patch based approach which provides the constraint of sparsity in the final solution. Using a data-fitting term, and sparsity prior term, the cost function for a single patch of PCA band $\beta$ can be written as,

$$
\epsilon_\beta = \arg\min_{z_p} ||y_p - A_p z_p||_2^2 + \lambda ||y_p - A_p D_{Hp} x_p||_2^2,
$$

(6.20)

where $y_p$ is the LR test patch, $x_p$ is sparse coefficient vector which is already estimated, $z_p$ is the SR patch to be estimated, $A_p$ is degradation matrix taking care of aliasing. Here $D_{Hp}$ is the dictionary of SR atoms that has to be estimated and $\lambda$ represents the weightage given to the sparsity term, chosen empirically. Considering overlapped SR patches, equation (6.20) is constructed for each patch and the final cost consists of sum of these. Note that for an LR test patch of size $4 \times 4$ (i.e. $16 \times 1$) we consider HR patch size of $10 \times 10$ (i.e. $100 \times 1$), instead of $8 \times 8$. This is because of the consideration of overlapped patches while constructing dictionary. This results in maintaining correlatedness among SR patches while avoiding abrupt variations at patch boundaries as mentioned in section 6.5.2.

For an image size of $128 \times 128$, with a patch size of $10 \times 10$ (vector of $100 \times 1$) having overlapping of 2 pixels in horizontal and vertical directions, we have 16 patches.
in each direction, giving us a total of 256 patches. Then the dictionary $D_{hp}$ will be of size $100 \times 255$ and the degradation matrix $A_p$ is of $16 \times 100$. Here $A_p$ consists of $q^4$ non-zero elements along each row, whose values and locations are determined using the estimated low pass wavelet filter coefficients for each PCA band separately as per the equation (6.14), where $I_{m,n}$ represents $Z_\beta$ and $B_{m1,n1}$ corresponds to $Y_\beta$. Note that all the patches are processed simultaneously in order to obtain the final solution.

The above cost function is convex. Hence it can be minimized using a simple optimization technique like gradient descent. In order to provide good initial guess and to speedup the convergence, the learned initial estimate is used as initialization. It may be noted that since we are regularizing the PCA transformed images, we expect a better spectral consistency in final solution. Inverse PCA gives us the SR image in spatial domain.

Before we proceed to the discussion on experimental results, we briefly explain the number of comparisons required in learning the initial estimate in our approach as this step adds to overall computational complexity of our approach. Proposed approach is divided into three steps: (i) estimation of optimum wavelet filter coefficients (ii) learning high frequency wavelet coefficients in order to obtain the initial estimate, and (iii) regularization. Estimation of wavelet filter coefficients is one time offline procedure, hence we have not considered it in complexity analysis. Regularization was carried out by using the available code [171], so we do not include it in this discussion. However, we discuss in detail the number of computations required for learning the high frequency wavelet coefficients, since it involves significant number of comparisons. It involves finding the number of comparisons required to obtain the detail wavelet coefficients when using a database of $Q$ HR training images. For a test image of size $M \times M$ we decompose the same into $W$ levels of wavelet transform, resulting in $3W$ subbands corresponding to horizontal (H), vertical (V), and diagonal (D) details and one subband having the coarser information. We need to learn detail coefficients for each of the coefficients in the coarse subband. For each of the coefficients in this subband, the best matching coefficients at finer level in the training database can be searched by comparing $\sum_{u=0}^{W-1} 2^{2u}$ coefficients within each of the detail subbands. Considering subband corresponding to horizontal details, we search for the best matching wavelet coefficients in the entire horizontal subband of all the HR training images. The size of the subband at $W^{th}$ level is $M/2^W \times M/2^W$ and
it consists of $(M/2^W)^2$ coefficients. These coefficients are compared with corresponding horizontal coefficients at all locations ($(M/2^W)^2$) in each of the HR training images $(Q)$ in the database. Thus, the number of comparisons required for learning horizontal details is $Q(M/2^W)^4 \sum_{u=0}^{W-1} 2^{2u}$. Similar comparisons are used to find detail coefficients of vertical and diagonal subbands. Hence the total number of comparisons required for learning all detail coefficients amounts to $3Q(M/2^W)^4 \sum_{u=0}^{W-1} 2^{2u}$. In our experiment of the single band image, we use two level decomposition $(W=2)$ of the test image of size $64 \times 64$ and learn detail coefficients using the database of $100$ training images $(Q=100)$. In this case, the number of comparisons required are $3 \times 100 \times (64/2^2)^4 \times (2^0 + 2^2)$. We would like to mention here that, although this involves significant number of comparisons of $98.304 \times 10^4$, it will not cause computational burden in our case due to the use of high performance computer and because of the process being non iterative.

6.6 Experiments and Result Analysis

In this section we show the effectiveness of the proposed method by conducting experiments on different data sets. Experiments are carried out on: (1) Single band natural images (2) Natural hyperspectral images, and (3) Remotely sensed HSIs (AVIRIS). Due to the lack of availability of the true LR-HR pairs of hyperspectral images, as a simple sanity check, the proposed SR approach is first tested on a single but wide band natural images. Data for this experiment include three sets of size $64 \times 64$, $128 \times 128$ and $256 \times 256$, captured by computer controlled camera. These data sets are used to test the effectiveness of our method in estimating the wavelet filter coefficients and learning of the high frequency details. The data sets used in the experiments on hyperspectral data constitute images with 31 and 224 spectral bands, respectively. Detailed analysis of the results is performed on 224 band AVIRIS HSI. We performed experimentations for $q = 2$ as well as for $q = 4$. Due to the space constraint, we are demonstrating results only for the case of $q = 4$. For all our experiments, the step size for gradient descent algorithm was chosen as $0.01$. The weightage to sparsity term was set as $\lambda = 0.23$ in the regularization equation (6.20) through a trial-and-error procedure for all experiments. Here, one can use a generalized cross-validation technique [156] to identify the optimum value of $\lambda$, but it is computationally expensive and it is specific to a given image only.
Figure 6.3: Randomly selected training image sets from the database. (a) LR images 64 × 64, (b) HR images 128 × 128 (for $q = 2$), and (c) HR images 256 × 256 (for $q = 4$)

We show the visual as well as quantitative comparison for experiments on all the three data sets. For quantifying the results on single band and 31 band natural HSI we used mean squared error (MSE) as a preliminary evaluation index. Definition of this measure is given in chapter 3 section 3.3.1 (see equation 3.15). For remotely sensed hyperspectral data, detailed quantitative evaluation of spatial and spectral fidelity is performed using different measures such as erreur relative globale adimensionnelle de synthese (ERGAS), spectral angle mapper (SAM), and $Q^2_n$. Brief review of ERGAS and SAM is given in chapter 4 section 4.4.1 (see equations 4.9 and 4.8), while that of $Q^2_n$ is given in chapter 5 section 5.5.1 (see equation 5.7 and 5.8).

Table 6.1: Performance comparison showing importance of initial estimate on single band “Ganapati” image for $q = 4$ in terms of MSE between groundtruth and initial SR, and groundtruth and final SR

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Bicubic interpolation as initial SR</th>
<th>Learned SR as initial SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial SR</td>
<td>0.0063</td>
<td>0.0046</td>
</tr>
<tr>
<td>Final SR</td>
<td>0.0055</td>
<td>0.0032</td>
</tr>
<tr>
<td>Reference</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
6.6 Experiments and Result Analysis

Figure 6.4: SR results for $q = 4$ showing importance of initial estimate. (a) LR image of size 64\(\times\)64, (b) Ground truth image of size 256\(\times\)256, (c) Bicubic interpolated image, (d) Learned SR image using estimated wavelet filter, (e) Regularization result when using bicubic interpolated image as initial SR estimate, and (f) Regularization result when using learned SR image as initial SR estimate (Proposed approach)

6.6.1 Experiments on Single band Natural Images

Here we used LR-HR grayscale image pairs captured by varying the optical zoom setting of a simple low cost camera. PCA is not used in this experiment as all the images correspond to single band only. Our database consist of LR images of size 64\(\times\)64 and the HR images of size 128\(\times\)128 and 256\(\times\)256, respectively. An LR image of size 64\(\times\)64 available in the database is used as a test image and the HR images in the database are used to obtain SR for $q = 2$ and $q = 4$, respectively. Note that the true HR of test image is not used in the experiments. They are used for computing the quantitative measure and for visual comparison only. Our database consist of 100 images of different scenes each having three different resolutions, that include indoor as well as outdoor scenes captured at different times. This results in a total of $100 \times 3 = 300$ images in the database. All the scenes are real world images captured by a computer controlled camera. To capture these images, a stable and isolated physical setup was used. Images were captured by triggering the camera using MATLAB program. The time difference of less than one
6.6 Experiments and Result Analysis

Table 6.2: Estimation of wavelet filter coefficients and comparison of MSE between true LR and reconstructed coarser images for single band images

<table>
<thead>
<tr>
<th>Image</th>
<th>q</th>
<th>Estimated c</th>
<th>MSE using Db4 coefficients [161]</th>
<th>MSE using the derived filter coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
<td>2</td>
<td>0.13</td>
<td>0.0099</td>
<td>0.0077</td>
</tr>
<tr>
<td>Text</td>
<td>2</td>
<td>0.22</td>
<td>0.0087</td>
<td>0.0081</td>
</tr>
<tr>
<td>Car</td>
<td>4</td>
<td>−0.01</td>
<td>0.0023</td>
<td>0.0015</td>
</tr>
<tr>
<td>Text</td>
<td>4</td>
<td>0.02</td>
<td>0.0039</td>
<td>0.0024</td>
</tr>
</tbody>
</table>

Table 6.3: Mean squared error comparison for SR results on single band images for $q = 4$

<table>
<thead>
<tr>
<th>Image</th>
<th>Bicubic interpolation [148]</th>
<th>Yang et al. [79]</th>
<th>Initial SR (Db4) Jiji et al. [161]</th>
<th>Initial SR (Proposed)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
<td>0.0079</td>
<td>0.0064</td>
<td>0.0067</td>
<td>0.0058</td>
</tr>
<tr>
<td>Text</td>
<td>0.0097</td>
<td>0.0078</td>
<td>0.0088</td>
<td>0.0074</td>
</tr>
</tbody>
</table>

millisecond was kept between two successive triggers. Mean correction was applied on the captured images to compensate for the illumination variations.

Figure 6.3 shows randomly selected training images in the database. First of all, to demonstrate the effectiveness of initial SR estimate in obtaining the final SR, we show the regularization results obtained by using two different images as initial SR estimates. Figure 6.4(a) and (b) show LR and groundtruth images of Lord “Ganapati”. Figure 6.4(c) shows the upsampled image obtained using the bicubic interpolation while Figure 6.4(d) displays learned initial SR estimate obtained using the proposed approach. When we use the bicubically interpolated image as initial SR estimate to learn the sparsity coefficients and perform regularization we obtain the image shown in Figure 6.4(e). Similarly use of Figure 6.4(d) as the initial SR estimate (which is more closer to groundtruth image) in regularization, we obtain the result shown in Figure 6.4(f). Comparing the images in Figure 6.4(e) and (f), it is clearly observed that the SR image of the proposed method has better details when compared to SR with bicubic interpolated image used as initial estimate. We can see that the $\psi$ shape on the forehead of “Ganapati” is clearly visible in Figure 6.4(f) when compared to that shown in Figure 6.4(e). This may be because of better sparseness obtained using the proposed approach for initial estimate. The benefit of using the learning in SR is also evident from the MSE values listed in Table 6.1. Observe that MSE is closer to reference value of 0, when we use the learned SR image as
initial estimate when compared to the bicubic interpolation.

Now, we analyze the results on the usefulness of filter coefficients estimation. For estimating the filter coefficients we make use of images shown in Figure 6.3 (a) and (b) for $q = 2$ and use images in Figure 6.3 (a) and (c) for $q = 4$. We selected 10 LR-HR image pairs having similar kinds of edge and texture details to estimate the wavelet filter coefficients. Table 6.2 shows the estimated values of $c$ for Car and Text images for $q = 2$ and 4, respectively. Note that $c$ gives us the optimum values of LP and HP filter coefficients as in equations (6.12) and (6.15), respectively. The MSE between the true LR and the coarser part reconstructed using the estimated wavelet filter coefficients and the standard Db4 wavelet for $q = 2$ and $q = 4$ are also given in Table 6.2. From this table we can see that as the resolution factor is changed the values of estimated filter coefficients are also changing and they are also dependent on the image. The MSE for the estimated filter coefficients is less when compared to using the fixed filter coefficients, indicating that estimating the filter coefficients has the advantage.

Finally, we discuss the results on SR for natural images. Figure 6.5 shows the SR results for the experimentation on Car and Text images, respectively for a resolution factor of 4. Figures 6.5(a) and (b) show the LR test images, and groundtruth images, respectively. Images displayed in Figure in 6.5(c) correspond to those expanded using bicubic interpolation. Figure 6.5(d) shows SR images obtained using Yang et al. method [79]. To show the effectiveness of estimated wavelet filter coefficients when compared to fixed Db4 wavelet, we show the initial SR obtained, using Jiji et al. [161] in Figure 6.5(e) and that obtained using the proposed method in Figure 6.5(f). We mention here that learning based method proposed by Jiji et al. [161] uses initial SR obtained using the fixed wavelet basis.

It can be observed that the fading appears in spokes of the wheel in image of Figure 6.5(c) and the blockiness is also visible in the spokes in Figures 6.5(d) and (e). The initial SR obtained using estimated wavelet filter coefficients compares well with the groundtruth as can be observed from Figure 6.5(f). Spreading and blockiness of characters are reduced considerably in the initial SR estimate of proposed approach. This is expected since we learn the wavelet filter coefficients as well as the high frequency details. Note that the artifacts are significantly reduced in the SR images of Figure 6.5(f) which shows the effectiveness of estimated wavelets. The quantitative comparison of these results is
shown in Table 6.3. It is clearly observed that the MSE between the true and the SR using the estimated wavelet in proposed method is significantly less when compared to bicubic interpolation, Yang et al. method [79], and initial SR obtained using fixed basis Db4 wavelet in Jiji et al. method [161].

Table 6.4: Effect of different PSFs on estimation of wavelet filter coefficients for 31-band Natural hyperspectral image for \( q = 4 \) (Here \( \kappa = 3 \))

<table>
<thead>
<tr>
<th>Band</th>
<th>Filtering kernel used</th>
<th>Estimated ( c )</th>
<th>MSE using Db4 coefficients [161]</th>
<th>MSE using estimated coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCA-1</td>
<td>NN</td>
<td>−0.22</td>
<td>0.0014</td>
<td>0.0012</td>
</tr>
<tr>
<td>PCA-2</td>
<td>NN</td>
<td>−0.17</td>
<td>0.0053</td>
<td>0.0044</td>
</tr>
<tr>
<td>PCA-3</td>
<td>NN</td>
<td>−0.18</td>
<td>0.0254</td>
<td>0.0230</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Gaussian*</td>
<td>−0.37</td>
<td>0.0020</td>
<td>0.0012</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Gaussian*</td>
<td>−0.29</td>
<td>0.0033</td>
<td>0.0031</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Gaussian*</td>
<td>−0.32</td>
<td>0.0124</td>
<td>0.0117</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Gaussian**</td>
<td>−0.19</td>
<td>0.0025</td>
<td>0.0017</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Gaussian**</td>
<td>−0.15</td>
<td>0.0222</td>
<td>0.0199</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Gaussian**</td>
<td>−0.17</td>
<td>0.0351</td>
<td>0.0295</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Motion(^1)</td>
<td>−0.20</td>
<td>0.0034</td>
<td>0.0030</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Motion(^1)</td>
<td>−0.23</td>
<td>0.0912</td>
<td>0.0901</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Motion(^1)</td>
<td>−0.19</td>
<td>0.0773</td>
<td>0.0741</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Motion(^2)</td>
<td>−0.26</td>
<td>0.0021</td>
<td>0.0016</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Motion(^2)</td>
<td>−0.24</td>
<td>0.0686</td>
<td>0.0674</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Motion(^2)</td>
<td>−0.29</td>
<td>0.0595</td>
<td>0.0561</td>
</tr>
</tbody>
</table>

Table 6.4: Effect of different PSFs on estimation of wavelet filter coefficients for 31-band Natural hyperspectral image for \( q = 4 \) (Here \( \kappa = 3 \))

<table>
<thead>
<tr>
<th>Band</th>
<th>Filtering kernel used</th>
<th>Estimated ( c )</th>
<th>MSE using Db4 coefficients [161]</th>
<th>MSE using estimated coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCA-1</td>
<td>NN</td>
<td>−0.22</td>
<td>0.0014</td>
<td>0.0012</td>
</tr>
<tr>
<td>PCA-2</td>
<td>NN</td>
<td>−0.17</td>
<td>0.0053</td>
<td>0.0044</td>
</tr>
<tr>
<td>PCA-3</td>
<td>NN</td>
<td>−0.18</td>
<td>0.0254</td>
<td>0.0230</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Gaussian*</td>
<td>−0.37</td>
<td>0.0020</td>
<td>0.0012</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Gaussian*</td>
<td>−0.29</td>
<td>0.0033</td>
<td>0.0031</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Gaussian*</td>
<td>−0.32</td>
<td>0.0124</td>
<td>0.0117</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Gaussian**</td>
<td>−0.19</td>
<td>0.0025</td>
<td>0.0017</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Gaussian**</td>
<td>−0.15</td>
<td>0.0222</td>
<td>0.0199</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Gaussian**</td>
<td>−0.17</td>
<td>0.0351</td>
<td>0.0295</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Motion(^1)</td>
<td>−0.20</td>
<td>0.0034</td>
<td>0.0030</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Motion(^1)</td>
<td>−0.23</td>
<td>0.0912</td>
<td>0.0901</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Motion(^1)</td>
<td>−0.19</td>
<td>0.0773</td>
<td>0.0741</td>
</tr>
<tr>
<td>PCA-1</td>
<td>Motion(^2)</td>
<td>−0.26</td>
<td>0.0021</td>
<td>0.0016</td>
</tr>
<tr>
<td>PCA-2</td>
<td>Motion(^2)</td>
<td>−0.24</td>
<td>0.0686</td>
<td>0.0674</td>
</tr>
<tr>
<td>PCA-3</td>
<td>Motion(^2)</td>
<td>−0.29</td>
<td>0.0595</td>
<td>0.0561</td>
</tr>
</tbody>
</table>

6.6.2 Experiments on Hyperspectral Data Sets

For hyperspectral images, the experiments are conducted on two different data sets. The first set consists of 31-band reflectance images of natural scene, having spectral range of 400nm – 700nm all acquired under the direct sunlight in clear or almost clear sky [149]. HSI cubes available in [172] and [173] are used in this experiment. Cropped region of “Scene 5” of hyperspectral images of natural scenes 2002 [172] is used as test data. Our second HSI data set is comprised of 224-bands of AVIRIS HSI cubes available in [174]. In this case cropped region of an urban area in Moffett Field is used as test data. After discarding few bands having low signal to noise ratio (SNR), 196 bands were used for super-resolving by a factor of 2 and 4, respectively. The band removal was based on
6.6 Experiments and Result Analysis

Table 6.5: Quantitative evaluation measures for SR of 31-band Natural hyperspectral image using different techniques for $q = 4$

<table>
<thead>
<tr>
<th>Quantitative Measures</th>
<th>Bicubic interpolation [148]</th>
<th>Jiji et al. [161]</th>
<th>Zhao et al. [61]</th>
<th>Yang et al. [79]</th>
<th>Proposed approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSE-PCA-1</td>
<td>0.0085</td>
<td>0.0067</td>
<td>0.0071</td>
<td>0.0069</td>
<td>0.0019</td>
</tr>
<tr>
<td>MSE-PCA-2</td>
<td>0.0116</td>
<td>0.0098</td>
<td>0.0126</td>
<td>0.0128</td>
<td>0.0029</td>
</tr>
<tr>
<td>MSE-PCA-3</td>
<td>0.0429</td>
<td>0.0102</td>
<td>0.0122</td>
<td>0.0196</td>
<td>0.0055</td>
</tr>
<tr>
<td>ERGAS</td>
<td>3.7358</td>
<td>2.7877</td>
<td>3.4723</td>
<td>4.3172</td>
<td>2.4568</td>
</tr>
<tr>
<td>SAM(Deg)</td>
<td>6.3577</td>
<td>4.9012</td>
<td>4.8340</td>
<td>6.0791</td>
<td>4.6200</td>
</tr>
<tr>
<td>Q2*</td>
<td>0.9610</td>
<td>0.9774</td>
<td>0.9662</td>
<td>0.9679</td>
<td>0.9810</td>
</tr>
</tbody>
</table>

visual inspection of the images. Care was taken to include bands having spectral range in accordance to that of the test image while creating training data.

The above data sets have high spatial dimensions and hence specific regions are cropped from them and experiments are carried out on the cropped regions. Here, we do not have the true LR-HR pairs of HSIs. Hence the low spatial resolution (LR) images were created from these cropped images by using filtering and downsampling operations. Remaining regions of original as well as other HSI cubes are used to generate training data sets. We used 5 sets of generated LR-HR training pairs cropped from the same HSI cube excluding the test image cube for estimating the filter coefficients. Use of same cube to create training LR-HR pairs ensures inclusion of large number of materials and objects of interest. Additional 15 HR training HSIs were included while estimating the initial HR for both the experiments on HSIs. These images are different for natural HSI and AVIRIS HSI. In order to evaluate the performance of our approach using quantitative measures, we need the groundtruth images. Since these images are not available, we consider original cropped HSIs of size $256 \times 256$ as ground truths and generated the LR HSIs of size $128 \times 128$ and $64 \times 64$ by applying downsampling operation by a factor of $q = 2$ and $q = 4$, respectively. The SR algorithm was then applied on these LR HSIs.

Each set in our training database has one LR image, an HR image with $q = 2$ and an HR image with $q = 4$. In order to restrict the maximum spatial frequency in the image we use low pass filtering operation before downsampling. The low pass filtering operation was performed and tested using five different kernels namely, nearest neighbor (NN), Gaussian filter with standard deviations of 0.5 and 0.8, horizontal motion blur (5 pixels) with no rotation, and with 30 degree rotation. For this purpose we used filter
masks of size $5 \times 5$. Table 6.4 shows the estimated values of $c$ and LR reconstruction error (i.e., error between true LR and LR obtained using the filter coefficients derived from $c$) for PCA bands 1 to 3 for 5 different masks for the 31-band natural HSI. We can see that as we change the filter mask, the estimated filter coefficients also change, thus making it clear that the degradation operation plays significant role while super-resolving. One can see that the values of MSE are significantly reduced when using the estimated filter coefficients. Low value of MSE shows that the use of estimated filter coefficients for SR purposes is better since in general the imaging operation may be modeled in different ways.

Table 6.6: Quantitative evaluation metrics of AVIRIS SR for $q = 4$

<table>
<thead>
<tr>
<th></th>
<th>ERGAS</th>
<th>SAM (Deg)</th>
<th>$Q^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bicubic interpolation [148]</td>
<td>4.9550</td>
<td>8.3634</td>
<td>0.9367</td>
</tr>
<tr>
<td>Jiji et al. [161]</td>
<td>2.8108</td>
<td>6.9554</td>
<td>0.9734</td>
</tr>
<tr>
<td>Zhao et al. [61]</td>
<td>4.8694</td>
<td>7.2628</td>
<td>0.9438</td>
</tr>
<tr>
<td>Yang et al. [79]</td>
<td>3.1952</td>
<td>7.8997</td>
<td>0.9665</td>
</tr>
<tr>
<td>Proposed approach</td>
<td>2.5349</td>
<td>6.5849</td>
<td>0.9773</td>
</tr>
<tr>
<td>Reference</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

We next consider the visual and quantitative assessment of SR using the hyperspectral images. Since Gaussian filter was performing better in terms of MSE, we used it prior to downsampling operation in order to generate LR hyperspectral images. Results of the proposed algorithm on 31-band natural HSI is presented in Figure 6.6 for $q = 4$. After taking the PCA we retained 3 images corresponding to the principal components with highest variance and applied the SR algorithm on them. Figure 6.6 displays the results for the PCA-I. Figure 6.6(a) and (b) display the LR test image and the ground truth image, respectively. The SR results obtained using different methods are shown in Figures 6.6(c-g). SR obtained using sparsity based regularization of the proposed approach shown in Figure 6.6(g) has sharper borders of white colored table on the top left corner. Visual inspection of images in Figures 6.6(c-f) indicate that these borders appear blurred in the bicubic interpolated image, SR image using Jiji et al. method [161], Zhao et al. method [61] and the method proposed by Yang et al. [79]. One can see that the SR image obtained using the proposed method displayed in Figure 6.6(g) compares well with the groundtruth. The letters in the text written on the ball are more clear in Figure 6.6(g). The method proposed by Zhao et al. [61] does not use dimensionality reduction.
algorithm such as PCA. Hence to display the results, PCA is applied on the SR result and the first three PCA components are retained for comparison purpose only. Quantitative comparisons for this experiment are given in Table 6.5. From the table we can see that the MSEs between the true and the estimated SR PCA components are significantly less for the proposed method. The use of estimated filter coefficients and regularization improves the results in our approach as evident from quantitative evaluation measures such as ERGAS, SAM and $Q^2_n$. These measures show that proposed approach better preserves spatial and spectral fidelity in the super-resolved images.

In Figure 6.7 the SR results on remotely sensed data acquired using AVIRIS hyperspectral imager is shown for a specific band 100. Results are listed for the first 3 PCA bands that include 99.3% of spectral variability of HSI. Figure 6.7(a) shows the LR test image and the original HSI band is displayed in Figure 6.7(b). From Figure 6.7(c) we can see that bicubic interpolation blurs the image when upsampled and the high frequency spatial details are lost. One may notice that bicubic interpolation in Figure 6.6(c) appears better than the result in Figure 6.7(c). This is because there is significant high frequency content in AVIRIS data when compared to natural HSI displayed in Figure 6.6 and the interpolation fails to preserve the high frequency details indicating that the interpolation techniques are not suitable for solving the SR problem. Quantitative comparisons given in Table 6.5 and Table 6.6 further proves this observation. Result of Zhao et al.[61] shown in Figure 6.7(e) is less blurred compared to bicubic interpolated image but it shows artifacts and has loss of high frequency details. Jiji et al. [161] approach results in improved visual quality than the interpolation and the method proposed by Zhao et al.[61], but the overall contrast of the image is not preserved as seen from Figure 6.7(d). Sparsity based SR result of Yang et al. [79] method shown in Figure 6.7(f) is visually better than SR obtained using Zhao et al. [61] (see Figure 6.7(e)), but it fails to preserve high frequency details as evident from vertical lines in the middle region of the image.

As seen from Figure 6.7(g), the use of estimated filter coefficients and sparsity regularization results in reduced artifacts and also takes care of preservation of high frequency details. It gives better visual quality closely resembling the groundtruth. We can see that the white patches visible in LR observation appear grayish in Jiji et al. method (See Figure 6.7(d)), but the result is improved in the proposed approach. One can clearly
discriminate the vertical lines appearing in the mid region of the image in Figure 6.7(g) indicating that edge details are very well preserved in the proposed approach.

As far as the quantitative comparison is concerned, it is clear from Table 6.6 that the proposed method provides scores that are closer to the reference values when compared to bicubic interpolation [148], Jiji et al. method [161], Zhao et al. approach [61] and Yang et al. approach [79]. Lower value of ERGAS in the proposed method indicates lesser global distortion in super-resolved HSI. Generally a value of ERGAS below 3 is believed to be an image with good quality [124]. We see that when compared to other approaches the remaining measures such as SAM, \( Q^2_n \) are better for the proposed method. Lower value of \( Q^2_n \) in Table 6.6 indicates minimum spatial as well as spectral distortions by the proposed approach. To further support the performance improvement in terms of spectral fidelity using our approach, we show the SAM error plot in Figure 6.8. The plot depicts the total count of pixels in super-resolved image having specific degree of spectral angle error. We can see that when compared to other methods, the proposed approach has maximum number of pixels having spectral distortion less than 5 degrees. Also the number of pixels having spectral error more than 27 degrees is quite less. Lower values for SAM indicate that the proposed method provides better spectral fidelity.

We would like to point out that compared to the SR method discussed in previous chapters, here the use of estimated wavelet filter coefficients in initial estimate as well as in the decimation matrix entries improves the performance of proposed method. It is to be noted that in the present method we use point spread function (PSF)/degradation with overlapping entries, while in previous method non-overlapping entries were considered to define PSF. Improvement in the performance of this approach is evident from the quantitative measure ERGAS given in Table 6.6 and Table 5.2. The value of ERGAS is 2.5349 for the approach proposed in this chapter while it is 2.9725 for the approach based on learned dictionaries and Gabor priors as discussed in chapter 5. We also found that as far as other quantitative measures are concerned the proposed approach performs better. Thus this method outperforms the other previous methods of super-resolution discussed in this thesis.

Finally, we discuss about the timing complexity of our method considering a resolution factor of 4 for SR of HSIs. All algorithms were implemented using MATLAB 7.0 on Intel(R) Core(TM) i3 CPU M380 having operating frequency of 2.53 GHz. Comparisons
of the running time of all the methods for different input images is given in Table 6.7 for each experiment. First two rows correspond to computation times of single band images, while third and fourth rows correspond to 31-band and 196-band HSIs. It is to be noted that in Zhao et al. [61] approach spectral regularization step is suitable only for HSI, hence experiments are not performed on Car and Text images and corresponding entities in Table 6.7 are left blank. For the proposed method, the time required for estimating the filter coefficients was 15 seconds which is a one time offline procedure. The time taken for detail coefficients learning i.e., initial SR estimate was only 4 seconds. Note that as already explained in section 6.5, this step involved significant number of comparisons and the use of high speed processor reduced the computing burden. Finally, regularization and inverse PCA was done in 27 seconds. Thus the total time required was about 46 seconds to obtain the final SR image as indicated in the table. From the table we see that time taken for SR of 31-band and 196-band HSIs is significantly higher than that of car and text images. This is because here the total time corresponds to super-resolution of more than one image. The difference in computation time for 31-band and 196-band HSIs in the proposed approach is due to the execution time difference in PCA and inverse PCA.

We see that the execution time of bicubic interpolation is much less when compared to other methods. Unfortunately, interpolation fails to retain high frequency details, hence it is not considered as SR technique. Time complexity to estimate SR image in Jiji et al. [161] approach is several hours. This is because the cost function used in their regularization is non-differentiable and hence the use of simulated annealing for cost minimization increases the time complexity. But the time for obtaining the initial estimate in their approach is less when compared to our approach because they do not estimate the filter coefficients. The method proposed by Zhao et al. [61] and Yang et al. [79] are quite expensive in terms of time requirements since the dictionary learning employed in both these approaches takes few hours. Besides, the method of Zhao et al. [61] super-resolve each band separately without using any dimensionality reduction. Hence, time complexity increases as the number of bands increases. For AVIRIS HSI their algorithm takes several hours. This is because the use of sparsity based regularization which is computationally expensive adds to their computational complexity. Another reason for increase in time is due to regularization done separately on each band. However, in the proposed approach computation of sparse coefficients is one time operation and
training of dictionary is also not required. In conclusion, for the proposed approach, spatial and spectral fidelity is better preserved with reduced time complexity. Further improvement in the speed can be achieved by implementing the algorithm using C or C++ using optimized code.

6.7 Conclusion

We have presented an SR approach for hyperspectral images based on the design of an adaptive wavelet basis. We estimate the wavelet filter coefficients using a database and use them in our learning based super-resolution. The decimation matrix entries for each significant spectral band is represented in terms of estimated low-pass wavelet filter coefficients. Use of sparsity based regularization and the use of optimum degradation matrix results in improved quality of super-resolved image when compared to other approaches. The advantage of the proposed technique is that there is no need of registration while learning. Quantitative comparison of score indices indicates that our method enhances spatial information without introducing significant spectral distortions. Experimental results show that the proposed approach outperforms other methods in terms reconstruction quality and computational complexity.

Adaptive wavelet basis will have positive impact on the subsequent hyperspectral image processing applications where high spatial and spectral resolution is desirable. We conclude that the wavelet basis can be tailored to take care of the variability in sensor characteristics. Our future work involves the incorporation of spectral mixing models in order to improve the estimation of filter coefficients and hence the SR.
Figure 6.5: Experimental results on single band images for $q = 4$. (a) LR test image of size $64 \times 64$, (b) Ground truth of size $256 \times 256$, (c) Bicubic interpolation [148], (d) SR image using Yang et al. method [79], (e) Initial SR image using Db4 wavelet [161], and (f) Initial SR estimate using proposed approach.
Figure 6.6: Experimental results on PCA-1 of 31-band natural HSI for $q = 4$. (a) LR test image of size $64 \times 64$, (b) Ground truth of size $256 \times 256$, (c) Bicubic interpolation [148], (d) SR image using Jiji et al. method [161], (e) SR image using Zhao et al. method [61], (f) SR image using Yang et al. method [79], and (g) SR image using the proposed approach.
6.7 Conclusion

Figure 6.7: Experimental results on 100th band of AVIRIS data for $q = 4$. (a) LR test image of size $64 \times 64$, (b) Ground truth of size $256 \times 256$, (c) Bicubic interpolation [148], (d) SR image using Jiji et al. method [161], (e) SR image using Zhao et al. method [61], (f) SR image using Yang et al. method [79], and (g) SR image using the proposed approach.

Figure 6.8: Spectral angle error for AVIRIS HSI for $q = 4$. 